An Introduction to the theory of (non-local) Dirichlet forms

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1 Definitions and examples

1.1 Basic notions

A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is an analytic object that can be used to construct and study certain Markov processes. Dirichlet forms use a quasi-sure analysis, meaning that we are permitted to ignore certain exceptional sets which are not visited by the process, which can sometimes have certain advantages.

A Dirichlet form is a generalization of the energy form $f \mapsto |\nabla f|^2 d\lambda$ introduced in the 19th century in connection to the Dirichlet principle (the solution to the Dirichlet problem minimises the Dirichlet energy).

Let $(X, \mathcal{B}, m)$ be a σ-finite measure space.

Definition 1.1. A *symmetric form* on $L^2(X, m)$ is a function $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ such that

(i) $\mathcal{D}(\mathcal{E})$ is dense in $L^2(X, m)$,

(ii) $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for all $u, v \in \mathcal{D}(\mathcal{E})$,

(iii) $\mathcal{E}(au + v, w) = a\mathcal{E}(u, w) + \mathcal{E}(v, w)$ for all $u, v, w \in \mathcal{D}(\mathcal{E})$ and $a \in \mathbb{R}$,

(iv) $\mathcal{E}(u, u) \geqslant 0$ for all $u \in \mathcal{D}(\mathcal{E})$.

For $\alpha > 0$ denote by $\mathcal{E}_\alpha$ a new symmetric form on $L^2(X, m)$ with domain $\mathcal{D}(\mathcal{E})$

$$\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)_{L^2}, \quad u, v \in \mathcal{D}(\mathcal{E})$$

and note that the forms $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are comparable for different $\alpha, \beta > 0$. Then the space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ is a pre-Hilbert space with inner product $\mathcal{E}_1$. A symmetric form $\mathcal{E}$ is said to be *closed* if $\mathcal{D}(\mathcal{E})$ is complete with respect to the norm induced by $\mathcal{E}_1$. The space $\mathcal{D}(\mathcal{E})$ is then a Hilbert space with inner product $\mathcal{E}_\alpha$ for every $\alpha > 0$.

Definition 1.2. A symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, m)$ is *closable* if

$$(u_n) \subset \mathcal{D}(\mathcal{E}),\ \mathcal{E}(u_n - u_m, u_n - u_m) \xrightarrow{n,m\to\infty} 0,\ ||u_n||_{L^2} \xrightarrow{n\to\infty} 0 \implies \mathcal{E}(u_n, u_n) \xrightarrow{n\to\infty} 0.$$

A symmetric form $(\widetilde{\mathcal{E}}, \mathcal{D}(\widetilde{\mathcal{E}}))$ is an *extension* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ if $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\widetilde{\mathcal{E}})$ and $\mathcal{E} = \widetilde{\mathcal{E}}$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$. We write $\mathcal{E} \leqslant \widetilde{\mathcal{E}}$.

Lemma 1.3. *A symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ possesses a closed extension if and only if $\mathcal{E}$ is closable.*

Proof. $\boxed{\Leftarrow}$ Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be closable and

$$\mathcal{L} = \{(u_n)_n \subset \mathcal{D}(\mathcal{E}) : (u_n)_n \text{ is a } \mathcal{E}_1 - \text{Cauchy sequence}\}.$$

Define $\sim$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ as

$$(u_n)_n \sim (u'_n)_n \;\Leftrightarrow\; \lim_n \mathcal{E}_1(u_n - u'_n, u_n - u'_n) = 0$$

and note that $\sim$ is an equivalence relation with quotient set $\widetilde{\mathcal{F}} := \mathcal{L}/_\sim$. Let

$$\widetilde{\mathcal{E}}(u, v) = \lim_n \mathcal{E}(u_n, v_n),\; u, v \in \widetilde{\mathcal{F}} \tag{1.1}$$

be a symmetric linear form with domain $\mathcal{D}(\widetilde{\mathcal{E}}) = \widetilde{\mathcal{F}}$. We will show that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a closed extension of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (called the smallest closed extension).

First, let's show that the limit in (1.1) exists. Note that for every $(u_n)_n \subset \mathcal{L}$

$$\left|\sqrt{\mathcal{E}_1(u_n, u_n)} - \sqrt{\mathcal{E}_1(u_m, u_m)}\right| \leqslant \sqrt{\mathcal{E}_1(u_n - u_m, u_n - u_m)} \xrightarrow{n,m\to\infty} 0,$$

so the sequence $(\mathcal{E}_1(u_n, u_n))_n$ is a Cauchy sequence in $\mathbb{R}$. Therefore, for all sequences $(u_n)_n$, $(v_n)_n$ in $\mathcal{L}$ the Cauchy-Schwartz inequality implies that

$$\begin{aligned}&|\mathcal{E}(u_n, v_n) - \mathcal{E}(u_m, v_m)| = |\mathcal{E}(u_n, v_n - v_m) + \mathcal{E}(u_n - u_m, v_m)|\\ &\leqslant \mathcal{E}_1(u_n, u_n)^{1/2}\mathcal{E}_1(v_n - v_m, v_n - v_m)^{1/2}\\ &+ \mathcal{E}_1(u_n - u_m, u_n - u_m)^{1/2}\mathcal{E}_1(v_m, v_m)^{1/2} \xrightarrow{n,m\to\infty} 0,\end{aligned}$$

i.e. the sequence $(\mathcal{E}(u_n, v_n))_n$ is Cauchy in $\mathbb{R}$ and therefore convergent.

Next, we will show that the form $\mathcal{E}$ is well defined by (1.1), i.e. that it does not depend on the choice of sequences $(u_n)_n \in [u]$ and $(v_n)_n \in [v]$. By using the same argument as above, we get that for all $(u_n)_n, (u'_n)_n \in [u]$ and $(v_n)_n, (v'_n)_n \in [v]$

$$\begin{aligned}&|\mathcal{E}(u_n, v_n) - \mathcal{E}(u'_n, v'_n)| = |\mathcal{E}(u_n, v_n - v'_n) + \mathcal{E}(u_n - u'_n, v'_n)|\\ &\leqslant \mathcal{E}_1(u_n, u_n)^{1/2}\mathcal{E}_1(v_n - v'_n, v_n - v'_n)^{1/2} + \mathcal{E}_1(u_n - u'_n, u_n - u'_n)^{1/2}\mathcal{E}_1(v'_n, v'_n)^{1/2} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Finally, let's show that $(\widetilde{\mathcal{E}}, \mathcal{D}(\widetilde{\mathcal{E}}))$ is a closed extension of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Obviously, $\mathcal{D}(\mathcal{E}) \subset \widetilde{\mathcal{F}}$ and $\mathcal{E} = \widetilde{\mathcal{E}}$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ (for $u \in \mathcal{D}(\mathcal{E})$ just take a sequence $u_n = u$, $n \in \mathbb{N}$). To show closedness, take a $\widetilde{\mathcal{E}}_1$-Cauchy sequence $(u_n)_n \subset \widetilde{\mathcal{F}}$ and for each u_n an approximate sequence $(u_{n,k})_k \subset \mathcal{D}(\mathcal{E})$. For every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that for all $k \geqslant k_n$, $\widetilde{\mathcal{E}}_1(u_n - u_{n,k_n}, u_n - u_{n,k_n}) < \frac{1}{n}$. One easily shows that the sequence $(u_{n,k_n})_n$ is $\widetilde{\mathcal{E}}_1$-Cauchy (and therefore $\mathcal{E}_1$-Cauchy). Let $u \in \widetilde{\mathcal{F}}$ be such that

$$\widetilde{\mathcal{E}}(u, u) = \lim_{n\to\infty} \mathcal{E}(u_{n,k_n}, u_{n,k_n}).$$

Function u is also the $\widetilde{\mathcal{E}}_1$-limit of the sequence $(u_n)_n$.

$\boxed{\Rightarrow}$ Let $\widetilde{\mathcal{E}} \geqslant \mathcal{E}$ be a closed extension of $\mathcal{E}$ and $(u_n)_n \subset \mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\widetilde{\mathcal{E}})$ such that $||u_n||_{L^2} \xrightarrow{n\to\infty} 0$ and $\widetilde{\mathcal{E}}_1(u_n - u_m, u_n - u_m) = \mathcal{E}_1(u_n - u_m, u_n - u_m) \xrightarrow{n,m\to\infty} 0$. Since $(\widetilde{\mathcal{E}}, \mathcal{D}(\widetilde{\mathcal{E}}))$ is closed, there exists a function $u \in \mathcal{D}(\widetilde{\mathcal{E}})$ such that $\widetilde{\mathcal{E}}_1(u_n - u, u_n - u) \xrightarrow{n\to\infty} 0$. This implies that $||u_n - u||_{L^2} \xrightarrow{n\to\infty} 0$ so $u = 0$. Therefore $\mathcal{E}(u_n, u_n) \xrightarrow{n\to\infty} 0$, i.e. $\mathcal{E}$ is closable. $\square$

Definition 1.4. A symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, m)$ is *Markovian* if for each $\varepsilon > 0$ there exists an increasing Lipschitz function $\phi_\varepsilon : \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$ with Lipschitz constant $K = 1$ such that $\phi_{\varepsilon|[0,1]} = \mathrm{id}$ and

$$u \in \mathcal{D}(\mathcal{E}) \;\Rightarrow \phi_\varepsilon(u) \in \mathcal{D}(\mathcal{E}) \text{ and } \mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \leqslant \mathcal{E}(u,u). \tag{1.2}$$

A *Dirichlet form* is a symmetric form on $L^2(X, m)$ which is closed and Markovian.

Remark 1.5. (a) We say that the unit contraction operates on a symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ if

$$u \in \mathcal{D}(\mathcal{E}),\; v = (u \vee 0) \wedge 1 \;\Rightarrow\; v \in \mathcal{D}(\mathcal{E}),\, \mathcal{E}(v,v) \leqslant \mathcal{E}(u,u). \tag{1.3}$$

A function $v \in L^2(X, m)$ is a normal contraction of $u \in L^2(X, m)$ if

$$|v(y) - v(x)| \leqslant |u(y) - u(x)|,\; |v(x)| \leqslant |u(x)|,\; x, y \in X.$$

We say that normal contractions operate on $\mathcal{E}$ if for every $u \in \mathcal{D}(\mathcal{E})$ and every normal contraction v of u

$$v \in \mathcal{D}(\mathcal{E}),\, \mathcal{E}(v,v) \leqslant \mathcal{E}(u,u). \tag{1.4}$$

Obviously (1.4) $\Rightarrow$ (1.3) $\Rightarrow$ (1.2). One can show that these conditions are equivalent when $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closed.

(b) Suppose that $\mathcal{E}$ is a closable Markovian symmetric form on $L^2(x, m)$. Then its smallest closed extension $\tilde{\mathcal{E}}$ is again Markovian and hence a Dirichlet form (see [1, Theorem 3.1.1.] for the proof using the corresponding resolvents). Note that not all extensions need to be Markovian, but there exists a maximal closed extension which is Markovian.

Example 1.6. Let $D \subset \mathbb{R}^d$ be a domain (open, connected set) and set

$$\begin{aligned} \mathcal{E}(u,v) &= \sum_{i,j=1}^{d} \int_D \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \nu_{ij}(dx) \\ &\quad + \iint_{D \times D \setminus \mathbf{d}} (u(y) - u(x))(v(y) - v(x)) J(dx, dy) \\ &\quad + \int_D u(x) v(x) \kappa(dx), \\ \mathcal{D}(\mathcal{E}) &= C_c^\infty(D), \end{aligned} \tag{1.5}$$

where

(i) $(\nu_{ij})_{1 \leqslant i,j \leqslant d}$ are Radon measures[1] such that for any $\xi \in \mathbb{R}^d$ and any compact $K \subset D$

$$\sum_{i,j=1}^{d} \xi_i \xi_j \nu_{i,j}(K) \geqslant 0,\; \nu_{ij}(K) = \nu_{ji}(K),\; 1 \leqslant i, j \leqslant d.$$

[1] A Radon measure is a Borel measure which is inner regular, outer regular and locally finite.

(ii) J is a positive symmetric Radon measure on $D \times D$ off the diagonal $\mathbf{d}$ such that for any compact set K and open set U, $K \subset U \subset D$

$$\iint_{K \times K \setminus \mathbf{d}} |y-x|^2 J(dx,dy) < \infty \text{ and } J(K, D \setminus U) < \infty.$$

(iii) κ is a positive Radon measure on D.

Under these conditions, the form $(\mathcal{E}, C_c^\infty(D))$ is a Markovian symmetric form.

- $\mathcal{E}(u,v)$ is well defined for $u, v \in C_c^\infty(D)$. One only needs to check that the second term in $\mathcal{E}(u,v)$ is finite. Let U be a relatively compact open set such that $\operatorname{supp} u, \operatorname{supp} v \subset U \subset D$. By decomposing the second term in the definition of $\mathcal{E}(u,v)$ we get

$$\begin{aligned}\iint_{D\times D\setminus \mathbf{d}} \dots &= \iint_{U\times U\setminus \mathbf{d}} \dots + \iint_{U \times D\setminus U} \dots + \iint_{D\setminus U \times U} \dots + \iint_{D\setminus U\times D\setminus U\setminus \mathbf{d}} \dots \\ &= I_1 + I_2 + I_3 + I_4\end{aligned}$$

That the integral I_1 is finite follows by applying the mean value theorem and condition (ii),

$$I_1 = \iint_{U\times U\setminus \mathbf{d}} (u(y)-u(x))(v(y)-v(x))J(dx,dy) \lesssim \iint_{U\times U\setminus \mathbf{d}} |y-x|^2 J(dx,dy) < \infty.$$

Next, by the second part of condition (ii) it follows that I_2 and I_3 are finite,

$$I_2 = \iint_{U\times D\setminus U} u(x)v(x)J(dx,dy) \leqslant ||u||_\infty ||v||_\infty J(\operatorname{supp} u \cup \operatorname{supp} v, D\setminus U) < \infty.$$

Finally, note that $I_4 = 0$.

- Note that the first condition in (i) assures positivity, i.e. $\mathcal{E}(u,u) \geqslant 0$ for $u \in C_c^\infty(D)$. For $\delta > 0$ there exists $n = n(\delta) \in \mathbb{N}$ and cubes $\{C_1, \dots, C_n\}$ of side length δ covering $\operatorname{supp} u$ and $\operatorname{supp} v$. Let η_k be a point from cube C_k and denote $\xi_k^i = \frac{\partial u(\eta_k)}{\partial x_i}$. By the Lebesgue theorem and condition in (i)

$$\sum_{i,j=1}^d \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\nu_{ij}(dx) = \lim_{\delta\to 0} \sum_{k=1}^{n(\delta)} \sum_{i,j=1}^d \xi_k^i \xi_k^j \nu_{ij}(C_k) \geqslant 0.$$

That the bilinear form $\mathcal{E}$ is symmetric follows analogously.

- Next we verify the Markovian property. Let ϕ_ε be a function from Definition 1.4. Then for every $u \in C_c^\infty(D)$, $\phi_\varepsilon(u) \in C_c^\infty(D)$ and

$$\begin{aligned}\mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) &= \sum_{i,j=1}^d \int_D |\phi'_\varepsilon(u(x)|^2 \frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}\nu_{ij}(dx) \\ &+ \iint_{D\times D\setminus \mathbf{d}} (\phi_\varepsilon(u(y)) - \phi_\varepsilon(u(x)))^2 J(dx,dy) \\ &+ \int_D \phi_\varepsilon(u(x))^2 \kappa(dx) \\ &\leqslant \mathcal{E}(u,u),\end{aligned}$$

since $0 \leqslant \phi'_\varepsilon(u(x)) \leqslant 1$, $|\phi_\varepsilon(u(x)) - \phi_\varepsilon(u(y))| \leqslant |u(x)-u(y)|$, $|\phi_\varepsilon(u(x))| \leqslant |u(x)|$.

Let m be a positive Radon measure with supp $m = D$. The Markovian symmetric form $(\mathcal{E}, C_c^\infty(D))$ is not a closed form in $L^2(D, m)$, but it is closable in many cases:

(1) When $\nu_{ij} = \lambda\delta_{ij}$, $J = 0$ and $\kappa = 0$, the form $(\mathcal{E}, C_c^\infty)$ is closable and the extension $(\mathcal{E}, H^1(D))$ is a Dirichlet form. Here, $H^1(D)$ is the Sobolev space of order 1 on D,

$$H^1(D) = \left\{ u \in L^2(D) : \frac{\partial u}{\partial x_i} \in L^2(D),\ 1 \leqslant i \leqslant d \right\},$$

where $\frac{\partial u}{\partial x_i}$ is the weak derivative. For every $\mathcal{E}_1$-Cauchy sequence $(u_n)_n \subset H^1(D)$ there exists a function $u \in L^2(D)$ such that $||u_n - u||_{L^2} \to 0$. Also, since the sequence $(\nabla u_n)_n$ is L^2-Cauchy, there exists a function $v \in L^2(D)$ such that $||\nabla u_n - v||_{L^2} \to 0$. The function v is the weak gradient of u, since for every $\varphi \in C_c^\infty(D)$

$$(v, \varphi)_{L^2} = \lim_n (\nabla u_n, \varphi)_{L^2} = -\lim_n (u_n, \nabla\varphi)_{L^2} = -(u, \nabla\varphi)_{L^2}.$$

This implies that $u \in H^1(D)$ is the $\mathcal{E}_1$-limit of $(u_n)_n$.

(2) When $\nu_{ij}(dx) = a_{ij}(x)dx$, $J = 0$, $\kappa = 0$ and the densities a_{ij} satisfy the uniform ellipticity condition,

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \gtrsim |\xi|^2,\ \xi \in \mathbb{R}^d,\ x \in D,$$

the form $(\mathcal{E}, C_c^\infty)$ is again closable.

(3) When $\nu_{ij} = 0$, $\kappa = 0$ and $J(dx, dy) = j(x, dy)m(dx)$ the form $(\mathcal{E}, C_c^\infty(D))$ is closable. Set

$$\mathcal{F} = \{u \in L^2(D, m) : \mathcal{E}(u, u) < \infty\}.$$

Then the extension $(\mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, C_c^\infty(D))$ is a Dirichlet form. To show closedness, let $(u_n)_n \subset \mathcal{F}$ be a $\mathcal{E}_1$-Cauchy sequence. Since $(u_n)_n$ converges in $L^2(D, m)$ there exists a subsequence $(u_{n_k})_k$ converging m-a.e. to a function $u \in L^2(D, m)$ (the function u can be defined for every $x \in D$ by extending it on this null set by 0). By Fatou's lemma it follows that

$$\begin{aligned}\lim_n \mathcal{E}(u - u_n, u - u_n) &= \lim_n \iint_{D\times D\setminus\mathbf{d}} \lim_k (u_{n_k}(y) - u_n(x))^2 j(x, dy)m(dx)\\ &\leqslant \lim_n \liminf_k \iint_{D\times D\setminus\mathbf{d}} (u_{n_k}(y) - u_n(x))^2 j(x, dy)m(dx) = 0.\end{aligned}$$

1.2 Closed forms and semigroups

Definition 1.7. (a) A family $(T_t)_{t>0}$ of linear operators with domain $L^2(X, m)$ is called a *strongly continuous semigroup* if the following conditions hold:

(i) (semigroup property) $T_tT_s = T_{t+s}$, $t, s > 0$.

(ii) (contraction property) $||T_tu||_{L^2} \leqslant ||u||_{L^2}$, $t > 0$, $u \in L^2(X, m)$.

(iii) $||T_tu - u||_{L^2} \xrightarrow{t\to 0} 0$, $u \in L^2(X, m)$.

(b) A family $(G_\alpha)_{\alpha>0}$ of symmetric linear operators with domain $L^2(X,m)$ is called a *strongly continuous resolvent* if the following conditions hold:

(i) (resolvent equation) $G_\alpha - G_\beta + (\alpha-\beta)G_\alpha G_\beta = 0$.

(ii) (contraction property) $||\alpha G_\alpha u||_{L^2} \leqslant ||u||_{L^2}$, $\alpha>0$, $u \in L^2(X,m)$.

(iii) $||\alpha G_\alpha u - u||_{L^2} \xrightarrow{\alpha\to\infty} 0$, $u \in L^2(X,m)$.

Remark 1.8. Let $(T_t)_{t>0}$ be a strongly continuous semigroup on $L^2(X,m)$.

(a) $(T_t)_{t>0}$ determines a strongly continuous resolvent $(G_\alpha)_{\alpha>0}$ on $L^2(X,m)$ by

$$G_\alpha u = \int_0^\infty e^{-\alpha t} T_t u dt.$$

(b) The generator A of $(T_t)_{t>0}$, defined by

$$\begin{aligned} Au &= \lim_{t\downarrow 0} \frac{T_t u - u}{t} \\ \mathcal{D}(A) &= \{u \in L^2(X,m) : Au \text{ exists as a strong limit}\}, \end{aligned}$$

is a non-positive definite self-adjoint operator. Analogously, one can define A through the corresponding strongly continuous resolvent $(G_\alpha)_{\alpha>0}$ by

$$\begin{aligned} Au &= \alpha u - G_\alpha^{-1} u \\ \mathcal{D}(A) &= G_\alpha(L^2(X,m)). \end{aligned}$$

This is well defined since G_α is invertible,

$$G_\alpha u = 0 \ \Rightarrow \ G_\beta u = 0, \forall \beta > 0 \ \Rightarrow \ u = \beta G_\beta u \xrightarrow{\beta\to\infty} 0,$$

and the definition does not depend on α (follows from the resolvent equation).

Theorem 1.9. *[1, Theorem 1.3.1] There is a one-to-one correspondence between the family of closed symmetric forms $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(X,m)$ and the family of non-positive definite self-adjoint operators $(A,\mathcal{D}(A))$ on $L^2(X,m)$. The correspondence is determined by:*

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \mathcal{D}(\sqrt{-A}) \\ \mathcal{E}(u,v) &= (\sqrt{-A}u, \sqrt{-A}v), \ \ u,v \in \mathcal{D}(\mathcal{E}). \end{aligned}$$

This correspondence can be also characterized by

$$\begin{aligned} \mathcal{D}(A) &\subset \mathcal{D}(\mathcal{E}) \\ \mathcal{E}(u,v) &= (-Au,v), \ \ u \in \mathcal{D}(A), \ v \in \mathcal{D}(\mathcal{E}). \end{aligned} \tag{1.6}$$

By using the spectral representation of $\mathcal{E}$ (through the spectral family associated with the operator $-A$ and (1.6)), one can easily show that $G_\alpha(L^2(X,m)) \subset \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}_\alpha(G_\alpha u, v) = (u,v)_{L^2}, \ \ \alpha>0, \ u \in L^2(X,m), \ v \in \mathcal{D}(\mathcal{E}). \tag{1.7}$$

Given (1.9), the closed symmetric form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ can be directly described in terms of the strongly continuous semigroup $(T_t)_{t>0}$ or resolvent $(G_\alpha)_{\alpha>0}$ corresponding to the non-positive definite self-adjoint operator A. Define the approximation forms $\mathcal{E}^{(t)}$ and $\mathcal{E}^{(\alpha)}$ determined by (T_t) and (G_α) as

$$\begin{aligned} \mathcal{E}^{(t)}(u,v) &= \frac{1}{t}(u - T_t u, v), \quad u,v \in L^2(X,m) \\ \mathcal{E}^{(\alpha)}(u,v) &= \alpha(u - \alpha G_\alpha u, v), \quad u,v \in L^2(X,m). \end{aligned}$$

Corollary 1.10. *([1, Lemma 1.3.4, Theorem 1.4.1.]) The closed symmetric form $\mathcal{E}$ corresponding to A can be defined as*

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \{u \in L^2(\mathbb{R}^n) : \lim_{t \downarrow 0} \mathcal{E}^{(t)}(u,u) < \infty\} \\ \mathcal{E}(u,v) &= \lim_{t \downarrow 0} \mathcal{E}^{(t)}(u,v), \quad u,v \in \mathcal{D}(\mathcal{E}) \end{aligned} \tag{1.8}$$

or analogously as

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \{u \in L^2(\mathbb{R}^n) : \lim_{\alpha \to \infty} \mathcal{E}^{(\alpha)}(u,u) < \infty\} \\ \mathcal{E}(u,v) &= \lim_{\alpha \to \infty} \mathcal{E}^{(\alpha)}(u,v), \quad u,v \in \mathcal{D}(\mathcal{E}) \end{aligned} \tag{1.9}$$

Definition 1.11. A linear operator S with domain $L^2(X,m)$ is called *Markovian* if

$$0 \leqslant u \leqslant 1 \ m\text{-a.e.} \ \Rightarrow \ 0 \leqslant Su \leqslant 1 \ m\text{-a.e.}$$

Lemma 1.12. *([1, Theorem 1.4.1.]) Let $\mathcal{E}$ be a closed symmetric form on $L^2(X,m)$. Then $\mathcal{E}$ is Markovian iff one of the following holds*

(a) *T_t is Markovian for all $t > 0$.*

(b) *αG_α is Markovian for all $\alpha > 0$.*

Take a symmetric Markov process $\mathbb{M} = ((M_t)_{t \geqslant 0}, (\mathbb{P}_x)_{x \in X})$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with values in X and transition probabilities $(p_t(x,\cdot))_{t>0, x \in X}$ on $(X, \mathcal{B}(X))$,

$$p_t(x,B) = \mathbb{P}_x(M_t \in B).$$

The transition probabilities are m-symmetric, i.e.

$$\int_X \int_X u(x)v(y)p_t(x,dy)m(dx) = \int_X \int_X u(y)v(x)p_t(x,dy)m(dx)$$

for all non-negative measurable functions u and v. The family of linear operators $(T_t)_{t>0}$ defined by

$$T_t u(x) = \int_X u(y)p_t(x,dy),$$

for all $u \in L^2(X,m)$ which are bounded. This operator can be extended to an operator on $L^2(X,m)$, because it satisfies the contraction property. Therefore, $(T_t)_{t>0}$ is a Markovian semigroup which is not necessarily strongly continuous. The corresponding semigroup will be strongly continuous if, for example, there exists $\mathcal{L} \subset \mathcal{B}_b(X) \cap L^1(X,m)$ dense in $L^2(X,m)$ such that for all $u \in \mathcal{L}$

$$T_t u(x) \xrightarrow{t \downarrow 0} u(x), \ m\text{-a.e. } x \in X.$$

This is, in turn, satisfied for example if the semigroup is Feller, i.e. $T_t(C_\infty) \subset C_\infty$ and

$$||T_t u - u||_\infty \xrightarrow{t \downarrow 0} 0, \ u \in C_\infty,$$

where $(C_\infty, ||\cdot||_\infty)$ is the space of continuous functions vanishing at infinity, equipped with uniform norm. Under these conditions there exists a unique Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ given by (1.9) corresponding to the symmetric Markov process $\mathbb{M}$.

In general, given a Dirichlet form on $L^2(X)$ it is not possible to construct a Feller transition kernel such that (1.9) holds. This is because functions $T_t u$ are defined m-a.e. for all $u \in L^2(X,m)$ and $t > 0$. But by introducing certain regularity to the Dirichlet form, we are able to choose good representatives of $T_t u$ which allow us to construct a Hunt process outside of some set of capacity zero.

1.3 Regularity

Let X be a locally compact separable metric space and m a positive Radon measure on X with supp $m = D$.

Definition 1.13. A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *regular* if it possesses a *core*, i.e. if there exists a subset $\mathcal{C}$ of $\mathcal{D}(\mathcal{E}) \cap C_c(X)$ such that

(i) $\mathcal{C}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm,

(ii) $\mathcal{C}$ is dense in $C_c(X)$ with respect to the uniform norm.

A core $\mathcal{C}$ of $\mathcal{E}$ is said to be *standard* if it is a dense linear subspace of $C_c(X)$ and $\phi_\varepsilon(u) \in \mathcal{C}$ for every $u \in \mathcal{C}$ and function ϕ_ε from Definition 1.4.

Remark 1.14. Assume that $\mathcal{E}$ is a closable Markovian symmetric form on $L^2(X, m)$ such that $\mathcal{D}(\mathcal{E})$ is a dense subalgebra of $C_c(X)$. Then the smallest closed extension $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ is a regular Dirichlet form possessing $\mathcal{D}(\mathcal{E})$ as its special standard core.

Example 1.15. Recall the closable Markovian symmetric form $(\mathcal{E}, C_c^\infty(D))$ from Example 1.6(1),

$$\mathcal{E}(u, v) = (\nabla u, \nabla v)_{L^2(D)}, \ u, v \in C_c^\infty(D).$$

- Let $H_0^1(D) = \overline{C_c^\infty(D)}^{\mathcal{E}_1}$. Obviously, $(\mathcal{E}, H_0^1(D))$ is the smallest closed extension of $(\mathcal{E}, C_c^\infty(D))$. By the previous remark, $(\mathcal{E}, H_0^1(D))$ is a regular Dirichlet form with a core $C_c^\infty(D)$. When D is additionally regular enough (e.g. a C^1 domain), $H_0^1(D) = \{u \in H^1(D) : tr_{\partial D}(u) = 0\}$, where $tr_{\partial D} : H^1(D) \to L^2(\partial D)$ is the trace operator.

- The extension $(\mathcal{E}, H^1(D))$ is a Dirichlet form, but generally not regular. It is a well known fact that in case $D = \mathbb{R}^d$, $C_c^\infty(\mathbb{R}^d)$ is $\mathcal{E}_1$-dense in $H^1(\mathbb{R}^d)$, and therefore $(\mathcal{E}, H^1(\mathbb{R}^d))$ is a regular Dirichlet form with a core $C_c^\infty(\mathbb{R}^d)$. Using Plancharel's theorem we can give an alternative description of the Dirichlet form $\mathcal{E}$. Given $u, v \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \mathcal{E}(u, v) &= \int_{\mathbb{R}^d} \nabla u \cdot \nabla v dx = (2\pi)^d \int_{\mathbb{R}^d} \widehat{\nabla u} \cdot \overline{\widehat{\nabla v}} d\xi = (2\pi)^d \int_{\mathbb{R}^d} i\xi \cdot \overline{i\xi}\, \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi \\ &= (2\pi)^d \int_{\mathbb{R}^d} |\xi|^2\, \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi \end{aligned} \tag{1.10}$$

A general representation theorem of regular Dirichlet forms is due to Beurling-Deny and LeJan from 1960s (see [1, Section 3.2]).

Theorem 1.16. *Any regular Dirichlet form $\mathcal{E}$ on $L^2(X, m)$ can be expressed as*

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \int_{X \times X \setminus \mathbf{d}} (u(x) - u(y))(v(x) - v(y)) J(dx, dy) + \int_X u(x) v(x) \kappa(dx), \tag{1.11}$$

for $u, v \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$. Here *(i)* $\mathcal{E}^{(c)}$ *is the* local part *of* $\mathcal{E}$*, i.e. a symmetric form with domain* $\mathcal{D}(\mathcal{E}^{(c)}) = \mathcal{D}(\mathcal{E}) \cap C_c(X)$ *which satisfies the strong local property:*

$$\mathcal{E}^{(c)}(u,v) = 0 \textit{ for all } u,v \in \mathcal{D}(\mathcal{E}^{(c)}) \textit{ such that } v \textit{ is constant on } U, \textit{ supp } u \subset U \subset X,$$

(ii) J *is a symmetric positive Radon measure on* $X \times X$ *off the diagonal* $\mathbf{d}$*, called the* jumping measure*,*

(iii) κ *is a positive Radon measure on* X *called the* killing measure.

Such $\mathcal{E}^{(c)}$*,* J *and* κ *are uniquely determined by* $\mathcal{E}$*.*

Remark 1.17. Let D be a domain in $\mathbb{R}^d$. Every closable Markovian symmetric form $(\mathcal{E}, C_c^\infty(D))$ on $L^2(D,m)$ can be uniquely expressed by the form (1.5) from Example 1.6.

2 Potential theory

Dirichlet forms give an axiomatic approach to potential theory, starting with the notion of energy. In this framework one can explore other potential-theoretical notions, e.g. capacities, (equilibrium) potentials, etc.

Let X be a LCS metric space, m a positive Radon measure such that supp $m = X$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a regular Dirichlet form on $L^2(X,m)$.

Definition 2.1. (i) $\mathcal{E}$*-capacity* (1-*capacity*) of a set is defined in the following way; for an open set $U \subset \mathbb{R}^n$

$$\mathrm{Cap}_{\mathcal{E}}(U) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{D}(\mathcal{E}),\ u \geqslant 1\, m\text{-a.e. on } U\},$$

and for $A \subset \mathbb{R}^n$ arbitrary set

$$\mathrm{Cap}_{\mathcal{E}}(A) = \inf\{\mathrm{Cap}_{\mathcal{E}}(U) : A \subset U \text{ open }\}.$$

(ii) We say that a statement depending on $x \in A$ holds $\mathcal{E}$-*quasi-everywhere* (q.e.) on A if there exists a set $N \subset A$ of zero $\mathcal{E}$-capacity such that the statement is true for every $x \in A \setminus N$.

(iii) Let u be a real valued function defined q.e. on X. We call u *quasi continuous* if for any $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\mathrm{Cap}_{\mathcal{E}}(G) < \varepsilon$ and $u_{|X\setminus G}$ is continuous.

(iv) A functions v is said to be a *quasi continuous modification* of a function $u \in L^2(X,m)$ if v is quasi continuous and $v = u$ m-a.e.

(v) A sequence $\{F_k\}_{k\in\mathbb{N}}$ of closed increasing sets such that $\mathrm{Cap}_{\mathcal{E}}(X \setminus F_k) \downarrow 0$, $k \uparrow \infty$ is called a *nest* on X.

Remark 2.2. (a) The present notion of capacity enables us to think of exceptional sets finer than sets of measure m zero. This is because, by definition, $m(A) \leqslant \mathrm{Cap}_{\mathcal{E}}(A)$ for all open $A \subset X$ such that $\{u \in \mathcal{D}(\mathcal{E}),\ u \geqslant 1\, m\text{-a.e. on } A\} \neq \emptyset$.

(b) By [1, (2.1.6)] capacity of any Borel set A can be calculated as

$$\mathrm{Cap}_{\mathcal{E}}(A) = \sup\{\mathrm{Cap}_{\mathcal{E}}(K) : K \subset A,\ K \text{ is compact}\}.$$

(c) Capacity $\mathrm{Cap}_{\mathcal{E}}$ is a Choquet capacity, i.e. increasing, continuous w.r.t. increasing sequences of sets, continuous w.r.t. decreasing sequences of compact sets.

Theorem 2.3. *[1, Theorem 2.1.3] Every function $u \in \mathcal{D}(\mathcal{E})$ admits a quasi-continuous modification $\tilde{u}$.*

Proof. For $u \in \mathcal{D}(\mathcal{E}) \cap C(X)$ and $\lambda > 0$ set $G = \{x \in X : |u(x)| > \lambda\}$. Note that G is open and that $\frac{|u|}{\lambda} \geqslant 1$ on G. Using the normal contraction property (1.4) we get that

$$\mathrm{Cap}_{\mathcal{E}}(G) \leqslant \mathcal{E}_1\left(\frac{|u|}{\lambda}, \frac{|u|}{\lambda}\right) \leqslant \frac{1}{\lambda^2}\mathcal{E}_1(u,u). \tag{2.1}$$

Now let $u \in \mathcal{D}(\mathcal{E})$. Since $\mathcal{E}$ is regular, there exists a sequence $(u_n)_n \subset \mathcal{D}(\mathcal{E}) \cap C_c(X)$ such that $\mathcal{E}_1(u_n - u, u_n - u) \to 0$. Without loss of generality we can assume that

$$\mathcal{E}_1(u_n - u_{n+1}, u_n - u_{n+1}) < 2^{-3n}.$$

Define a sequence of sets $G_n = \{x \in X : |u_n(x) - u_{n+1}(x)| > 2^{-n}\}$, $n \in \mathbb{N}$. Then by (2.1),

$$\mathrm{Cap}(G_n) \leqslant 2^{-2n}\mathcal{E}_1(u_n - u_{n+1}, u_n - u_{n+1}) < 2^{-n}.$$

A sequence of sets $\{F_n\}_n$, $F_n := \bigcap_{k=n}^{\infty} G_k^c$, is a nest and for all $k, m > N \geqslant n$ and $x \in F_n$

$$|u_k(x) - u_m(x)| \leqslant \sum_{i=N+1}^{\infty} |u_i(x) - u_{i+1}(x)| < \frac{1}{2^N}.$$

This means that for each $n \in \mathbb{N}$ the sequence $(u_k)_k$ is uniformly convergent. Set $F = \bigcup_{n=1}^{\infty} F_n$ and define

$$\widetilde{u}(x) = \lim_{n\to\infty} u(x),\ x \in F.$$

Since $\mathrm{Cap}_{\mathcal{E}}(F) = 0$, $\widetilde{u}_{|F_n}$ is continuous and $u = \widetilde{u}$ m-a.e., $\widetilde{u}$ is a quasi-continuous modification of u. □

3 Regular Dirichlet forms and symmetric Markov processes

Definition 3.1. A $\mathbb{F}$-adapted stochastic process $\mathbb{M} = ((M_t)_{t\geqslant 0},\ (\mathbb{P}_x)_{x\in X})$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with state space X is a *Hunt process* (w.r.t. the right-continuous minimal completed admissible filtration $\mathbb{F} = (\mathcal{F}_t)_{t>0}$) if the following hold:

(i) $x \to \mathbb{P}_x(X_t \in B)$ is measurable for all $t > 0$ and $B \in \mathcal{B}(X)$,

(ii) $\mathbb{M}$ is a strong Markov process, i.e. for every stopping time T, M_T is $(\mathcal{F}_T, \mathcal{B}(X))$-measurable and for every $B \in \mathcal{B}(X)$

$$\mathbb{P}_x(M_{T+t} \in B|\mathcal{F}_T) = \mathbb{P}_{M_T}(M_t \in B)\ \mathbb{P}_x\text{-a.s. on } \{T < \infty\},$$

(iii) $\mathbb{M}$ is right-continuous, i.e.

$$\lim_{s\downarrow t} M_s = M_t,\ \forall t\ \mathbb{P}_x\text{-a.s.}$$

(iv) $\mathbb{M}$ is quasi left-continuous, i.e. for all stopping times T and $(T_n)_n$ such that $T_n \uparrow T$ a.s.

$$\lim_{n\to\infty} M_{T_n} = M_T, \ \mathbb{P}_x\text{-a.s. on } \{T < \infty\}.$$

Remark 3.2. (a) Note that quasi left-continuity does not imply left-continuity, because the set

$$A = \left\{ \lim_{s_n \uparrow t} M_{s_n} = M_t \right\}$$

depends on the choice of sequence $(s_n)_n$, $s_n \uparrow t$.

(b) If $(\mathbb{P}_x(M_t \in \cdot))_{x,t}$ are sub-probability measures, we can preform a one-point compactification of X by introducing a cemetery state $\partial \notin X$ and redefine P_x to be a probability measure on $X \cup \{\partial\}$. See, for example [1, Section 7.2].

(c) The transition probability measures $(p_t(x, \cdot))_{t \geqslant 0, x \in X}$ for $\mathbb{M}$ are given by

$$p_t(x, B) = \mathbb{P}_x(M_t \in B), \ t > 00, \ B \in \mathcal{B}(X), \ x \in X.$$

(d) We only consider symmetric Hunt processes with transition probabilities which are m-symmetric, i.e.

$$\int_X \int_X u(x)v(y)p_t(x, dy)m(dx) = \int_X \int_X u(y)v(x)p_t(x, dy)m(dx)$$

for all non-negative measurable functions u and v. The family of linear operators $(T_t)_{t>0}$ defined by

$$T_t u(x) = \int_X u(y)p_t(x, dy),$$

for all $u \in L^2(X, m)$ which are bounded. This operator can be extended to an operator on $L^2(X, m)$, because it satisfies the contraction property. One can show that this is a strongly continuous semigroup. We can also define the corresponding strongly continuous resolvent in the following way

$$G_\alpha f(x) = \int_0^\infty e^{-\alpha t} T_t f(x) dt = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha t} f(M_t) dt \right], \ f \in L^2(X, m), \ \alpha > 0.$$

(e) For $A \in \mathcal{F}$ and a nonnegative measurable function h such that $||h||_{L^1(X,m)} = 1$ let $\mathbb{P}_{h \cdot m}$ be the probability measure with respect to the initial distribution $h(x)m(dx)$,

$$\mathbb{P}_{h \cdot m}(A) = \int_X \mathbb{P}_x(A) h(x) m(dx).$$

Definition 3.3. Two symmetric Hunt processes $\mathbb{M}^{(1)}$ and $\mathbb{M}^{(2)}$ are *equivalent* if their transition probabilities $(p_t^{(1)}(x, \cdot))_{t,x}$ and $(p_t^{(2)}(x, \cdot))_{t,x}$ coincide outside of a common *properly exceptional set* N, i.e. a set such that $m(N) = 0$ and

$$T_t^{(i)}(u 1_{N^c}) = 1_{N^c} T_t^{(i)} u \quad m\text{-a.e.}$$

for any $u \in L^2(X, m)$ and $i = 1, 2$.

Theorem 3.4. *([1, Theorem 4.2.8, Theorem 7.2.1]) Given a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ there exists a symmetric Hunt process $\mathbb{M}$ with Dirichlet form $\mathcal{E}$. Two symmetric Hunt processes $\mathbb{M}^{(1)}$ and $\mathbb{M}^{(2)}$ possessing a common regular Dirichlet form are equivalent.*

References

[1] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes.* de Gruyter, 2nd edition, Berlin, 2010.