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Meerschaert, Sikorskii : Stochastic models for fractional calculus, 2019

Miller, Ross : An introduction to the fractional calculus and fractional differential equations

Deng, Schilling : Exact asymptotic formulas for the heat kernels of space and time-fractional equations, 2019

Meerschaert, Baeumer : Stochastic solutions for fractional Cauchy problems, 2001

FRACTIONAL CALCULUS

Goal: Introduce basic notions from fractional calculus, and then from the point of view of probability, connect them with <u>anomalous diffusions</u>.

- traditional diffusion equation

$$\frac{\partial u}{\partial t} = L u(x) \qquad L \text{ local operator}$$

$$L = \Delta$$

- fractional diffusion equation (term from mathematical physics) replaces the time/space derivatives with their fractional analogues

  → here we use the term fractional rather freely, since we are going to look at rather general nonlocal derivatives

Rather quickly after introducing differentiation of integer order $\frac{d^n f}{dx^n}$ Leibniz was prompted by L'Hôpital to give sense of the apropriate fractional derivative for $n = \frac{1}{2}$ (1690s) :

$$d^{1/2}(x) = x\sqrt{\frac{dx}{x}}$$

$$\frac{d^{1/2} x}{dx^{1/2}} = \sqrt{x}$$

- Euler, Lagrange, Laplace (integral represent.)
- Lacroix 1810s gives an example of R-L derivative of order 1/2 of the identity :

$$\frac{d^{1/2} x}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

using the Legendre's generalised factorial (gamma function) for $y = x^m$

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \qquad \begin{array}{l} m = 1 \\ n = 1/2 \end{array}$$

· Fourier 1820s uses the integral representation of f

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(a)\,da \int_{-\infty}^{+\infty} \cos[y(x-a)]\,dy$$

and

$$\frac{d^n}{dx^n}\cos[y(x-a)] = y^n \cos\left[y(x-a) + \frac{n\pi}{2}\right]$$

Now taking α instead of n

$$\frac{d^\alpha f}{dx^\alpha}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(a)\,da \int_{-\infty}^{+\infty} y^\alpha \cos\left[y(x-a) + \frac{\alpha\pi}{2}\right] dy$$

· Liouville (1830s)

By noting that

$$D^m e^{ax} = a^m e^{ax}$$

define the fractional derivative of order α as an operator such that

$$D^\alpha e^{ax} = a^\alpha e^{ax}$$

so now we can define $D^\alpha f$ for all functions of the form

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}.$$

Motivated by this he continues to

define the fractional order derivative for $x^{-\beta}$, $\beta > 0$:

$$x^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty a^{\beta-1} e^{-xa}\, da$$

change of variable

$$\Rightarrow D^\alpha x^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty a^{\beta-1} \cdot (-a)^\alpha e^{-xa}\, da$$

$$= \frac{(-1)^\alpha \Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}$$

(& Peacock)

↳ Louville system for $x^{-\beta}$ and Lacroix for x^β give seemingly differing approaches
(& Abel)

⟶ both are now incorporated in a more general system

$$\Gamma(\beta) = \int_0^\infty a^{\beta-1} e^{-a}\, da$$

the gamma function (generalised factorial)

- Here we start with the approach by Grünwald

$$\frac{\partial^n f}{\partial x^n} = \lim_{h\to 0} \frac{\Delta_h^n f(x)}{h^n}$$

right derivative approach

where $\Delta_h f(x) = f(x) - f(x-h)$

and $\Delta_h^n = \Delta_h(\Delta_h^{n-1})$

By iteration

$$\Delta_h^n f(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j f(x-jh)$$

$$= (I-B)^n f(x)$$

where B is the backward shift operator

$$Bf(x) = f(x-h).$$

Now define the fractional difference operator for $\alpha > 0$

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha-j+1)}$$

$$\Delta^\alpha f(x) = (I-B)^\alpha f(x)$$

generalized binomial theorem

$$= \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x-jh)$$

$\hookrightarrow$ $\Delta_h^\alpha \Delta_h^\beta = \Delta_h^{\alpha+\beta}$ iterative form and

$$\frac{d^\alpha f(x)}{dx^\alpha} = \lim_{h\to 0} \frac{\Delta^\alpha f(x)}{h^\alpha}.$$

(GRÜNWALD-LETNIKOV finite difference form)

Grünwald fractional derivative.

↳ no assumptions on f in order to define it, but calculating the limit in concrete cases seems rather difficult

- One can also look at it from the Fourier transform approach:

Since $\mathcal{F}\left(\frac{d^n f}{dx^n}\right)(\xi) = \int_{-\infty}^{+\infty} e^{-i\xi x} \frac{d^n f}{dx^n}(x)\,dx$

$$= (i\xi)^n \mathcal{F}f(\xi)$$

one can define $\frac{d^\alpha f}{dx^\alpha}$ as the Fourier inverse of $(i\xi)^\alpha \mathcal{F}f(x)$

i.e. $\frac{d^\alpha f}{dx^\alpha}(x) = \frac{1}{2\pi}\int e^{i\xi x} (i\xi)^\alpha \mathcal{F}f(\xi)\,d\xi$

first integral form representation

Under suitable conditions
(f & derivatives of f up to some integer order $n > 1+\alpha$ exist and are abs. integrable, f bdd)

Grünwald f.d. $\Longleftrightarrow$ $\mathcal{F}$ f.d.

- Starting with the Grünwald-Letnikov finite difference form we can obtain several integral "representations" or (depending on the conditions imposed on f) variations of the fractional derivative. For $\alpha \in (0,1)$

$$\Delta_h^\alpha f(x) = \sum_{j=0}^\infty \underbrace{\binom{\alpha}{j}(-1)^j}_{(-1)^j \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha-j+1)} \sim \frac{-\alpha}{\Gamma(1-\alpha)} j^{-1-\alpha}} f(x-jh)$$

$$j \to \infty$$

so instead of

$$\frac{\Delta_h^\alpha f(x)}{h^\alpha} = \frac{1}{h^\alpha}\Big[f(x) + \underbrace{\sum_{j=1}^\infty \binom{\alpha}{j}(-1)^j}_{=-1} f(x-hj) \Big]$$

$$= \frac{1}{h^\alpha} \sum_{j=1}^\infty \binom{\alpha}{j}(-1)^j \big[f(x-hj) - f(x) \big]$$

we can consider

$$h^{-\alpha}\sum_{j=1}^{\infty}\left(f(x)-f(x-jh)\right)\frac{\alpha}{\Gamma(1-\alpha)}\,j^{-1-\alpha}$$

$$=\sum_{j=1}^{\infty}\left(f(x)-f(x-jh)\right)\frac{\alpha}{\Gamma(1-\alpha)}\,(jh)^{-1-\alpha}h$$

$$\xrightarrow{h\to 0}\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\left(f(x)-f(x-y)\right)y^{-1-\alpha}\,dy = Lf(x)$$

(considered by Hermann Weyl)

This motivates the so called <u>generator form</u> for $\frac{\partial^\alpha f}{\partial x^\alpha}$ which is given in terms of the generator of the standard α-stable subordinator ($\alpha\in(0,1)$) whose Laplace exponent is

$$\phi(\lambda)=\lambda^{\alpha}=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\left(1-e^{-\lambda y}\right)y^{-1-\alpha}\,dy.$$

This in turn justifies the approximations made in the previous steps, since we know that

$$\mathcal{F}(Lf)(\xi)=(i\xi)^{\alpha}\mathcal{F}f$$

where L is the generator of the α-stable subordinator

For $\alpha \in (1,2)$ the generator form would be

$$D_M^{\alpha} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^{\infty} \left[f(x-y) - f(x) + y f'(x) \right] \frac{dy}{y^{1+\alpha}}$$

↳ generator of the 1-dim spectrally positive α-stable Lévy process, it creeps downwards

$$\mathbb{P}\left(X_{\tau_{[0,x]}} = x \right) > 0$$

due to compensation part in the Lévy-Itô decomposition

(the implicit generalization to every $\alpha > 0$ is due to Marchaud

→ this is why this form is known as the Marchaud fractional derivative)

• If we preform now the integration by parts for the generator form for $\alpha \in (0,1)$, by taking

$$u = f(x) - f(x-y) \qquad (\text{wrt } y)$$
$$dv = \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy$$

we get

$$D_c^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{\partial f}{\partial x}(x-y) \frac{dy}{y^\alpha}$$

which is the Caputo form of the fractional derivative (note that for this equivalence we needed e.g. $f \in C^1$ bounded).

For $\alpha \in (1,2)$ we would get

$$D_c^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)} \int_0^\infty \frac{\partial^2 f}{\partial x^2}(x-y)\, y^{1-\alpha} dy$$

(apply partial integration twice)

$$\alpha = \lfloor\alpha\rfloor + r \qquad = \frac{1}{\Gamma(1-r)} \int_0^\infty \frac{\partial^{\lfloor\alpha\rfloor+1} f}{\partial x^{\lfloor\alpha\rfloor+1}}(x-y)\, y^{-r} dy$$

- What if we can move the derivative on f in the Caputo form ($\alpha \in (0,1)$) outside of the integral? We obtain the RIEMANN-LIOUVILLE FORM

$$D^{\alpha}_{RL} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^{\infty} f(x-y) \frac{dy}{y^{\alpha}}$$

$$= \frac{d}{dx} I^{1-\alpha}_{RL} f(x)$$

In general D^{α}_{C} and D^{α}_{RL} do not need to coincide

$$D^{\alpha}_{C} \mathbb{1}_{(0,\infty)}(x) = 0$$

$$D^{\alpha}_{RL} \mathbb{1}_{(0,\infty)}(x) = \frac{1}{\Gamma(1-\alpha) x^{\alpha}} \qquad x > 0$$

$$= L \, \mathbb{1}_{(0,\infty)}(x)$$

The general relation

$$D^{\alpha}_{C} f(x) = D^{\alpha}_{RL} f(x) - f(0) \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$$

(if $D^{\alpha}_{C} f$ exists it is equal to!)

- Looking at the Caputo fractional derivative for $\alpha \in (0,1)$

$$D_C^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} (f(y) - f(0)) (x-y)^{-\alpha} dy$$

one gets an immediate idea for possible generalisations : instead of the function $\omega(x) = \frac{1}{\Gamma(1-\alpha)} x^{-\alpha}$, $x > 0$

take any unbounded nonincreasing function $\omega : (0, \infty) \to [0, \infty)$ such that the measure μ induced by $-\omega$

$\hookrightarrow \omega(x) = \mu(x, \infty)$

satisfies

$$\int_0^\infty (1 \wedge x) \mu(dx) < \infty$$

(in the case of the Caputo derivative

$$\mu(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1} dx \Big)$$

$$D_{GC}^\alpha f(x) = \frac{d}{dx} \int_{-\infty}^{x} (f(y) - f(0)) \, \omega(x-y) dy$$

$\mu \to$ the Lévy measure of a subordinator

Time - fractional diffusion equation

∂_t^α ... will denote the Caputo derivative D_c^α in time

$\alpha \in (0,1)$

$$\partial_t^\alpha p(t,x) = \Delta p(t,x) = \frac{\partial^2}{\partial x^2} p(t,x)$$

Let us try to approach the solution of this equation using the transform method (Fourier in space, Laplace in time)

In the classical setting

$$\partial_t p(t,x) = \Delta p(t,x)$$

applying the Fourier transform (in space) leads to

$$\partial_t \mathcal{F}p(t,\xi) = -\xi^2 \mathcal{F}p(t,\xi)$$

Now applying the Laplace transform (in time) we arrive to $\rightarrow$ $\mathcal{F}p(\xi,0) = 1$ since $B_0 = 0$

$$s\,\mathcal{L}\mathcal{F}p(s,\xi) - 1 = -\xi^2 \mathcal{L}\mathcal{F}p(s,\xi)$$

$$\Rightarrow \quad \mathcal{L}\mathcal{F}p(s,\xi) = \frac{1}{s+\xi^2}$$

The inverse Laplace transform gives us that

$$\mathcal{F}p(t,\xi) = e^{-\xi^2 t}$$

Fourier transform of the heat kernel for X_t

and the inverse Fourier transform

$$p(t,x) = \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-x^2}{4t}}.$$

How does this procedure work in the fractional setting? Again

$$\partial_t^\alpha \mathcal{F}p(t,\xi) = -\xi^2 \mathcal{F}p(t,\xi).$$

Assuming $\mathcal{F}p(0,\xi)=0$ and applying the LT

$$\boxed{s^\alpha \mathcal{L}\mathcal{F}p(s,\xi) - s^{\alpha-1}} = -\xi^2 \mathcal{L}\mathcal{F}p(s,\xi)$$

$$\mathcal{L}\,\partial_t^\alpha u(s) = \int_0^\infty e^{-st}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{du}{dt}(t-y)\frac{dy}{y^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{dy}{y}\int_0^\infty e^{-st}u'(t-y)\,dt$$

$$= \frac{1}{\Gamma(1-\alpha)}\left(s\mathcal{L}u(s) - u(0)\right)\int_0^\infty e^{-ys}\frac{dy}{y^\alpha}$$

$$= \left(s\mathcal{L}u(s) - u(0)\right)\cdot\frac{1}{\Gamma(1-\alpha)}\cdot\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}$$

$u = \mathcal{F}p(\cdot,\xi)$

$$\Rightarrow \mathcal{L}\mathcal{F}p(s,\xi) = \frac{s^{\alpha-1}}{s^\alpha + \xi^2}$$

Now by inverting the Laplace transform we arrive to the notion of the MITTAG-LEFFLER function which is defined via the power series

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+\alpha j)}$$

which converges absolutely $\forall z \in \mathbb{C}$.

Note that when $\alpha = 1$, $\Gamma(1+j) = j!$

so $E_1(z) = e^z$.

One can show that the LT of $f(t) = E_\alpha(-\lambda t^\alpha)$ is equal to

$$\mathcal{L}f(s) = \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{\Gamma(1+\alpha j)} s^{-\alpha j-1} \Gamma(1+\alpha j)$$

$$= s^{-1} \sum_{j=0}^{\infty} (-\lambda s^{-\alpha})^j = s^{-1} \frac{1}{1+\lambda s^{-\alpha}}$$

$$= \frac{s^{\alpha-1}}{s^\alpha + \lambda} \quad \text{for} \quad s^\alpha > |\lambda|$$

Therefore

$$\mathcal{F}p(t,\xi) = E_\alpha(-\xi^2 t^\alpha).$$

Obviously, this heat kernel won't fall into the class of Lévy or Lévy type kernels $\longrightarrow$ in order to invert it we will need to consider the appropriate stochastic model for this "time-fractional diffusion".

[DS, Corollary 2.4]

Meerschaert / Baeumer

$$p(t,x) \asymp \begin{cases} \dfrac{1}{2\Gamma(1-\frac{\alpha}{2})}\, t^{-\alpha/2}, & |x-y|^{-2/\alpha}\, t \to \infty \\[2ex] |x-y|^{\frac{1-\alpha}{2-\alpha}}\, t^{\frac{-\alpha}{2(2-\alpha)}}\, e^{-c|x-y|^{\frac{2}{2-\alpha}}\, t^{\frac{-\alpha}{2-\alpha}}}, & |x-y|^{-2/\alpha}\, t \to 0 \end{cases}$$

strong result by Saichev, Zaslavsky

One can show that inverting the Fourier transform from the last page we get

$$p(t,x) = \frac{t}{\alpha}\int_0^\infty \frac{1}{\sqrt{4\pi s}}\, e^{\frac{-x^2}{4s}}\, g_\alpha\!\left(\frac{t}{s^{1/\alpha}}\right) \frac{1}{s^{1+1/\alpha}}\, ds$$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi (t/s)^\alpha}}\, e^{\frac{-x^2}{4(t/s)^\alpha}}\, g_\alpha(s)\, ds$$

where $\mathcal{L} g_\alpha(s) = e^{-s^\alpha}$ (density of the stable sub.)