Searching for a clique in a huge sparse graph

Vinko Petričević

JOSIP JURAJ STROSSMAYER UNIVERSITY OF OSIJEK

Faculty of Electrical Engineering, Computer Science and Information Technology Osijek

e-mail: vinko.petricevic@ferit.hr

Introduction

A clique is a fully connected subset of an undirected graph. Finding the largest clique is NP complete problem (it takes exponential time to solve the problem).

However, if we know that the largest clique is not too large, then it is a polynomial problem. If the graph is also *very sparse*, then it does not have to be a polynomial of high degree, and it can be solved up to some predetermined limits.

Let's look at an ancient problem: *If in the set of numbers* $\{1, 3, 8\}$ *we multiply each of its two different elements and add the number* 1 *to the product, the result will be the square of a number:*

$$1 \cdot 3 + 1 = 2^2, \qquad 1 \cdot 8 + 1 = 3^2, \qquad 3 \cdot 8 + 1 = 5^2.$$

The curiosity of mathematicians has been tickled by similar sets for millennia. Solving such problems, the coming centuries gave birth to many new mathematical disciplines. With the advent of computers and supercomputers, many new interesting sets have been found (and are still being found).

Diophantine sets

For an integer n, a set of m distinct nonzero integers with the property that the product of any two of its distinct elements plus n is a square, is called a Diophantine m-tuple with the property $D(n)$ or $D(n)$-m-tuple. The $D(1)$-m-tuples (with rational elements) are called simply (rational) Diophantine m-tuples, and have been studied since the ancient time.

The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus. Fermat found the first Diophantine quadruple in integers $\{1, 3, 8, 120\}$. Euler proved that there exist infinitely many rational Diophantine quintuples; in particular, he was able to extend the integer Diophantine quadruple found by Fermat to the rational quintuple

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

In 1969, using linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport proved that if d is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then d has to be 120 (see [3]). This result motivated the conjecture that there does not exist a Diophantine quintuple in integers. The conjecture has been proved recently by He, Togbé and Ziegler (see [19]).

On the other hand, it is not known how large can a rational Diophantine tuple be. In 1999, Gibbs [18] found the first example of rational Diophantine sextuple

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

In 2017, Dujella, Kazalicki, Mikić and Szikszai (see [9]) proved that there are infinitely many rational Diophantine sextuples. Recently, Dujella, Kazalicki and Petričević in [13] proved that there are infinitely many rational Diophantine sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares, and in [11] they proved that there are infinitely many Diophantine sextuples containing two regular quadruples and one regular quintuple. No example of a rational Diophantine septuple is known.

Sets with $D(n)$ properties have also been extensively studied. It is easy to show that there are no integer $D(n)$-quadruples if $n \equiv 2 \pmod 4$, and it is know that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there is at least one $D(n)$-quadruple [6]. Recently, Bonciocat, Cipu and Mignotte (see [2]) proved that there are no $D(-1)$-quadruples (as well as $D(-4)$-quadruples) thus leaving the existence of $D(n)$-quadruples in the remaining six sporadic cases open.

Dražić and Kazalicki [4] described rational $D(n)$-quadruples with fixed product of elements in terms of points on certain elliptic curves. It is not known if there is a rational Diophantine $D(n)$-quintuple for every n, and no example of rational $D(n)$-sextuple is known if n is not a perfect square.

One can also study m-tuples that have $D(n)$-property for more than one n. Adžaga, Dujella, Kreso and Tadić [1] presented several families of Diophantine triples which have $D(n)$-property for two distinct n's with $n \neq 1$ as well as some Diophantine triples which are $D(n)$-sets for three distinct n's with $n \neq 1$. Dujella and Petričević in [14] proved that there are infinitely many (essentially different) integer quadruples which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples with $n_1 \neq n_2$, and in [15] showed that the same thing is true for three distinct n's (since the elements of their quadruples are squares one of n's is equal to zero). Dujella, Kazalicki and Petričević in [10] proved that there are infinitely many $D(n)$-quintuples with square elements, while Dujella, Franušić and Petričević (see [7]) showed that there exists some quituples in $\mathbb{Z}[\sqrt{D}]$.

For more information on Diophantine m-tuples see the new book of prof. Dujella: *Diophantine m-tuples and Elliptic Curves* [5].

Associated graph

In order to find (up to some predetermined limit) D-sets with m elements, it is natural to first find sets with $m-1$ elements. In order not to find the same D-pair more than once, it is natural to imagine (and remember) pairs as edges in a graph.

In the initial example, we have a graph with edges $(1, 3)$, $(1, 8)$ and $(3, 8)$. However, in such a graph there would be, for example, also edges $(2, 4)$, $(1, 15)$, $(3, 5)$ and many others. In any case, the largest clique corresponds to the largest Diophantine set.

In some Diophantine sets, matching pairs are not easy to find (but GPU cards can often help us here), so the graph is not too big, and even the largest cliques are not difficult to find ([17], [8]).

Sometimes it is not too difficult to find pairs, even the largest cliques, but we have to generate a lot of different graphs ([16], [14], [15], even [10]).

In some cases, matching pairs are not difficult to find, but that's why the matching graph becomes huge very quickly ([13], [11], [12], and the last part of [10]).

In all these cases, we have great help from supercomputers, and for huge graphs such calculations would be impossible without *High Performance Computing*.

An example of a calculation

If for fix n we would like to search for integer $D(n)$ sets, i.e. search for pairs $k, l \in \mathbb{N}$ such that $k \cdot l + n = x^2$, we should go through certain interval of numbers x, and connected all divisors of $x^2 - n$ in the graph and then searched for the largest cliques.

When we are searching for rational sets, x goes through all rational numbers (up to a predetermined size of numerator and denominator), we need to find all divisors of numerator and denominator of $x^2 - n$. But now for every $\frac{k_p}{k_q} \cdot \frac{l_p}{l_q} = x^2 - n$, not only $\frac{k_p}{k_q}$ and $\frac{l_p}{l_q}$ makes a pair, but also all the numbers obtained by multiplying the numerator of the first fraction and the denominator of the second by some number i, and the denominator of the first and the numerator of the second fraction by another number j (specially, they should be coprime). In any case, the pair $k, l \in \mathbb{Q}$ will generate an infinite number of pairs, so even here we generate them up to some predetermined limit.

For example, if we search up to 2^{17} (that is, if we also look at negative numbers, and discard the fractions that will be shortened, there will be $\approx 2\frac{3}{\pi^2}(2^{17})^2 \approx 10.5$ billions), for different n, the graph will have from a few tens of millions to almost a billion edges. If we write each edge with 4 `longs`, the graph itself takes up, for example, 30TB of data on storage. After that, the data needs to be sorted and organized more intelligently, then it takes up, for example, about 20TB. Since we don't have a computer with 20TB of memory (and a little more), that data still needs to be divided into blocks of, for example, of 1.5TB (if we want to process the data on a computer with 4TB of working memory), and process each block with each one. Fortunately, on such HPC systems, the data storage is extremely fast, so this is almost like using memory. In any case, for example on HPC *Bura* or Supek, the largest such calculations lasts about a week.

If we increase the power by 1 (in this case up to 2^{18}), depending on the problem we are looking at, we will need about five times more storage and about six times more time for the calculation.

In contrast to the mentioned procedure, in the paper [10] we obtained the pairs by finding for $b \in \mathbb{N}$ all Pythagorean triangles with the length of one leg b, and then connecting the divisors of the remaining leg. After finding all cliques, we go to the next b. And only in the calculation at the end of this paper we combined all these graphs into one huge graph, and search for the largest clique using the same procedure.

Future works

In addition to the above results, we also found a large number of rational $D(1)$ almost-septuples (only one edge is missing to be a clique), but not a single septuple. We also calculated a large number of ratinal $D(n)$-quintuples, a lot of near-sextupes, but not a single sextuple. We have also found many other interesting examples that have yet to be studied, and the infrastructure allows us to count up to considerably larger numbers, and we have no doubt that there will be many more good results.

References

[1] N. Adžaga, A. Dujella, D. Kreso and P. Tadić, *Triples which are D(n)-sets for several n's*, J. Number Theory **184** (2018), 330–341.

[2] N. C. Bonciocat, M. Cipu and Maurice Mignotte, *There is no Diophantine $D(-1)$–quadruple*, J. London Math. Soc. **105** (2022), 63–99.

[3] A. Baker and H. Davenport, *The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[4] G. Dražić and M. Kazalicki, *Rational $D(q)$-quadruples*, Indag. Math. (N.S.), Volume **33**, Issue 2, March 2022, 440–449.

[5] A. Dujella, Diophantine m-tuples and Elliptic Curves, Springer, Cham, 2024.

[6] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65** (1993), 15–27.

[7] A. Dujella, Z. Franušić, V. Petričević, *Formulas for Diophantine quintuples containing two pairs of conjugates in some quadratic fields*, Period. Math. Hungar. **85** (2022), 303–311.

[8] A. Dujella, I. Gusić, V. Petričević and P. Tadić, *Strong Eulerian triples*, Glas. Mat. Ser. III **53** (2018), 33–42.

[9] A. Dujella, M. Kazalicki, M. Mikić and M. Szikszai, *There are infinitely many rational Diophantine sextuples*, Int. Math. Res. Not. IMRN **2017 (2)** (2017), 490–508.

[10] A. Dujella, M. Kazalicki, V. Petričević, *D(n)-quintuples with square elements*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **115** (2021), Article 172, (10pp)

[11] A. Dujella, M. Kazalicki and V. Petričević, *Rational Diophantine sextuples containing two regular quadruples and one regular quintuple*, Acta Mathematica Spalatensia, **1** (2021), 19–27.

[12] A. Dujella, M. Kazalicki and V. Petričević, *Rational Diophantine sextuples with strong pair*, preprint. (9pp)

[13] A. Dujella, M. Kazalicki and V. Petričević, *There are infinitely many rational Diophantine sextuples with square denominators*, J. Number Theory **205** (2019), 340–346.

[14] A. Dujella and V. Petričević, *Diophantine quadruples with the properties $D(n_1)$ and $D(n_2)$*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **114** (2020), Article 21.

[15] A. Dujella and V. Petričević, *Doubly regular Diophantine quadruples*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **114** (2020), Article 189.

[16] A. Dujella and V. Petričević, *On the largest element in D(n)-quadruples*, Indag. Math. (N.S.) **30** (2019) 1079–1086.

[17] A. Dujella and V. Petričević, *Strong Diophantine triples*, Experimental Mathematics **17** (2008), 83–89.

[18] P. Gibbs, *Some rational Diophantine sextuples*, Glas. Mat. Ser. III **41** (2006), 195–203.

[19] B. He, A. Togbé and V. Ziegler, *There is no Diophantine quintuple*, Trans. Amer. Math. Soc. **371** (2019), 6665–6709.