Fusion rules for the $\beta - \gamma$ system and Lie superalgebra $\mathfrak{gl}(1|1)$

based on **On fusion rules and intertwining operators for the Weyl vertex algebra, J. Math. Phys. 60 (2019), no. 8, 081701, 18 pp.,**

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PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU ZNANSTVENOG CENTRA IZVRSNOSTI ZA KVANTNE I KOMPLEKSNE SUSTAVE TE REPREZENTACIJE LIEJEVIH ALGEBRI

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Abstract

We describe fusion rules in the category of weight modules for the Weyl vertex algebra and explicitly construct their intertwining operators. This way, we confirm a Verlinde type conjecture. We present a connection between irreducible weight modules for the Weyl vertex algebra and for the affine Lie superalgebra $\widehat{\mathfrak{gl}}(1|1)$.

Introduction

In the theory of vertex algebras and conformal field theory, determination of fusion rules is one of the most important problems. By a result of Y. Z. Huang [7] for a rational vertex algebra, fusion rules can be determined by using the Verlinde formula. However, although there are certain versions of Verlinde formula for a broad class of non-rational vertex algebras, so far there is no proof that it holds.

In this paper, we study the case of the Weyl vertex algebra (mathematical literature), or $\beta-\gamma$ system (physical literature). Its Verlinde type conjecture for fusion rules was given by S. Wood and D. Ridout in [2]. We explicitly construct the associated intertwining operators and therefore confirm this conjecture.

Let M be the Weyl vertex algebra, ρ the so called *spectral flow automorphism* and let $\mathcal{K}$ be the category of weight M–modules such that the operators $\beta(n)$, $n \geq 1$, act locally nilpotent on each module N in $\mathcal{K}$. Our main result is the following theorem:

Theorem. *Assume that $\lambda, \mu, \lambda+\mu \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:*

(i) $\rho_{\ell_1}(M) \times \rho_{\ell_2}(M) = \rho_{\ell_1+\ell_2}(M)$,

(ii) $\rho_{\ell_1}(M) \times \rho_{\ell_2}(\widetilde{U(\lambda)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda)})$,

(iii) $\rho_{\ell_1}(\widetilde{U(\lambda)}) \times \rho_{\ell_2}(\widetilde{U(\mu)}) = \rho_{\ell_1+\ell_2}(\widetilde{U(\lambda+\mu)}) + \rho_{\ell_1+\ell_2-1}(\widetilde{U(\lambda+\mu)})$.

Here we will prove the statement (iii) of our theorem and for this purpose it is sufficient and necessary to prove the following:

(i) $\dim I\begin{pmatrix} \rho_s(\widetilde{U(\lambda)}) \\ \rho_{s_1}(\widetilde{U(\lambda_1)})\ \rho_{s_2}(\widetilde{U(\lambda_2)}) \end{pmatrix} \leq 1$,

(ii) $\dim I\begin{pmatrix} \rho_s(\widetilde{U(\lambda)}) \\ \rho_{s_1}(\widetilde{U(\lambda_1)})\ \rho_{s_2}(\widetilde{U(\lambda_2)}) \end{pmatrix} = 1 \iff \lambda = \lambda_1+\lambda_2,\ s = s_1+s_2$,

or $\lambda = \lambda_1+\lambda_2$, $s = s_1+s_2-1$.

Therefore, we will have to construct the intertwining operators arising from our fusion rule, and then also prove that these are the only ones possible.

Construction of intertwining operators

Using a proposal of Verlinde formula for non-rational VOAs, these fusion rules were also obtained by S. Wood and D. Ridout [2], so we proved the Verlinde type of conjecture for fusion rules.

We construct the intertwining operators using the lattice VOA. Let L be the lattice

$$L = \mathbb{Z}\alpha + \mathbb{Z}\beta,\ \langle\alpha,\alpha\rangle = -\langle\beta,\beta\rangle = 1, \quad \langle\alpha,\beta\rangle = 0,$$

and $V_L = M_{\alpha,\beta}(1) \otimes \mathbb{C}[L]$ the associated lattice vertex superalgebra, where $M_{\alpha,\beta}(1)$ is the Heisenberg vertex algebra generated by fields $\alpha(z)$ and $\beta(z)$ and $\mathbb{C}[L]$ is the group algebra of L. We have its vertex subalgebra

$$\Pi(0) = M_{\alpha,\beta}(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha+\beta)] \subset V_L.$$

Then M is embedded in $\Pi(0)$ via the injective vertex algebra homomorphism $f: M \to \Pi(0)$ such that

$$f(a) = e^{\alpha+\beta},\ f(a^*) = -\alpha(-1)e^{-\alpha-\beta}.$$

Let us consider irreducible $\Pi(0)$–modules

$$\Pi_r(\lambda) = \Pi(0).e^{r\beta+\lambda(\alpha+\beta)}.$$

Using lattice VOA results [1], we have an intertwining operator of type

$$\begin{pmatrix} \Pi_{r_1+r_2}(\lambda+\mu) \\ \Pi_{r_1}(\lambda)\ \Pi_{r_2}(\mu) \end{pmatrix}$$

for the vertex algebra $\Pi(0)$, and we consider its restriction to M.

We have proved that:

$$I\begin{pmatrix} \Pi_{r_1+r_2}(\lambda+\mu) \\ \Pi_{r_1}(\lambda)\ \Pi_{r_2}(\mu) \end{pmatrix} \cong I\begin{pmatrix} \rho_{l_1+l_2-1}(\widetilde{U(\lambda+\mu)}) \\ \rho_{l_1}(\widetilde{U(\lambda)})\ \rho_{l_2}(\widetilde{U(\mu)}) \end{pmatrix},$$

so we have constructed an intertwining operator in the category of weight M–modules. By using the Weyl vertex algebra automorphism g which maps $a \mapsto -a^*$, $a^* \mapsto a$, on this intertwining operator we get the second intertwining operator

$$\begin{pmatrix} \rho_{l_1+l_2}(\widetilde{U(\lambda+\mu)}) \\ \rho_{l_1}(\widetilde{U(\lambda)})\ \rho_{l_2}(\widetilde{U(\mu)}) \end{pmatrix}.$$

Vertex algebra $V_1(\mathfrak{gl}(1|1))$

For the calculation of our fusion rules we will use the affine Lie superalgebra $\hat{\mathfrak{g}} = \widehat{gl(1|1)} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ with the commutation relations

$$[x(n), y(m)] = [x,y](n+m) + n\delta_{n+m,0}(x|y)K,$$

and its associated simple affine vertex algebra $V_1(\mathfrak{gl}(1|1))$, because the fusion rules are known for the latter [4].

Let $\mathcal{V}_{r,s}$ be the Verma module for the Lie superalgebra $\mathfrak{g}$ generated by the vector $v_{r,s}$ such that $Nv_{r,s} = rv_{r,s}$, $Ev_{r,s} = sv_{r,s}$. Let $\widehat{\mathcal{V}}_{r,s}$ denote the Verma module of level 1 induced from the irreducible $gl(1|1)$–module $\mathcal{V}_{r,s}$.

Proposition. *Let $r_1, r_2, s_1, s_2 \in \mathbb{C}$, $s_1, s_2, s_1+s_2 \notin \mathbb{Z}$. Then*

$$\dim I\begin{pmatrix} \widehat{\mathcal{V}}_{r_3,s_3} \\ \widehat{\mathcal{V}}_{r_1,s_1}\ \widehat{\mathcal{V}}_{r_2,s_2} \end{pmatrix} \leq 1.$$

Assume that there is a non-trivial intertwining operator $\begin{pmatrix} \widehat{\mathcal{V}}_{r_3,s_3} \\ \widehat{\mathcal{V}}_{r_1,s_1}\ \widehat{\mathcal{V}}_{r_2,s_2} \end{pmatrix}$ in the category of $V_1(\mathfrak{g})$–modules. Then $s_3 = s_1+s_2$ and $r_3 = r_1+r_2$, or $r_3 = r_1+r_2-1$.

Let F be the Clifford vertex algebra. We define the following vertex superalgebra:

$$S\Pi(0) = \Pi(0) \otimes F \subset V_L,$$

and its irreducible modules

$$S\Pi_r(\lambda) = \Pi_r(\lambda) \otimes F = S\Pi(0).e^{r\beta+\lambda(\alpha+\beta)}.$$

Let $\mathcal{U} = M \otimes F$. Then $\mathcal{U}$ is a $\hat{\mathfrak{g}}$–module of level 1 and we have the following gradation:

$$\mathcal{U} = \bigoplus \mathcal{U}^{\ell}, \quad E(0)|_{\mathcal{U}^{\ell}} = \ell\ \mathrm{Id}.$$

Proposition. *[6] We have:*

$$V_1(\mathfrak{g}) \cong \mathcal{U}^0 = Ker_{M\otimes F} E(0).$$

Calculation of fusion rules

Theorem. *Assume that $r \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:*

(i) $S\Pi_r(\lambda)$ is an irreducible $M \otimes F$–module,

(ii) $S\Pi_r(\lambda)$ is a completely reducible $\widehat{gl(1|1)}$–module:

$$S\Pi_r(\lambda) \cong \bigoplus_{s\in\mathbb{Z}} U(\hat{\mathfrak{g}}).e^{r(\beta+\gamma)+(\lambda+s)(\alpha+\beta)} \cong \bigoplus_{s\in\mathbb{Z}} \widehat{\mathcal{V}}_{r+\frac{1}{2}(\lambda+s),-\lambda-s}.$$

By using the following natural isomorphism of the spaces of intertwining operators:

$$\mathrm{I}_{M\otimes F}\begin{pmatrix} S\Pi_{r_3}(\lambda_3) \\ S\Pi_{r_1}(\lambda_1)\ S\Pi_{r_2}(\lambda_2) \end{pmatrix} \cong \mathrm{I}_M\begin{pmatrix} \Pi_{r_3}(\lambda_3) \\ \Pi_{r_1}(\lambda_1)\ \Pi_{r_2}(\lambda_2) \end{pmatrix},$$

and the previous theorem, we obtain the fusion rules result in the category of modules for the Weyl vertex algebra M.

Corollary. *Assume that $\lambda_1, \lambda_2, \lambda_1+\lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$, $r_1, r_2, r_3 \in \mathbb{Z}$. There exists a non-trivial intertwining operator of type $\begin{pmatrix} \Pi_{r_3}(\lambda_3) \\ \Pi_{r_1}(\lambda_1)\ \Pi_{r_2}(\lambda_2) \end{pmatrix}$ in the category of M–modules if and only if $\lambda_3 = \lambda_1+\lambda_2$ and $r_3 = r_1+r_2$ or $r_3 = r_1+r_2-1$.*

The fusion rules in the category of weight M–modules are given by

$$\Pi_{r_1}(\lambda_1) \times \Pi_{r_2}(\lambda_2) = \Pi_{r_1+r_2}(\lambda_1+\lambda_2) \oplus \Pi_{r_1+r_2-1}(\lambda_1+\lambda_2).$$

Forthcoming and Related Research

In our future work we would like to consider generalized weight modules such that their weight spaces are all ∞–dimensional and extend our work to $\mathfrak{gl}(n|m)$. We will also try to include Whittaker modules into the fusion category.

Some related research includes [3] and [5].

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