Searching for elliptic curves with high rank in the PARI/GP software package

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Let E be an elliptic curve.

It is well known that the group $E(\mathbb{Q})$ is a finitely generated Abelian group (Mordell 1922).

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \times \mathbb{Z}^r.$$

In 1928, Weil generalized it to Abelian varieties over number fields.

In 1978, Mazur proved that there are exactly 15 possible torsion groups for elliptic curves over $\mathbb{Q}$:

$$\begin{aligned} \mathbb{Z}/k\mathbb{Z}, &\quad \text{for } k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}, &\quad \text{for } k = 2, 4, 6, 8. \end{aligned}$$

Kenku and Momose (1988) and Kamienny (1992) proved that in quadratic fields there can occur precisely the following 26 groups:

$$\begin{aligned} \mathbb{Z}/k\mathbb{Z}, &\quad 1 \leq k \leq 18, k \neq 17, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2k\mathbb{Z}, &\quad 1 \leq k \leq 6, \\ \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3k\mathbb{Z}, &\quad k = 1, 2, \\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}. & \end{aligned}$$

Motivation

The questions concerning the rank of elliptic curves are much more difficult than those which concern the torsion group, and satisfactory answers are still unknown. For a long time, it was believed that the rank can be arbitrarily large, i.e. that for any $M \in \mathbb{N}$, there is an elliptic curve E over $\mathbb{Q}$ such that $rank(E) \geq M$. However, some recent papers provide different heuristic arguments on why the rank might actually be bounded.

Details on the record curves can be found on the web pages
`https://web.math.pmf.unizg.hr/~duje/tors/tors.html`
`https://web.math.pmf.unizg.hr/~duje/tors/torsquad.html`

In the last two years, we have found curves with the same rank as the record for $E(\mathbb{Q})$ and torsion group $\mathbb{Z}/9\mathbb{Z}$.

In $E(\mathbb{K})$ we broke the record for torsion groups $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

In the book *Number theory* of prof. Dujella, can be found formulas for elliptic curves with given torsion group.

The problem is that we need to calculate the rank for large number of curves. In various math packages there are commands that we can use to calculate the rank more or less well.

In the newer versions of PARI/GP software package, they added the command `ellrank` which in some situations calculate the rank of the elliptic curve very well. They also added support for multithreading, which allowed us to use supercomputers.

Torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

In torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ previous record was held by M. Jukić Bokun. In the paper *On the rank of elliptic curves over $\mathbb{Q}(\sqrt{-3})$ with torsion groups $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$* she have found rank 7 elliptic curve. We have improved her results and found elliptic curves over the field $\mathbb{Q}(\sqrt{-3})$ with ranks equal to 8 for torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.
We used some similar tools. Our main tool for calculating the rank over $\mathbb{Q}(\sqrt{-3})$ (the only field where $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ can appear) was also the fact that if E is an elliptic curve over $\mathbb{Q}$, then the rank of over $\mathbb{Q}(\sqrt{-3})$ is given by

$$rank(E(\mathbb{Q}(\sqrt{-3}))) = rank(E(\mathbb{Q})) + rank(E_{-3}(\mathbb{Q})),$$

where E_{-3} is the (-3)-twist of E over $\mathbb{Q}$.

We also started with a family of elliptic curves $E(s,t)$:

$$y^2 + s(ts^2 - 12)xy + 4t(144s^2 - 24ts^4 + 432ts + 432t^2 + t^2s^6 - 36t^2s^3)y = x^3.$$

Rank 8 curves

For $(s, t) \in \{(2/7, -45/134), (4/7, -630/923), (-6/7, 945/998)\}$ we get elliptic curve

$$y^2 + y = x^3 + x^2 + 8804205157156465x + 383138550586901919299306,$$

with 4 independent points of infinite order:

$$[5258382019765/324, 12058283186746703749/5832],$$
$$[17139433097850/54289, 74437432142069283479/12649337],$$
$$[-688615622270135/18386944, 2338818831165669722299/78843215872],$$
$$[-79419500492675/3020644, 1918045489966876551529/5249879272],$$

while (-3)-twist also have 4 independent points:

$$[364239808947/2209, 278659347206075988/103823],$$
$$[1338962052421/100, 1549699239583785869/1000],$$
$$[181213268371821/643204, 3022985780907365891361/515849608],$$
$$[63972838783, 16180712076509801].$$

Rank 8 curves

For $(s, t) = (-17/7, 345/784)$ we get elliptic curve

$$y^2 + xy + y = x^3 - x^2 - 6175786126492675209 77x \\ - 181043813810219242475416950387 1,$$

with 5 independent points:

$$[-2253623821475783/192721, 1651921157504066970247 54/84604519],$$
$$[6365701073973751/106929, 4591160060638760209431 56/34965783],$$
$$[82283478289207/81, 746171021611085246684/729],$$
$$[48974552317, 9241713515007306],$$
$$[1129566488961424 3/344569, 7341163151411916216039 68/202262003],$$

while (-3)-twist have 3 independent points:

$$[69881507983, 13141268234384 08],$$
$$[729852143293/9, 309555491671516223/27],$$
$$[8427306357071/100, 13378211660467663051/1000].$$

Torsion group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

In torsion group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ previous record was also held by Duje and M. Jukić Bokun (A. Dujella and M. Jukic Bokun, *On the rank of elliptic curves over* $\mathbb{Q}(i)$ *with torsion group* $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$). For $\mathbb{K} = \mathbb{Q}(i)$ (the only field where $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ can appear), we are also looking of curves and (-1)-twist and family with torsion group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$:

$$y^2 + 4xy + (-64t^4 + 4)y = x^3 + (-16t^4 + 1)x^2.$$

We also used this family, and found elliptic curve with the record rank.

Rank 8 curve

For $t = 180/6643$ we get elliptic curve

$$y^2 = x^3 - x^2 - 126428563078457191959734976240 0x - 546907189853176652858460972620151389392800000,$$

with 4 independent points of infinite order:

$$[\frac{18434717483122731307680720}{784728169}, \frac{7905830688503157573136937066504573952 0}{21982590198197}],$$
$$[\frac{20557575647759080585800 0}{19351201}, \frac{926640573483023154209448975633120 00}{85125933199}],$$
$$[\frac{-13331002960857887983139 04}{2019513721}, \frac{130400646832541745425498266660089 60}{90754927108019}],$$
$$[\frac{-30724541040755906490339963 1}{465158100625}, \frac{16672539824494902460966518939675019528}{317249453578765625}],$$

while (-1)-twist have also 4 independent points:

$$[469572437338020, 8659141260066207245250],$$
$$[\frac{78622798756666773932 37}{17606416}, \frac{5831526765432388269277761480088 47}{73876521536}],$$
$$[177864765470100, 3432790861144055914 50],$$
$$[\frac{44749064535467856027498 0}{2884441849}, \frac{22220310008210701689236734902375750}{154914718384243}].$$

Torsion group $\mathbb{Z}/9\mathbb{Z}$ - rank 4 curves

In this torsion group, there were known 3 curves with rank 4, one from Fisher (2009), and two from van Beek (2015).

We have found 3 new curves.

$$y^2 + xy = x^3 - 84456994632996966740292855431 55x + 9446705591085118541016112920676157302177853025.$$

Independent points of infinite order are:

$$[10126558784331150, 9810050966998329335149 05],$$
$$[\frac{440967195740498845510}{194481}, \frac{3791112809372033023823906318545}{85766121}],$$
$$[\frac{118540192086626977 90}{8649}, \frac{1698432356408016607124620 0765}{804357}],$$
$$[\frac{328539759252576683735}{206116}, \frac{548759545844829191241645301655}{93576664}].$$

Torsion group $\mathbb{Z}/9\mathbb{Z}$ - rank 4 curves

$$y^2 + xy + y = x^3 - x^2 - 395752220478604696220020199256740467 7x$$

$$+ 302998861458514211864465346429825262328144785209493030 1.$$

Independent points of infinite order are:

$$[\frac{27874035597863902879 9}{324}, \frac{29853757503942644211138347041 43}{5832}],$$

$$[\frac{9341065321450103340 9}{64}, \frac{3084881866320886921746875895 23}{512}],$$

$$[\frac{19029226654785323829049791}{16982641}, \frac{3423877156972834911022891319624984864}{69985463561}],$$

$$[\frac{35800244364608335766528552863903525412900232471885917 71}{92885212048803856202352901751408941 21},$$

$$\frac{35379277977384751025092882844234659759886850651086816874574493227432324517604950656}{2830869604376382427661147615657525535352092852130534141 9}].$$

Pari was unable to find the last point, but Dujella find it using Magma.

Torsion group $\mathbb{Z}/9\mathbb{Z}$ - rank 4 curves

$$y^2 + xy = x^3 - 234034538757080655826172526109047402\,90x + 5674047213421910419156498953038386247082807655585641210\,0.$$

Independent points of infinite order are:

$$[118399436087458096\,2, 553991773424666561282538937\,2],$$

$$[-\frac{2124831061815491209344}{625}, -\frac{1538973368260499532264948442482 54}{/15625}],$$

$$[-\frac{253835742468176051393615142187898 70}{11541875487800641}, -\frac{12248431721078582644888104112139415756148879568434740}{1239979705847820288430561}],$$

we were unable to find the fourth point, but R. Rathbun did it:

$$[\frac{11039168903707688516615907465345318179937903715079971086630}{254035585383892857719517730338266639041}, \frac{115305096157281597385665226536060378122280796148861607272988571226063433382495853763501 0}{404894460876686382414459398802541944342411044212198616016 1}].$$

We have found many curves which could be a good candidate for huge rank. However, many of them had not big rank. To reduce them we used W. Stein's implementation of Bober's algorithm, which easily can be implement to work in multithreading (J. W. Bober: *Conditionally bounding analytic ranks of elliptic curves*).

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