Ruin probabilities for competing claim processes *

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Abstract

Let $C_1, C_2, \ldots, C_m$ be independent subordinators with finite expectations and denote their sum by C. Consider the classical risk process $X(t) = x + ct - C(t)$. The ruin probability is given by the well known Pollaczek-Hinchin formula. If ruin occurs, however, it will be caused by a jump of one of the subordinators whose sum constitutes C. Formulae for the probability that ruin is caused by C_i are derived. These formulae can be extended to perturbed risk processes of the type $X(t) = x + ct - C(t) + Z(t)$ where Z is a Lévy process with mean 0 and no positive jumps.

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1 Introduction.

The classical Cramér-Lundberg risk process X is defined as $X(t) = x + ct - C(t)$ where C is a compound Poisson process with i.i.d. jumps with distribution F and the intensity of the underlying Poisson process equal to λ. The value $x \geq 0$ is referred to as the initial capital and c as the premium rate. The ruin probability is defined as

$$\vartheta(x) = \mathbb{P}(X(t) < 0 \text{ for some } t \geq 0)\,.$$

Let μ be the mean of the distribution F. Then $\mathbb{E}(C(1)) = \lambda\mu$. Usually one assumes the net profit condition $\lambda\mu < c$ since otherwise ruin is certain. The ruin probability is given by the classical Pollaczek-Hinchin formula

$$1 - \vartheta(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n F_I^{n*}(x)\,, \tag{1.1}$$

where $\rho = 1 - \lambda\mu/c$ and

$$F_I(x) = \frac{1}{\mu} \int_0^x (1 - F(u))\,\mathrm{d}u$$

is the integrated tail distribution of F. See e.g. [RSST].

Risk processes can be generalised in many ways. In Section 2 of this paper the compound Poisson process is replaced by a sum of subordinators. More precisely, $C(t) = C_1(t) + \cdots + C_m(t)$, where the C_i are independent subordinators with finite expectations. One can think of the C_i as independent risk portfolios competing to cause ruin. Formulae for the overall ruin probability given the initial capital $x \geq 0$ are well known. One can, however, ask about probabilities that ruin will be caused by an individual risk portfolio. These probabilities are given in Section 2. For related computations involving subordinators rather than risk processes see [Win].

Many recent papers in risk theory consider further generalisations of the classical risk process. An interesting generalisation arises when the risk process is perturbed by an independent Lévy process Z with no positive jumps. Dufresne and Gerber [DG] consider the case when $Z = \varsigma W$ for $\varsigma > 0$ and W being standard Brownian motion. Furrer [Fur] takes a stable process of index $\alpha \in (1, 2)$ with no positive jumps. These authors derive various forms of Pollaczek-Hinchin type formulae for such perturbed processes. Huzak, Perman, Šikić and Vondraček [HPSV] give explicit expressions for ruin probabilities for general perturbed risk process. For perturbed processes ruin can either occur by a jump of one of the subordinators C_i, or at an instant when none of the C_i's have a jump. Section 3 gives formulae for these individual ruin probabilities. The computations will rely on results derived in [HPSV]. The methods are more general than those used in Section 2.

In Section 4 explicit formulae are given for ruin probabilities when the perturbation is a multiple of Brownian motion and the risk processes are compound Poisson.

2 Ruin probabilities.

Let $C_1, \ldots, C_m$ be independent subordinators without drift with Lévy measures $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$. Assume that

$$\mathbb{E}(C_i(1)) = \int_0^\infty x\, \Lambda_i(\mathrm{d}x) < \infty \tag{2.1}$$

for $i = 1, \ldots, m$. Denote $C = C_1 + \cdots + C_m$. It is clear that the Lévy measure Λ of the subordinator C is the sum of the Lévy measures of individual summands. Let $\mu_i = \mathbb{E}(C_i(1))$ for $i = 1, \ldots, m$ and $\mu = \mu_1 + \cdots + \mu_m$. Define the risk process $X(t) = ct - C(t)$. As usual assume the net profit condition $c > \mu$ to avoid the trivial case of certain ruin.

Let $(\mathcal{F}_t)$ be the natural filtration generated by the subordinators C_i augmented in the usual way. The dual $\widehat{X}(t) = -ct + C(t)$ is a Lévy process with respect to $(\mathcal{F}_t)$. We always assume right continuous paths with left limits. For $x \geq 0$ let

$$\hat{\tau}_x = \inf\{t \geq 0 \colon \widehat{X}(t) > x\}\,. \tag{2.2}$$

Define $\Delta C(t) = C(t) - C(t-)$ and similarly $\Delta C_i(t) = C_i(t) - C_i(t-)$ for $i = 1, \ldots, m$.

The computations will be based on two known results. The first is a remarkable formula for conditional probabilities of ruin sometimes referred to as the *magic formula*. See [Tak], p. 37, for details.

Theorem 2.1 *Let $\widehat{X}$ be the dual risk process with $\widehat{X}(0) = 0$. Then*

$$\mathbb{P}\big(\sup_{0 \leq s \leq t} \widehat{X}(s) > 0 \,\big|\, \widehat{X}(t)\big) = 1 - \left(-\frac{\widehat{X}(t)}{ct}\right)_+ . \tag{2.3}$$

By continuity in probability of Lévy processes one can replace $\widehat{X}(t)$ in the above formula by $\widehat{X}(t-)$. The other computational means comes from fluctuation theory for Lévy processes. See [Ber], p. 190.

Lemma 2.2 *Let $\widehat{X}$ be the dual risk process with $\widehat{X}(0)=0$. Let $\hat{T}_y = \inf\{t \geq 0\colon \widehat{X}(t) = y\}$. Then the measures $t\,\mathbb{P}(\hat{T}_y \in \mathrm{d}t)\,\mathrm{d}y$ and $(-y)\,\mathbb{P}(\widehat{X}(t) \in \mathrm{d}y)\,\mathrm{d}t$ coincide on $[0,\infty) \times (-\infty, 0]$.*

In more explicit terms the above equality of measures implies that for any measurable $f\colon [0,\infty) \times (-\infty,0] \to \mathbb{R}_+$

$$\int\limits_{(-\infty,0]} \mathrm{d}y \int\limits_{[0,\infty)} f(t,y)\, t\,\mathbb{P}(\hat{T}_y \in \mathrm{d}t) = \int\limits_{[0,\infty)} \mathrm{d}t \int\limits_{(-\infty,0]} f(t,y)\,(-y)\,\mathbb{P}(\widehat{X}(t) \in \mathrm{d}y)\,. \tag{2.4}$$

Lemma 2.2 is all that is needed from the power of fluctuation theory to derive ruin probabilities in this section.

Let us turn to ruin probabilities. Let $y \leq 0$ and $x > 0$. Intuitively, one can compute

$$\begin{aligned}
&\mathbb{P}\big(\hat{\tau}_0 \in \mathrm{d}t, \widehat{X}(t-) \in \mathrm{d}y, \widehat{X}(t) \in \mathrm{d}x\big) \\
&\quad = \mathbb{P}(\widehat{X}(t-) \in \mathrm{d}y)\,\mathbb{P}(\hat{\tau}_0 \in \mathrm{d}t, \Delta C(t) \in -y + \mathrm{d}x \,|\widehat{X}(t-) \in \mathrm{d}y)\,.
\end{aligned}$$

By Markov property the events $\{\hat{\tau}_0 \in \mathrm{d}t\}$ and $\{\Delta C(t) \in -y + \mathrm{d}x\}$ are conditionally independent given $\{\widehat{X}(t-) \in \mathrm{d}y\}$. Moreover, the jump $\Delta C(t)$ and $\widehat{X}(t-)$ can be thought of as independent. One gets

$$\begin{aligned}
&\mathbb{P}\big(\hat{\tau}_0 \in \mathrm{d}t, \widehat{X}(t-) \in \mathrm{d}y, \widehat{X}(t) \in \mathrm{d}x\big) \\
&\quad = \mathbb{P}(\widehat{X}(t-) \in \mathrm{d}y)\,\mathbb{P}(\hat{\tau}_0 \in \mathrm{d}t|\widehat{X}(t-) \in \mathrm{d}y)\,\mathbb{P}(\Delta C(t) \in -y + \mathrm{d}x) \\
&\quad = \mathbb{P}(\widehat{X}(t) \in \mathrm{d}y)\,\frac{(-y)}{ct}\,\Lambda(-y + \mathrm{d}x)\,.
\end{aligned}$$

Formula (2.3) has been used to obtain the last line. Integrating over $(0,\infty)$ with respect to t and using Lemma 2.2 one gets the severity of ruin formula which is well known in the case when the claim process is compound Poisson, cf. [RSST], p. 164,

$$\mathbb{P}(\hat{\tau}_0 < \infty, \widehat{X}(\hat{\tau}_0 -) \in \mathrm{d}y, \widehat{X}(\hat{\tau}_0) \in \mathrm{d}x) = \frac{1}{c}\,\Lambda(-y + \mathrm{d}x)\,\mathrm{d}y\,.$$

Note that $\mathbb{P}(\hat{\tau}_0 > 0) = 1$ since $C(t)/t \to 0$ as $t \downarrow 0$ for arbitrary subordinators without drift.

A rigorous argument is based on master formulae for Poisson point processes. The vector valued process $\mathbf{\Delta C} = (\Delta C_1(t), \ldots, \Delta C_m(t))$ can be seen as a Poisson point process on $[0, \infty)^m \cup \{\partial\}$ with respect to $(\mathcal{F}_t)$ where ∂ is an added point representing the value of $\mathbf{\Delta C}$ whenever none of the subordinators have a jump. Denote the characteristic measure of this process by Υ. By definition $\Upsilon(\partial) = 0$. Further, Υ is concentrated on positive coordinate axes only and the restriction of Υ to the i-th positive axis is equal to Λ_i.

Let $\mathcal{H}(t, \cdot)$ be a non-negative predictable process taking values in the space of measurable functions on $[0, \infty)^m \cup \{\partial\}$ and such that $\mathcal{H}(t, \partial) = 0$. The master formula for Poisson point processes asserts that

$$\mathbb{E}\left(\sum_{0<t<\infty} \mathcal{H}(t, \mathbf{\Delta C}(t))\right) = \mathbb{E}\left(\int_0^\infty \mathrm{d}t \int_{[0,\infty)^m} \mathcal{H}(t, \epsilon)\, \Upsilon(\mathrm{d}\epsilon)\right). \tag{2.5}$$

See [RY], p. 452 for details. In order to properly formulate the results in this section we need to prove that ruin can only occur by a jump of one of the subordinators taking the process strictly over the level x. Define

$$\widehat{S}(t) = \sup_{0 \le s \le t} \widehat{X}(s) \quad \text{and} \quad \widehat{S}(\infty) = \sup_{s \ge 0} \widehat{X}(s). \tag{2.6}$$

Lemma 2.3 *Let $\widehat{X}$ be the dual risk process and assume $\widehat{X}(0) = 0$. For $x \ge 0$ one has $\mathbb{P}(\hat{\tau}_x > 0) = 1$ and $\mathbb{P}(\hat{\tau}_x < \infty, \widehat{X}(\hat{\tau}_x -) \le x = \widehat{X}(\hat{\tau}_x)) = 0$.*

Proof: For a subordinator witout drift one has $C(t)/t \to 0$ as $t \downarrow 0$. See [Ber], p. 84. Hence $\widehat{X}(t)/t \to -c < 0$ as $t \downarrow 0$. It follows that $\mathbb{P}(\hat{\tau}_x > 0) = 1$. To prove the second assertion note that $\hat{\tau}_x$ is a stopping time. Conditionally on $\{\hat{\tau}_x < \infty\}$ the process $(\widehat{X}(\hat{\tau}_x + s) - \widehat{X}(\hat{\tau}_x) \colon s \ge 0)$ has the law of $(\widehat{X}(s) \colon s \ge 0)$. Applying the first assertion for $x = 0$ it follows that $\widehat{X}(\hat{\tau}_x + s) - \widehat{X}(\hat{\tau}_x) < 0$ for small enough values of s. By the very definition of $\hat{\tau}_x$, however, there are arbitrarily small values $s > 0$ for which $\widehat{X}(\hat{\tau}_x + s) > x$. This forces $\widehat{X}(\hat{\tau}_x) > x$ which proves the second assertion. □

Using the strong Markov property Lemma 2.3 asserts that the points of increase of $\widehat{S}$ are discrete. More precisely, if whe denote

$$\sigma^{(1)} = \inf\{t > 0 \colon \widehat{S}(t) > 0\} \quad \text{and} \quad \sigma^{(k)} = \inf\{t > \sigma^{(k-1)} \colon \widehat{S}(t) > \widehat{S}(\sigma^{(k-1)})\}$$

with the usual convention $\inf \emptyset = \infty$ we have $0 < \sigma^{(1)} < \sigma^{(2)} < \cdots$. Furthermore, the differences $\sigma^{(1)}, \sigma^{(2)} - \sigma^{(1)}, \ldots$ are i.i.d random variables. As $\widehat{X}$ drifts to $-\infty$ only finitely many $\sigma^{(k)}$ are finite. This also proves that the absolute supremum $\widehat{S}(\infty)$ will be attained by a jump of one of the subordinators $C_1, C_2, \ldots, C_m$.

The quantities of interest in this section will be the probabilities of ruin caused by individual subordinators. We define the probability that ruin is caused by a jump of subordinator C_i by

$$\vartheta_i(x) = \mathbb{P}\big(\hat{\tau}_x < \infty, \Delta C(\hat{\tau}_x) = \Delta C_i(\hat{\tau}_x)\big)\,. \tag{2.7}$$

The probabilities are well defined as jumps of subordinators do not occur simultaneously with probability 1. Furthermore, Lemma 2.3 ensures that there is a jump at the instant of ruin. As mentioned in the introduction, the subordinators C_i can be thought of as independent risk portfolios. The main result of this section is the following:

Theorem 2.4 *Let $\widehat{X}$, $\widehat{S}$ and $C_1, C_2, \ldots, C_m$ and $\hat{\tau}_x$ be defined as at the beginning of this section.*

(i) Let $\widehat{X}(0) = 0$. For $y \le 0$, $x > 0$ and $i = 1, 2, \ldots, m$ one has

$$\mathbb{P}\Big(\hat{\tau}_0 < \infty, \widehat{X}(\hat{\tau}_0-) \in \mathrm{d}y, \widehat{X}(\hat{\tau}_0) \in \mathrm{d}x, \Delta C(\hat{\tau}_0) = \Delta C_i(\hat{\tau}_0)\Big) \tag{2.8}$$

$$= \frac{1}{c}\,\Lambda_i(-y + \mathrm{d}x)\,\mathrm{d}y\,,$$

and consequently

$$\mathbb{P}\Big(\hat{\tau}_0 < \infty, \Delta C(\hat{\tau}_0) = \Delta C_i(\hat{\tau}_0)\Big) = \frac{\mu_i}{c}\,. \tag{2.9}$$

(ii) Let $\gamma = \inf\{t > 0\colon \widehat{X}(t) = \widehat{S}(\infty)\}$. Then

$$\mathbb{P}\Big(\hat{\tau}_0 < \infty, \Delta C(\gamma) = \Delta C_i(\gamma)\Big) = \frac{\mu_i}{c}\,. \tag{2.10}$$

Proof: To prove the first formula take

$$\mathcal{H}(t, \epsilon) = 1(\widehat{X}(t-) \in \mathrm{d}y, \widehat{S}(t-) \le 0)\,1(\epsilon_i \in -y + dx)\,.$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in [0, \infty)^m$. The left side in the (2.5) is the desired probability. The right side transforms into

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^\infty \mathrm{d}t \int_{[0,\infty)^m} \mathcal{H}(t,\epsilon)\,\Upsilon(\mathrm{d}\epsilon)\right) \\
&= \mathbb{E}\bigg(\int_0^\infty 1(\widehat{X}(t-) \in dy, \hat{S}(t-) \le 0)\Upsilon(\epsilon_i \in -y + dx)\bigg) \\
&= \int_0^\infty \mathrm{d}t\,\mathbb{P}(\widehat{X}(t-) \in \mathrm{d}y, \widehat{S}(t-) \le 0)\,\Lambda_i(-y + \mathrm{d}x) \\
&= \int_0^\infty \mathrm{d}t\,\mathbb{P}(\widehat{X}(t-) \in \mathrm{d}y)\frac{(-y)}{ct}\,\Lambda_i(-y + \mathrm{d}x)\,.
\end{aligned}
$$

The last line follows by Theorem 2.1. By the equality of measures in Lemma 2.2, and since $\mathbb{P}(\hat{T}_y < \infty) = 1$, it follows that

$$
\int_0^\infty \mathrm{d}t\,\mathbb{P}(\widehat{X}(t-) \in \mathrm{d}y)\frac{(-y)}{ct}\,\Lambda_i(-y + \mathrm{d}x) = \frac{1}{c}\,\Lambda_i(-y + \mathrm{d}x)\,\mathrm{d}y
$$

which proves (2.8). Integrate by parts to get

$$
\mu_i = \int_0^\infty \Lambda_i(u, \infty)\,\mathrm{d}u\,.
$$

Integrate (2.8) over y to get

$$
\mathbb{P}\big(\hat{\tau}_0 < \infty, \widehat{X}(\hat{\tau}_0) \in \mathrm{d}x, \Delta C(\hat{\tau}_0)) = \Delta C_i\hat{\tau}_0)\big) = \frac{1}{c}\,\Lambda_i(x, \infty)\,\mathrm{d}x\,. \tag{2.11}
$$

Integration over $(0, \infty)$ with respect to x yields (2.9) concluding the proof of the first part of the theorem.

To prove the second part first note that by Lemma 2.3 the supremum will be attained by a jump of one of the subordinators. Denote $A_i = \{\hat{\tau}_0 < \infty, \Delta C(\hat{\tau}_0) = \Delta C_i(\hat{\tau}_0)\}$ and $B_i = \{\hat{\tau}_0 < \infty, \Delta C(\gamma) = \Delta C_i(\gamma)\}$. By the strong Markov property at $\hat{\tau}_0$ one has

$$
\begin{aligned}
\mathbb{P}(B_i) &= \mathbb{P}(B_i, \gamma = \hat{\tau}_0) + \mathbb{P}(B_i, \gamma > \hat{\tau}_0) \\
&= \mathbb{P}(A_i)\mathbb{P}(\hat{\tau}_0 = \infty) + \mathbb{P}(\hat{\tau}_0 < \infty)\mathbb{P}\big(B_i\big)\,.
\end{aligned}
$$

Integrating over $(0, \infty)$ with respect to x in (2.11) and adding over $i = 1, \dots, m$ one finds that $\mathbb{P}(\hat{\tau}_0 = \infty) = 1 - \mu/c$. Note that by (2.9), $\mathbb{P}(A_i) = \mu_i/c$. A straightforward calculation concludes the proof. □

Remark 2.5 *Note that adding over i in (2.8) gives the severity of ruin formula when the claim process is C.*

Theorem 2.4 says that conditionally on $\{\hat{\tau}_0 < \infty\}$ the supremum $\widehat{S}(\infty)$ will be achieved by a jump of C_i with probability μ_i/μ. Turning to ruin probabilities, recall that $\vartheta(x)$ is given by the Pollaczek-Hinchin formula (1.1). This formula, however, does not give the probability $\vartheta_i(x)$ that ruin will be caused by a jump of individual subordinators. Let $J := \widehat{X}(\hat{\tau}_0)$ on $\{\hat{\tau}_0 < \infty\}$. From (2.11) it follows that

$$\mathbb{P}(\hat{\tau}_0 < \infty, J \in \mathrm{d}x, \Delta C(\hat{\tau}_0) = \Delta C_i(\hat{\tau}_0)) = \frac{1}{c}\,\Lambda_i(x,\infty)\,\mathrm{d}x\,.$$

By the strong Markov property one can compute

$$\begin{aligned}\vartheta_i(x) &= \mathbb{P}\big(\hat{\tau}_0 < \infty, J > x, \Delta C(\hat{\tau}_0) = \Delta C_i(\hat{\tau}_0)\big) \\ &\qquad + \int_{(0,x]} \mathbb{P}\big(\hat{\tau}_0 < \infty, J \in \mathrm{d}u\big)\,\vartheta_i(x-u)\,.\end{aligned} \tag{2.12}$$

Denote

$$H(x) = \frac{1}{\mu}\int_0^x \Lambda(u,\infty)\,\mathrm{d}u \qquad \text{and} \qquad H_i(x) = \frac{1}{\mu_i}\int_0^x \Lambda_i(u,\infty)\,\mathrm{d}u\,.$$

and let $\bar{H}_i(x) = 1 - H_i(x)$. Let $\rho_i = \mu_i/c$ and $\rho = \mu/c$. Rewriting (2.12) one gets

$$\vartheta_i(x) = \rho_i\,\bar{H}_i(x) + \rho\int_0^x \vartheta_i(x-u)\,H(\mathrm{d}x) \tag{2.13}$$

which is a defective renewal equation. See [RSST], p. 213. The solution is given by

$$\vartheta_i(x) = \sum_{k=0}^{\infty} \rho_i\,\rho^k(\bar{H}_i * H^{*k})(x)\,. \tag{2.14}$$

In case of only one subordinator this formula translates into the standard formula (1.1).

3 Extension to perturbed risk processes.

As mentioned in the introduction the Cramér-Lundberg process has been generalised in many directions. Here the focus will be on perturbing the risk process by an independent Lévy process with finite expectation and no positive jumps. It will be shown that most of the formulae in section 2 can be generalised to this case. In particular formulae that ruin is caused by a jump of the particular subordinator will be derived, as well as the probability that ruin is not caused by a jump of any subordinator.

Let again $C_1, C_2, \dots, C_m$ be independent subordinators and C their sum. Let $\mu_i = \mathbb{E}(C_i(1))$ and $\mu = \mu_1 + \cdots + \mu_m$ as before. Let Z be an independent spectrally negative Lévy process such that $\mathbb{E}(Z(t)) = 0$ for each $t \geq 0$. Note that the assumption $\mathbb{E}(Z(t)) = 0$ does not reduce the generality as any drift can simply be "moved" to the premium rate c. The existence of the expectation is necessary as for spectrally negative Lévy processes $\lim_{t\to\infty} \widehat{X}(t) = -\infty$ if and only if $\mathbb{E}(\widehat{X}(1))$ exists and is strictly negative. See [DM] for details. If Π_Z denotes the Lévy measure of Z the assumptions imply that Π_Z is concentrated on $(-\infty, 0)$ and satisfies the integrability condition

$$\int_{(-\infty,-1)} |x| \, \Pi_Z(\mathrm{d}x) < \infty \, . \tag{3.1}$$

This in turn implies that the Laplace exponent of Z is given by

$$\psi_Z(\beta) = \frac{\varsigma^2 \beta^2}{2} + \int_{(-\infty,0)} (e^{-\beta x} - 1 - \beta x) \, \Pi_Z(\mathrm{d}x)$$

where the existence of the integral is ensured by (3.1). Furthermore, the net profit condition $\mu = \mu_1 + \cdots + \mu_m < c$ will be assumed. Define

$$X(t) = ct - C(t) + Z(t) \tag{3.2}$$

and let $\widehat{X} = -X$ be the dual process. Assume right continuous paths with left limits.

Formulae for the ruin probabilities for the perturbed process were given in [HPSV]. Here the formulae will be extended in several directions: formulae for the probability that ruin will be caused by a jump of C_i will be derived. As ruin can also occur when none of the subordinators has a jump leading to ruin, one can ask about the probability of such an event. Recall that $\hat{\tau}_x = \inf\{t > 0\colon \widehat{X}(t) > x\}$ is the entrance time into (x, ∞). For $x \geq 0$ define

$$\vartheta_i(x) = \mathbb{P}\big(\hat{\tau}_x < \infty, \widehat{X}(\hat{\tau}_x-) < x, \Delta C_i(\hat{\tau}_x) > x - \widehat{X}(\hat{\tau}_x-)\big)$$

and

$$\bar{\vartheta}(x) = \mathbb{P}\big(\hat{\tau}_x < \infty, \mathbf{\Delta C}(\hat{\tau}_x) = \partial\big) \, .$$

Furthermore, define

$$\sigma_i = \inf\{t > 0\colon \Delta C_i(t) > \widehat{S}(t-) - \widehat{X}(t-)\}$$

for $i = 1, 2, \ldots, m$, with the usual convention $\inf \emptyset = \infty$. In words, σ_i is the first time a new supremum of the dual process $\widehat{X}$ is attained by a jump of the subordinator C_i. Let

$$\sigma = \sigma_1 \wedge \cdots \wedge \sigma_m \,.$$

It is a consequence of Theorem 4.1 in [HPSV] that $\mathbb{P}(\sigma > 0) = 1$. Recall that

$$\widehat{S}(t) = \sup_{0 \le s \le t} \widehat{X}(s) \qquad \text{and} \qquad \widehat{S}(\infty) = \sup_{s \ge 0} \widehat{X}(s)\,.$$

Denote $\rho := \mu/c$, and let $G(y) = \mathbb{P}(\widehat{S}(\sigma-) \le y | \sigma < \infty)$.

Proposition 3.1 *Let $\widehat{X}$ be the dual risk process. Let $x > 0$ and $y > 0$. The expected occupation measure of the reflected process $\widehat{S} - \widehat{X}$ up to the stopping time $\sigma \wedge \hat{\tau}_y$ is given by*

$$\mathbb{E}\bigg(\int_0^{\hat{\tau}_y \wedge \sigma} 1_{(\widehat{S}(t) - \widehat{X}(t) \le x)} \,\mathrm{d}t \bigg) = \frac{1}{\mu}\, \mathbb{P}(\sigma < \infty)\, G(y)\, x\,.$$

Furthermore, $\mathbb{P}(\sigma < \infty) = \rho$.

See [HPSV], Proposition 4.2 and Corollary 4.5 for a proof.

The master formula for Poisson processes can now be used to derive several probabilities. Define

$$J_i := \Delta C_i(\sigma_i) - (\widehat{S}(\sigma_i-) - \widehat{X}(\sigma_i-))$$

on $\{\sigma_i < \infty\}$ and $J_i := 0$ else. Define

$$J := \Delta C(\sigma) - (\widehat{S}(\sigma-) - \widehat{X}(\sigma-))$$

on $\{\sigma < \infty\}$ and 0 else.

Proposition 3.2 *For $x, y, z > 0$*

$$\begin{aligned} &\mathbb{P}(\sigma < \infty, \widehat{S}(\sigma-) \le y, \widehat{S}(\sigma-) - \widehat{X}(\sigma-) > z, J_i > x) = \\ &= \frac{1}{\mu}\, \mathbb{P}(\sigma < \infty)\, G(y) \int_{x+z}^{\infty} \Lambda_i(u, \infty)\, \mathrm{d}u\,. \end{aligned} \tag{3.3}$$

Proof: Define

$$\mathcal{H}(t,\epsilon) = 1_{(\widehat{S}(t-)<y)} 1_{(\widehat{S}(t-)-\widehat{X}(t-)>z)} 1_{(t\le\sigma)} 1_{(\epsilon_i-(\widehat{S}(\sigma-)-\widehat{X}(\sigma-))>x)} .$$

The functional is predictable with respect to $(\mathcal{F}_t)$. The left hand side in the master formula (2.5) is the desired probability. For the right hand side compute

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^\infty \mathrm{d}t \int_{[0,\infty)^m} \mathcal{H}(t,\epsilon)\,\Upsilon(\mathrm{d}\epsilon)\Big) \\
&= \mathbb{E}\Big(\int_0^\sigma \mathrm{d}t\, 1_{(\widehat{S}(t-)<y)} 1_{(\widehat{S}(t-)-\widehat{X}(t-)>z)}\, \Lambda_i\big(x+\widehat{S}(\sigma-)-\widehat{X}(\sigma-),\infty\big)\Big) \\
&= \mathbb{E}\Big(\int_0^{\sigma\wedge\hat{\tau}_y} \mathrm{d}t\, 1_{(\widehat{S}(t-)-\widehat{X}(t-)>z)}\, \Lambda_i\big(x+\widehat{S}(\sigma-)-\widehat{X}(\sigma-),\infty\big)\Big) \\
&= \frac{1}{\mu}\,\mathbb{P}(\sigma<\infty)\,G(y) \int_0^\infty 1_{(z,\infty)}(u)\Lambda_i(x+u,\infty)\,\mathrm{d}u \\
&= \frac{1}{\mu}\,\mathbb{P}(\sigma<\infty)\,G(y) \int_{x+z}^\infty \Lambda_i(u,\infty)\,\mathrm{d}u\,.
\end{aligned}$$

Note that continuity in probability of Lévy processes was used to conclude that the expected occupation measure of $\widehat{S}(t-)-\widehat{X}(t-)$ is the same as the expected occupation measure of $\widehat{S}(t)-\widehat{X}(t)$. Proposition 3.1 was used to pass from the third to the fourth line. □

Let us now turn to the computation of ruin probabilities ϑ_i and $\bar{\vartheta}$ for perturbed risk processes. We will assume that the perturbation is not of the form $Z(t)=at-U(t)$ for a compound Poisson process or subordinator U. This case has been dealt with in Section 2. With these assumptions $\widehat{X}$ is of unbounded variation and hence the interval $(0,\infty)$ is regular for $\widehat{X}$. This implies that $\mathbb{P}(\widehat{S}(t)>0)=1$ for all $t>0$. As we know $\mathbb{P}(\sigma>0)=1$, one can infer that $\vartheta_i(0)=0$ and $\bar{\vartheta}(0)=1$. Using the strong Markov property for $\widehat{X}$ we have for $x>0$

$$\begin{aligned}
\vartheta_i(x) = \mathbb{P}(\sigma<\infty, \widehat{S}(\sigma-)\le x, J_i > x-\widehat{S}(\sigma-)) \qquad (3.4)\\
+\int_{(0,x]} \mathbb{P}(\sigma<\infty,\widehat{S}(\sigma)\in\mathrm{d}u)\,\vartheta_i(x-u)\,.
\end{aligned}$$

Let

$$H(x) = \frac{1}{\mu}\int_0^x \Lambda(u,\infty)\,\mathrm{d}u \quad \text{and} \quad H_i(x) = \frac{1}{\mu_i}\int_0^x \Lambda_i(u,\infty)\,\mathrm{d}u\,.$$

Note that by Proposition 3.2 one has

$$H(x) = \mathbb{P}(J \le x | \sigma < \infty) \quad \text{and} \quad H_i(x) = \mathbb{P}(J_i \le x | \sigma_i = \sigma < \infty)\,.$$

Proposition 3.2 will be used to write the probabilities in (3.4) more explicitly. Note that conditionally on $\{\sigma < \infty\}$, the random variables J_i and $\widehat{S}(\sigma-)$ are independent. We get

$$\begin{aligned} &\mathbb{P}(\sigma < \infty, \widehat{S}(\sigma-) \le x, J_i > x - \widehat{S}(\sigma-)) = \\ &\qquad = \frac{\mu_i}{\mu}\,\mathbb{P}(\sigma < \infty) \int_{(0,x]} G(\mathrm{d}u)\,(1 - H_i(x-u))\,, \end{aligned} \tag{3.5}$$

and for $0 < u \le x$

$$\begin{aligned} &\mathbb{P}(\sigma < \infty, \widehat{S}(\sigma) \in \mathrm{d}u) = \\ &= \int_{(0,u]} \mathbb{P}(\sigma < \infty, \widehat{S}(\sigma-) \in \mathrm{d}v, J \in (\mathrm{d}u - v)) \\ &= \mathbb{P}(\sigma < \infty)\Big(\int_{(0,u]} G(\mathrm{d}v)\,H(\mathrm{d}u - v)\Big) \end{aligned} \tag{3.6}$$

Recall that $\rho = \mathbb{P}(\sigma < \infty)$. Define $\rho_i := \mu_i/c$ and

$$\bar{H}_i(x) = 1 - H_i(x) \quad \text{and} \quad \bar{H}(x) = 1 - H(x)\,. \tag{3.7}$$

Equation (3.4) is transformed into

$$\begin{aligned} \vartheta_i(x) = \rho_i &\int_{(0,x]} G(\mathrm{d}u)\,\bar{H}_i(x-u) \;+ \\ &+ \rho \int_{(0,x]} \Big(\int_{(0,u]} G(\mathrm{d}v)\,H(\mathrm{d}u - v)\Big)\,\vartheta_i(x-u)\,. \end{aligned} \tag{3.8}$$

To solve (3.8) Laplace transforms will be used. Some care is needed with definitions. Let

$$\mathcal{L}G(\beta) = \int_0^\infty e^{-\beta x}\,G(\mathrm{d}x) \quad \text{and} \quad \mathcal{L}H(\beta) = \int_0^\infty e^{-\beta x}\,H(\mathrm{d}x)\,.$$

For functions ϑ_i and $\bar{H}_i$ we define using the same notation

$$\mathcal{L}\vartheta_i(\beta) = \int_0^\infty e^{-\beta x}\,\vartheta_i(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{L}\bar{H}_i(\beta) = \int_0^\infty e^{-\beta x}\,\bar{H}_i(x)\,\mathrm{d}x\,.$$

Taking Laplace transforms on both sides of (3.8) one finally gets

$$\mathcal{L}\vartheta_i = \rho_i\, \mathcal{L}G \cdot \mathcal{L}\bar{H}_i + \rho\, \mathcal{L}G \cdot \mathcal{L}H \cdot \mathcal{L}\vartheta_i\,.$$

Solving for $\mathcal{L}\vartheta_i$ and inverting gives the probability that ruin is caused by C_i:

$$\vartheta_i(x) = \sum_{k=0}^{\infty} \rho_i\, \rho^k\, (\bar{H}_i * H^{k*} * G^{(k+1)*})(x)\,. \tag{3.9}$$

Remark 3.3 *The convolutions in the sum (3.9) are to be interpreted as*

$$(\bar{H}_i * H^{k*} * G^{(k+1)*})(x) = \int_{(0,x]} \bar{H}_i(x-u)\, (H^{k*} * G^{(k+1)*})(\mathrm{d}u)\,.$$

As usual, also, H^{0*} and G^{0*} are understood as δ_0.

Remark 3.4 *The sum*

$$\vartheta_J(x) = \sum_{k=1}^{m} \vartheta_i(x)$$

gives the overall probability that ruin will be caused by a jump of a subordinator. Taking into account that $\sum_{i=1}^{m} \rho_i \bar{H}_i = \rho \bar{H}$ one obtains

$$\vartheta_J(x) = \sum_{k=0}^{\infty} \rho^{k+1} (\bar{H} * H^{k*} * G^{(k+1)*})(x)\,.$$

Formula (3.9) can only be of use if the functions G, H and H_i are known as explicitly as possible. For the distribution of $\widehat{S}(\sigma-)$ conditionally on $\{\sigma < \infty\}$ note the proposition:

Proposition 3.5 *The function G is the distribution function of the variable*

$$\sup_{t\geq 0} (-ct - Z(t))\,.$$

It is an infinitely divisible distribution with Laplace exponent

$$\int_0^{\infty} e^{-\beta y}\, G(\mathrm{d}y) = \frac{c\beta}{c\beta + \psi_Z(\beta)}$$

where ψ_Z is the Laplace exponent of Z.

See [HPSV], Corollary 4.9. Recall that $\vartheta(x) = \mathbb{P}(X(t) < 0 \text{ for some } t > 0)$.

Proposition 3.6 *The ruin probability for X is given by the formula*

$$1 - \vartheta(x) = (1-\rho) \sum_{k=0}^{\infty} \rho^k \, (G^{(k+1)*} * H^{k*})(x) \,, \;\; x \geq 0 \,.$$

See [HPSV], Proposition 3.6. for a proof. Summarizing we have:

Theorem 3.7 *Let $X(t) = ct - C(t) + Z(t)$ be the risk process. Let $x \geq 0$. We have:*

(i) The probabilities that ruin will be caused by a jump of the subordinator C_i are given by

$$\vartheta_i(x) = \sum_{k=0}^{\infty} \rho_i \, \rho^k \, (\bar{H}_i * H^{k*} * G^{(k+1)*})(x) \,.$$

for $i = 1, 2, \ldots, m$.

(ii) The probability that ruin will occur but will not be caused by a jump of one of the subordinators equals

$$\bar{\vartheta}(x) = \sum_{k=0}^{\infty} \rho^k (\bar{G} * H^{k*} * G^{k*})(x) \,.$$

where $\bar{G}(y) = 1 - G(y)$.

Proof: Part (i) has already been proved. For part (ii) recall the definition of $\bar{H}$ from (3.7). Compute

$$\begin{aligned}
\vartheta_J(x) + \bar{\vartheta}(x) = & \\
= \;& \sum_{k=0}^{\infty} \rho^{k+1} (H^{k*} * G^{(k+1)*})(x) - \sum_{k=0}^{\infty} \rho^{k+1} (H^{(k+1)*} * G^{(k+1)*})(x) + \\
& + \sum_{k=0}^{\infty} \rho^k (H^{k*} * G^{k*})(x) - \sum_{k=0}^{\infty} \rho^k (H^{k*} * G^{(k+1)*})(x) \\
= \;& 1 - (1-\rho) \sum_{k=0}^{\infty} \rho^k (H^{k*} * G^{(k+1)*})(x) \\
= \;& \vartheta(x) \,.
\end{aligned}$$

Here $\vartheta(x)$ is the overall ruin probability given in Proposition 3.6. This proves the assertion. $\square$

Remark 3.8 *Note that the question how the overall supremum $\widehat{S}(\infty)$ will be attained has not been asked. Because of unbounded variation of $\widehat{X}$ it will not be achieved by jump of C a.s.*

4 Example.

This section contains an explicit example of computations with formulae from Section 3. In particular, we compute ruin probabilities for processes perturbed by a multiple of Brownian motion.

Example 4.1 *Let W be standard Brownian motion and let $\varsigma > 0$. Consider the risk process*

$$X(t) = ct - \sum_{i=1}^{m} C_i(t) + \varsigma W(t)\,.$$

Let us first compute the distribution G. By Proposition 3.5 this distribution does not depend on the subordinators. As $Z = \varsigma W$ one has $\psi_Z(\beta) = \varsigma^2\beta^2/2$. Denote $\gamma = 2c/\varsigma^2$. The Laplace transform of G is given by

$$\mathcal{L}G(\beta) = \frac{\gamma}{\gamma + \beta}$$

so G is an exponential distribution with parameter γ.

Let consider the case when the C_i are compound Poisson processes with Lévy measures $\Lambda_i(\mathrm{d}x) = \mu_i\gamma^2 e^{-\gamma x}\,\mathrm{d}x$. A straightforward computation gives

$$H_i(\mathrm{d}x) = \gamma e^{-\gamma x}\,\mathrm{d}x \quad \textit{and} \quad H(\mathrm{d}x) = \gamma e^{-\gamma x}\,\mathrm{d}x\,.$$

Note that $\bar{H}_i(x) = e^{-\gamma x}$. By elementary properties of gamma distributions

$$(H^{k*} * G^{(k+1)*})(\mathrm{d}u) = \frac{\gamma^{2k+1}}{\Gamma(2k+1)}\,u^{2k}e^{-\gamma u}\,\mathrm{d}u$$

and hence

$$
\begin{aligned}
\vartheta_i(x) &= \sum_{k=0}^{\infty} \rho_i\, \rho^k \int_0^x e^{-\gamma(x-u)}\, \frac{\gamma^{2k+1}}{\Gamma(2k+1)}\, u^{2k} e^{-\gamma u}\, \mathrm{d}u \\
&= \rho_i\, \gamma e^{-\gamma x} \int_0^x \cosh(\gamma\sqrt{\rho}\, u)\, \mathrm{d}u \\
&= \frac{\rho_i}{\sqrt{\rho}}\, e^{-\gamma x} \sinh(\gamma\sqrt{\rho}\, x)\,.
\end{aligned}
$$

The sum C has Lévy measure $\Lambda(\mathrm{d}x) = \mu\gamma^2 e^{-\gamma x}\,\mathrm{d}x$. The overall probability that ruin will be caused by a jump of one of the subordinators is given by

$$\vartheta_J(x) = \sqrt{\rho}\, e^{-\gamma x} \sinh(\gamma\sqrt{\rho}\, x)\,.$$

By a similar computation using Theorem 3.7 (ii),

$$\bar{\vartheta}(x) = e^{-\gamma x}\, \cosh(\gamma\sqrt{\rho}\, x)\,.$$

The overall ruin probability is the sum

$$\vartheta(x) = e^{-\gamma x}[\sqrt{\rho}\, \sinh(\gamma\sqrt{\rho}\, x) + \cosh(\gamma\sqrt{\rho}\, x)]\,.$$

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