Basic potential theory of certain nonsymmetric strictly α-stable processes

by

ZORAN VONDRAČEK[1]
Department of Mathematics
University of Zagreb
Bijenička c. 30
10000 Zagreb, Croatia
E-mail: vondra@math.hr

Abstract

We study potential theoretic properties of strictly α-stable processes whose Lévy measure is comparable to that of a symmetric α-stable process. We show the existence, continuity and strict positivity of transition densities and Green function of the process killed upon exiting a bounded domain. We further show that the exit distributions of the process from a domain satisfying the uniform volume condition have a density. The density is used to establish a representation of regular harmonic functions of the process. Finally, we indicate that the Harnack inequality is true for nonnegative harmonic functions.

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1 Introduction

A symmetric (rotation invariant) α-stable process on $\mathbb{R}^d$, $0 < \alpha \leq 2$, is a Lévy process whose transition density with respect to Lebesgue measure is determined by its Fourier transform

$$\int_{\mathbb{R}^d} e^{ix\cdot\xi} p(t,x)dx = e^{-t|\xi|^\alpha} .$$

For $\alpha = 2$ this process is (essentially) a Brownian motion, while for $0 < \alpha < 2$ it is a pure jump process whose Lévy measure has a density $|x|^{-(d+\alpha)}$ relative to the d-dimensional Lebesgue measure. The infinitesimal generator of the latter process is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ which is non-local.

In the last several years remarkable progress has been made in understanding fine properties of symmetric α-stable processes. Most of the effort revolved around extending potential theoretic properties of Brownian motion to symmetric stable processes. The results that were obtained include estimates of the Green function and the Poisson kernel of symmetric α-stable processes on $C^{1,1}$ domains ([10]), boundary Harnack principle on Lipschitz (and more general) domains ([5], [7], [21]), identification of the Martin boundary of the domain with its Euclidean boundary for a wide class of domains (including Lipschitz domains) ([6], [11]), conditional gauge theorem, and intrinsic ultracontractivity for the killed semigroup ([9], [12]).

In this paper we study the potential theory of a nonsymmetric strictly α-stable process whose Lévy measure is comparable with the Lévy measure of the symmetric α-stable process. To be more precise, we assume that the spherical part of the corresponding Lévy measure has a density with respect to the surface measure which is bounded and bounded away from zero. Although the resulting process need not be a pure jump process anymore, comparability of Lévy measures suggests qualitatively similar path properties of the two type of processes. It is, therefore, conceivable, that the potential theoretic properties are also similar. And indeed, by collecting several facts on stable processes, applying time-honored methods in the Brownian motion theory (e.g. [1], [13]), and using some newly developed methods for symmetric stable processes ([5], [10]), we were able to show the following facts: (1) The process killed upon exiting a bounded domain has a jointly continuous, strictly positive transition density; (2) The Green function of the killed process is jointly continuous off the diagonal, and comparable with the Green function of the symmetric stable process away from the boundary; (3) There exists a jointly continuous Poisson kernel (in case of a bounded domain satisfying the uniform volume condition) serving as the density of the exit distribution. Moreover, we point out that the method developed in [2] in order to show the Harnack inequality for spatially nonhomogeneous pure jump processes whose jump kernels are comparable to those of symmetric stable processes, can be directly applied to the processes we consider. This leads to the Harnack inequality for nonnegative harmonic functions in a bounded open set.

The potential theoretic properties we show may be regarded as basic, compared with the finer properties proved in the above mentioned papers. Those finer properties strongly rely on sharp estimates of the Green function and the Poisson kernel. In case of symmetric stable processes these estimates are derived from the explicit formulae for the Green function and the Poisson kernel for the ball (see [4]). The lack of such formulae for nonsymmetric case will require new methods for proving corresponding sharp estimates.

2 Preliminaries

In this section we describe processes that we will be studying, state known properties and list conditions that will be assumed in our results. The reference to Lévy processes is [20].

Let $X = (X_t, \mathbb{P}^x)$ be a Lévy process in $\mathbb{R}^d$, $d \geq 2$, with the generating triplet (A, ν, γ). More precisely, the characteristic function of the distribution

μ of X_1 (under $\mathbb{P}^0$) is

$$\hat{\mu}(z) = \exp\left\{-\frac{1}{2}(z, Az) + i(\gamma, z) + \int_{\mathbb{R}^d} (e^{i(z,x)} - 1 - i(z, x)1_{\{|x|\le 1\}}(x))\nu(dx)\right\},$$

$z \in \mathbb{R}^d$, where A is a symmetric nonnegative definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty,$$

called the Lévy measure of X, and $\gamma \in \mathbb{R}^d$.

The process X is strictly α-stable if the following scaling property holds: For every $a > 0$, $(X_{at},\, t \ge 0) \stackrel{d}{=} (a^{1/\alpha}X_t,\, t \ge 0)$. If the probability measure is not explicitly mentioned, we always mean $\mathbb{P}^0$. The scaling property is equivalent to the fact that for every $a > 0$, $\hat{\mu}(z)^a = \hat{\mu}(a^{1/\alpha}z)$. For $\alpha = 2$, one gets Brownian motion. We will consider the case when $0 < \alpha < 2$. In that case $A = 0$, and there is a finite measure λ on the unit sphere $S = \{x \in \mathbb{R}^d : |x| = 1\}$ such that

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-(1+\alpha)} dr$$

for every Borel set B in $\mathbb{R}^d$. The measure λ is called the spherical part of the Lévy measure ν. To be more precise, the following holds true:

(i) Let $0 < \alpha < 1$. Then X is strictly α-stable if and only if

$$\hat{\mu}(z) = \exp\left\{\int_S \lambda(d\xi) \int_0^\infty (e^{i(z,r\xi)} - 1) r^{-(1+\alpha)} dr\right\}.$$

(ii) Let $\alpha = 1$ and $\nu \ne 0$. Then X is strictly α-stable if and only if

$$\hat{\mu}(z) = \exp\left\{\int_S \lambda(d\xi) \int_0^\infty (e^{i(z,r\xi)} - 1 - i(z, r\xi)1_{(0,1]}(r)) r^{-2} dr + i(z, \gamma)\right\}.$$

and $\int_S \xi\, \lambda(d\xi) = 0$.

(iii) Let $1 < \alpha < 2$. Then X is strictly α-stable if and only if

$$\hat{\mu}(z) = \exp\left\{ \int_S \lambda(d\xi) \int_0^\infty (e^{i(z,r\xi)} - 1 - i(z,r\xi)) r^{-(1+\alpha)} dr \right\} .$$

Note that when the spherical part λ is equal to the surface measure σ on the unit sphere, the corresponding process X is the symmetric α-stable process (for $\alpha = 1$ we have $\gamma = 0$). From now on, we assume that X satisfies (i), (ii) or (iii).

It is known that the distribution μ of X_1 has a smooth density. Let $p(t,x)$ denote the density of X_t. Then the following scaling property is valid: For every $a > 0$,

$$p(t,x) = p(at, a^{1/\alpha}x)a^{d/\alpha} . \tag{2.1}$$

Our main assumption concerns the form of the spherical part λ of the Lévy measure ν. We will assume that λ has a density with respect to the surface measure σ which is bounded and bounded away from zero. More precisely, we assume that there exist $\phi : S \to (0, +\infty)$ and $\kappa > 0$ such that

$$\phi = \frac{d\lambda}{d\sigma} \text{ and } \kappa \le \phi(\xi) \le \kappa^{-1}, \ \forall \xi \in S . \tag{2.2}$$

It immediately follows that the Lévy measure ν has a density $f(x) = \phi(x/|x|)|x|^{-(d+\alpha)}$ with respect to the d-dimensional Lebesgue measure, and

$$\kappa |x|^{-(d+\alpha)} \le f(x) \le \kappa^{-1} |x|^{-(d+\alpha)} \tag{2.3}$$

for every $x \in \mathbb{R}^d \setminus \{0\}$.

One consequence of the assumption (2.2) is that X is a process of type A in the terminology of [22], which implies that the densities are strictly positive: $p(t,x) > 0$ for all $t > 0$ and all $x \in \mathbb{R}^d$ (see [17] and [22]).

Another important consequence of the assumption (2.2) is the more recent estimate for the density of X_t: There exists a finite, positive constant C such that

$$p(t,x) \le Ct, \quad \text{for every } x \in \mathbb{R}^d \text{ such that } |x| = 1 . \tag{2.4}$$

Let us make a few remarks about this estimate. Since the density $p(1,x)$ is uniformly bounded in $x \in \mathbb{R}^d$, (2.4) is the estimate of the density for small t. It is a special case of Theorem 3 in [16], and spelled out under the assumption (2.2) as Example 2 of the same paper. Let us point out that the implicit constant in Theorem 3 in [16] may depend on $x \in \mathbb{R}^d$. Dependence of the constant on x enters through (34) in [16], and implies dependence of constant on x in Lemma 10 (in [16] the corresponding variable is denoted by y). A careful reading of the proof of Theorem 3 reveals that under the assumption (2.2), the constant may be chosen independently of x as long as $|x| = 1$. The estimate (2.4)is used to prove the following result.

Proposition 2.1 *The function $(t,x) \mapsto p(t,x)$ is uniformly continuous and bounded on the set $\{(t,x) : t > 0, |x| \geq \psi\}$ for every fixed $\psi > 0$.*

Proof: By the scaling property (2.1), uniform boundedness of $p(1,x)$, and (2.4), it easily follows that there exists a constant $\widetilde{C} > 0$ depending on ψ, such that

$$
\begin{aligned}
p(t,x) &\leq \widetilde{C}t, \quad |x| \geq \psi,\ t > 0, && (2.5)\\
p(t,x) &\leq \widetilde{C}t^{-d/\alpha}, \quad x \in \mathbb{R}^d,\ t > 0, && (2.6)\\
p(t,x) &\leq \widetilde{C}t|x|^{-(d+\alpha)}, \quad x \in \mathbb{R}^d \setminus \{0\},\ t > 0. && (2.7)
\end{aligned}
$$

Given $\epsilon > 0$, the first two estimates imply that there exist $t_0, T_0 \in (0,\infty)$ such that

$$p(t,x) < \epsilon, \quad \text{for } 0 < t < t_0 \text{ or } t > T_0, \quad \text{and } |x| \geq \psi\,. \tag{2.8}$$

From the third estimate it follows that there exists $\Psi > \psi$ such that

$$p(t,x) \leq \widetilde{C}T_0|x|^{-(d+\alpha)} < \epsilon, \quad \text{for all } t \leq T_0, \text{ and all } |x| > \Psi\,. \tag{2.9}$$

Since $p(t,x)$ is jointly continuous, it is uniformly continuous on the set $[t_0, T_0] \times \{x \in \mathbb{R}^d : \psi \leq |x| \leq \Psi\}$. Together with (2.8) and (2.9) this proves the uniform continuity of $p(t,x)$ on the set $\{(t,x) : t > 0, |x| \geq \psi\}$. Boundedness follows from (2.8) and uniform continuity. ■

Similarly to the upper estimate (2.4), there is a lower estimate for the transition density: There exist a constant $c > 0$ and $t_0 > 0$ such that

$$p(t,x) \geq ct, \quad \text{for every } x \in \mathbb{R}^d \text{ such that } |x| = 1, \quad \text{and all } t \leq t_0\,. \tag{2.10}$$

Again, this is proved in [16] (see Theorem 2) with a constant c possibly depending on x, and without explicit t_0. But, a slight modification of the proof of Lemma 5.3 in [15] reveals that the constant c and t_0 can be chosen independently of x for $|x| = 1$. Scaling property (2.1), strict positivity of the density, and (2.10) imply that for every $\psi > 0$ there exist a constant $\widetilde{c} > 0$ and $\widetilde{t_0} > 0$ (depending on ψ) such that

$$p(t,x) \geq \widetilde{c}\, t|x|^{-(d+\alpha)}, \quad \text{for all } |x| \geq \psi \text{ and all } 0 < t \leq \widetilde{t_0}\,. \tag{2.11}$$

Note that (2.7) and (2.11) give the following bounds on density $p(1,x)$ for large x (with, perhaps, different constants)

$$\widetilde{c}|x|^{-(\alpha+d)} \leq p(1,x) \leq \widetilde{C}|x|^{-(\alpha+d)} \tag{2.12}$$

This is a significant improvement of Theorem 1 in [19]. Let us note that for a symmetric α-stable process there actually exists the limit $\lim_{|x|\to\infty} p(1,x)|x|^{\alpha+d}$ (see [3]).

At the end of this section let us note that the dual process $\widehat{X} = -X$ has the transition density $\widehat{p}(t,x) = p(t,-x)$ and satisfies the same assumptions as X.

3 Transition density of the killed process

In this section we closely follow the presentation from [9].

Let $(P_t,\, t \geq 0)$ denote the transition semigroup of $X = (X_t, \mathbb{P}^x)$. Then $p(t,x,y) := p(t, y-x)$, $t \geq 0$, $x, y \in \mathbb{R}^d$, is the transition kernel of the semigroup (P_t). This function is strictly positive and jointly continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and satisfies the scaling property

$$p(t,x,y)=p(1,t^{-1/\alpha}x,t^{-1/\alpha}y)t^{-d/\alpha}\,.$$

Therefore the semigroup (P_t) has both the Feller and the strong Feller property.

For any set $D\subset\mathbb{R}^d$, let $\tau_D=\inf\{t>0:\ X_t\notin D\}$ denote the first exit time of X from D. A boundary point $z\in\partial D$ is regular for D if $\mathbb{P}^z(\tau_D=0)=1$, and D is said to be regular if every boundary point of D is regular. A boundary point $z\in\partial D$ is said to satisfy the exterior cone condition, if there exists a cone C with vertex z such that $C\cap B(z,r)\subset D^c$ for some $r>0$. Here $B(z,r)=\{x\in\mathbb{R}^d:\ |x-z|<r\}$. An open set D is said to satisfy the uniform exterior cone condition if every boundary point $z\in\partial D$ satisfies the exterior cone condition with the same aperture of the cone.

Proposition 3.1 *Let $z\in\partial D$ satisfy the exterior cone condition. Then z is regular for D.*

Proof: For $r>0$ let $\tau_r=\inf\{t>0:\ X_t\notin B(0,r)\}$. Under the $\mathbb{P}^0$ probability, the scaling property implies that for all $b>0$ and $r>0$, $X_{\tau_{br}}\stackrel{d}{=}bX_{\tau_r}$. Therefore, if C is a cone with vertex at the origin and $C_r=C\cap\{x\in\mathbb{R}^d:\ |x|\geq r\}$, then $r\mapsto\mathbb{P}^0(X_{\tau_r}\in C_r)$ is a constant function. With this at hand, the proof of Theorem 2.2 from [9] carries over to our setting. ∎

Let D be a bounded domain. Since the process X is transient (e.g. [20], Theorem 37.8 and Theorem 3.18), $\tau_D<\infty$, $\mathbb{P}^x$ a.s. for all $x\in\mathbb{R}^d$. Adjoin the cemetery point Δ to D and define the killed process X^D by

$$X_t^D(\omega)=\begin{cases} X_t(\omega) & \text{if}\quad t<\tau_D(\omega)\\ \Delta & \text{if}\quad t\geq\tau_D(\omega)\end{cases}$$

This process is killed upon leaving D. For $t\geq 0$, $x\in\mathbb{R}^d$, and $f\in L^\infty(D)$, we define the transition operators $(P_t^D:\ t\geq 0)$ by

$$P_t^D f(x)=\mathbb{E}^x[f(X_t):\ t<\tau_D]\,.$$

Let $C_b(D)$ be the space of bounded continuous functions on D and $C_0(D)$ the space of continuous functions on $\bar{D}$ that vanish on ∂D. Since X is a doubly Feller process, the standard arguments (see [13], Section 2.1) imply that $P_t^D f \in C_b(D)$ for $t > 0$ and $f \in L^\infty$. Moreover, if D is regular, then $P_t^D f \in C_0(D)$ for $f \in C_0(D)$, and X^D on D has both the Feller and the strong Feller property.

We now want to show that P_t admits a continuous transition density. For $t > 0$ and $x, y \in \mathbb{R}^d$, let

$$r^D(t, x, y) = \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t]\,,$$

and

$$p^D(t, x, y) = p(t, x, y) - r^D(t, x, y)\,.$$

For the dual process $\widehat{X} = -X$, we analogously define $(\widehat{P}_t : t > 0)$ and $\widehat{p}^D(t, x, y)$.

Theorem 3.2 *Let $D \subset \mathbb{R}^d$ be a bounded domain. Then the following properties are true:*

(1) For every nonnegative Borel measurable function f on $\mathbb{R}^d$, and any $t > 0, x \in \mathbb{R}^d$,

$$P_t^D f(x) = \int_{\mathbb{R}^d} p^D(t, x, y) f(y)\, dy\,.$$

(2) The function $(t, x, y) \mapsto p^D(t, x, y)$ is continuous on $(0, \infty) \times (\mathbb{R}^d \setminus \partial D) \times (\mathbb{R}^d \setminus \partial D)$.

(3) For all $t > 0, x, y \in \mathbb{R}^d$, it holds that

$$p^D(t, x, y) = \widehat{p}^D(t, y, x)\,.$$

(4) The function $p^D(t, \cdot, \cdot)$ is strictly positive on $D \times D$.

(5) For any $t, s > 0, x, y \in \mathbb{R}^d$

$$p^D(t+s,x,y) = \int_{\mathbb{R}^d} p^D(t,x,z)p^D(s,z,y)\,dz\,.$$

(6) For any $t > 0, y \in D$ *and a regular point* $z \in \partial D$,

$$\lim_{D\ni x\to z} p^D(t,x,y) = 0\,.$$

Remark: The analogous theorem for symmetric α-stable process is stated and proved in [9] as Theorem 2.4. The symmetry of the transition density is replaced by the duality relation (3).

Proof: Properties (1), (2), (3), (5) and (6) can be proved by applying the arguments from [1], Section II.4, or [13], Section 2.2. The uniform continuity and boundedness of $p(t,x,y)$ shown in Proposition 2.1, as well as estimates from the proof of that proposition, are crucially used in several places. Moreover, in proving (6), we need the following property (see Proposition 6.1(5)): If $g : D \to \mathbb{R}$ is bounded and continuous at z, and $H_D g(x) := \mathbb{E}^x[g(X_{\tau_D}]$, then

$$\lim_{D\ni x\to z} H_D g(x) = g(z)\,. \tag{3.1}$$

Property (4) can be proved in the same way as in Theorem 2.4 from [9]. In this part we have to use estimates (2.6) and (2.11). Symmetry of the transition density is used in [9] at one point in the proof to conclude that $p^D(t,x,y) > 0$ for any $0 \le t \le t_1$, and $(x,y) \in (B(a,3r)\setminus B(a,2r)) \times B(a,r)$. In our situation this follows from (3) and the fact that the analogous proof gives that $\hat{p}^D(t,y,x) > 0$ for any $0 \le t \le t_1$, and $(y,x) \in B(a,r) \times (B(a,3r)\setminus B(a,2r))$. ■

If $x \in \bar{D}^c$, then $\mathbb{P}^x(\tau_D = 0) = 1$, and therefore $r^D(t,x,y) = \mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y)] = p(t,x,y)$, implying $p^D(t,x,y) = 0$ for all $y \in \mathbb{R}^d$. Similarly, for $x \in \bar{D}^c, y \in \mathbb{R}^d$, $\widehat{p}^D(t,x,y) = 0$. By (3) in the previous theorem, $p^D(t,y,x) = \widehat{p}^D(t,x,y) = 0$ for $x \in D^c$. Hence $p^D(t,x,y) = 0$ if $x \in D^c$ or $y \in D^c$.

4 Green function of the killed process

For $x \in \mathbb{R}^d$ let

$$u(x) = \int_0^\infty p(t,x)\,dt\,.$$

By using the scaling property (2.1) it easily follows that for $x \neq 0$

$$u(x) = |x|^{\alpha-d}\,u(x/|x|)\,, \tag{4.1}$$

and $u(0) = +\infty$.

Proposition 4.1 *The function $x \mapsto u(x)$ is finite and continuous on $\mathbb{R}^d \setminus \{0\}$, and continuous in the extended sense on $\mathbb{R}^d$.*

Proof: Let us first show that $u(x) < \infty$ for $x \neq 0$. By (4.1), it suffices to consider points on the unit sphere S. For $|x| = 1$, we have that $p(t,x) \le Ct$, $t > 0$ and $p(t,x) \le \widetilde{C}t^{-d/\alpha}$ (see (2.4) and (2.6)). Therefore,

$$u(x) = \int_0^1 p(t,x)\,dt + \int_1^\infty p(t,x)\,dt \le \int_0^1 Ct\,dt + \int_1^\infty \widetilde{C}t^{-d/\alpha}\,dt < \infty\,.$$

To prove continuity, let $x \neq 0$ and let $\psi = |x|/2$. Let $(x_n : n \ge 1)$ be a a sequence converging to x such that $|x_n| \ge \psi$. By (2.5), $p(t,x_n) \le \widetilde{C}t$, and by (2.6), $p(t,x_n) \le \widetilde{C}t^{-d/\alpha}$. By splitting the integral $\int_0^\infty p(t,x_n)\,dt = \int_0^1 p(t,x_n)\,dt + \int_1^\infty p(t,x_n)\,dt$, applying the dominated convergence theorem to both parts, and using continuity of $p(t,\cdot)$, we get that

$$u(x_n) = \int_0^\infty p(t,x_n)\,dt \longrightarrow \int_0^\infty p(t,x)\,dt = u(x)\,, \quad \text{as } n \to \infty\,.$$

Let $x_n \to 0$. By Fatou's lemma

$$\liminf_{n\to\infty} u(x_n) = \liminf_{n\to\infty} \int_0^\infty p(t,x_n)\,dt \ge \int_0^\infty p(t,0)\,dt = u(0) = +\infty\,,$$

proving extended continuity at 0. ∎

Since u is strictly positive and continuous on the unit sphere S, there exists a constant $k \in (0, \infty)$ such that $k \leq u(x) \leq k^{-1}$ for all $x \in S$. It follows from (4.1) that

$$k|x|^{\alpha-d} \leq u(x) \leq k^{-1}|x|^{\alpha-d} \quad \text{for every } x \in \mathbb{R}^d\,. \tag{4.2}$$

Let U denote the potential of the process X, i.e., the semigroup $(P_t : t > 0)$. That is, for a nonnegative Borel function f on $\mathbb{R}^d$,

$$Uf(x) = \int_0^\infty P_t f(x)\, dt = \mathbb{E}^x \int_0^\infty f(X_t)\, dt\,.$$

By defining $u(x, y) = u(y - x)$, $x, y \in \mathbb{R}^d$, it immediately follows that

$$Uf(x) = \int_{\mathbb{R}^d} u(x, y)\, dy\,.$$

The function $(x, y) \mapsto u(x, y)$ is the Green function of X. It is finite and jointly continuous on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\}$, and continuous in the extended sense on $\mathbb{R}^d \times \mathbb{R}^d$. Moreover, by (4.1),

$$u(x, y) = |x - y|^{\alpha-d}\, u\left(\frac{y - x}{|y - x|}\right)\,.$$

Let $G(x, y)$ denote the Green function of the symmetric α-stable process. Then $G(x, y) = c(d, \alpha)|x-y|^{\alpha-d}$ with $c(d, \alpha) = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)\Gamma(\alpha/2)^{-1}$. Together with (4.2) this implies the following important estimate on the Green function of X (with a different constant k):

$$kG(x, y) \leq u(x, y) \leq k^{-1}G(x, y) \quad \text{for all } x, y \in \mathbb{R}^d\,. \tag{4.3}$$

Let D be a bounded domain in $\mathbb{R}^d$ and X^D the process killed upon exiting D. The potential of X^D is defined by

$$U_D f(x) = \int_0^\infty P_t^D f(x)\, dt = \mathbb{E}^x \int_0^\infty f(X_t^D)\, dt\,,$$

for a nonnegative Borel f defined on D. Let us define

$$u_D(x, y) = \int_0^\infty p^D(t, x, y)\, dt\,.$$

Note that by Theorem 3.2 (5), $u_D(x, y) > 0$ for all $x, y \in D$.

Theorem 4.2 *The following properties are true:*

(1) For every nonnegative Borel measurable function f on D,

$$U_D f(x) = \int_D u_D(x,y) f(y)\, dy\,.$$

(2) The function $u_D(\cdot,\cdot)$ is strictly positive, finite and continuous on $D \times D \setminus \{(x,x) : x \in D\}$, and continuous in the extended sense on $D \times D$.

(3) For all $x, y \in D$,

$$u_D(x,y) = u(x,y) - \mathbb{E}^x[u(X_{\tau_D}, y)]\,.$$

(4) For all $x, y \in D$,

$$u_D(x,y) = \widehat{u_D}(y,x)\,.$$

(5) For any $y \in D$ and a regular point $z \in \partial D$,

$$\lim_{D \ni x \to z} u_D(x,y) = 0\,.$$

Proof: Assertion (1) follows from Theorem 3.2 and definition of u_D. Finiteness and continuity of u_D on $D \times D \setminus \{(x,x) : x \in D\}$ can be proved in the same way as in Proposition 4.1.

Note that for $x, y \in D$

$$\begin{aligned} \int_0^\infty r^D(t,x,y)\, dt &= \int_0^\infty \mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y), t > \tau_D]\, dt \\ &= \mathbb{E}^x \int_{\tau_D}^\infty p(t-\tau_D, X_{\tau_D}, y)\, dt = \mathbb{E}^x \int_0^\infty p(s, X_{\tau_D}, y)\, dt \\ &= \mathbb{E}^x[u(X_{\tau_D}, y)]\,. \end{aligned}$$

Let $\delta = \mathrm{dist}(y, D^c)$. Then by (4.2)

$$\begin{aligned} \mathbb{E}^x[u(X_{\tau_D}, y)] &= \mathbb{E}^x[u(y - X_{\tau_D})] \\ &\le \mathbb{E}^x[|y - X_{\tau_D}|^{\alpha-d} k^{-1}] \\ &\le \delta^{\alpha-d} k^{-1} < \infty \end{aligned}$$

Therefore, $\int_0^\infty r^D(t,x,y)\,dt < \infty$, for all $x,y \in D$. Hence,

$$\begin{aligned} u_D(x,y) &= \int_0^\infty (p(t,x,y) - r^D(t,x,y))\,dt \\ &= \int_0^\infty p(t,x,y)\,dt - \int_0^\infty r^D(t,x,y)\,dt \\ &= u(x,y) - \mathbb{E}^x[u(X_{\tau_D},y)] \end{aligned}$$

proving assertion (3). In particular, $u_D(x,x) = +\infty$, for all $x \in D$. The extended continuity of u_D is now proved as is Proposition 4.1. Assertion (4) is a consequence of (3) in Theorem 3.2. Finally, assertion (5) follows from (3) by using (3.1). ■

Corollary 4.3 *Let $z \in \partial D$ be regular. Then*

$$\lim_{D \ni x \to z} \mathbb{E}^x(\tau_D) = 0\,.$$

Proof: Let us first note that $\mathbb{E}^x(\tau_D) = \int_D u_D(x,y)\,dy$. Further, by (4.2) and Theorem 4.2(3) , we have that $u_D(x,y) \le k^{-1}|x-y|^{\alpha-d}$, for all $x,y \in D$. Let us fix a small $\delta > 0$. For $x \in B(z,\delta/2) \cap D$,

$$\begin{aligned} \int_{B(z,\delta)\cap D} u_D(x,y)\,dy &\le k^{-1} \int_{B(z,\delta)} |x-y|^{\alpha-d}\,dy \\ &= k^{-1} \int_{B(x,\delta/2)} |x-y|^{\alpha-d}\,dy + k^{-1} \int_{B(z,\delta)\setminus B(x,\delta/2)} |x-y|^{\alpha-d}\,dy \end{aligned}$$

For the first integral above we have

$$\int_{B(x,\delta/2)} |x-y|^{\alpha-d}\,dy = \frac{c(d)}{\alpha}\left(\frac{\delta}{2}\right)^{\alpha}.$$

The second integral can be estimated as

$$\begin{aligned} \int_{B(z,\delta)\setminus B(x,\delta/2)} |x-y|^{\alpha-d}\,dy &\le \int_{B(z,\delta)\setminus B(x,\delta/2)} (\delta/2)^{\alpha-d}\,dy \\ &\le (\delta/2)^{\alpha-d}|B(z,\delta)| \\ &\le c(d)2^{d-\alpha}\delta^\alpha\,. \end{aligned}$$

Therefore,

$$\int_{B(z,\delta)\cap D} u_D(x,y)\,dy \le c_{4.1}\delta^\alpha \tag{4.4}$$

Further, for $x \in B(z,\delta/2)$ and $y \in D \setminus B(z,\delta)$, it holds that $u_D(x,y) \le k^{-1}|x-y|^{\alpha-d} \le k^{-1}(\delta/2)^{\alpha-d}$. Hence, the dominated convergence theorem and Theorem 4.2(5) imply that

$$\lim_{D\ni x\to z} \int_{D\setminus B(z,\delta)} u_D(x,y)\,dy = 0\,.$$

Together with (4.4), this implies that

$$\limsup_{D\ni x\to z} \int_D u_D(x,y)\,dy \le c_{4.1}\delta^\alpha\,.$$

Since $\delta > 0$ was arbitrary, the claim follows. ■

Let $G_D(x,y)$ denote the Green function of the symmetric α-stable process killed upon exiting D. For a $C^{1,1}$ domain D there exist quite precise estimates for G_D (see [10]). If one could compare $u_D(\cdot,\cdot)$ with $G_D(\cdot,\cdot)$ in $D\times D$, those estimates would transfer to estimates for u_D. In the next theorem we will show that u_D is comparable with G_D away from the boundary of D, or more explicitly, that u_D is comparable with $|x-y|^{\alpha-d}$ away from the boundary

For $\delta > 0$ let $D_\delta = \{x \in D:\ \mathrm{dist}(x, D^c) > \delta\}$. Recall the estimate (4.3):

$$kG(x,y) \le u(x,y) \le k^{-1}G(x,y) \ \text{ for all } x,y \in \mathbb{R}^d\,.$$

Theorem 4.4 *There exists a positive constant $c_{4.1}$ depending on δ, such that for all $x,y \in D_\delta$*

$$c_{4.1}|x-y|^{\alpha-d} \le u_D(x,y) \le c_{4.1}^{-1}|x-y|^{\alpha-d}\,. \tag{4.5}$$

Proof: Let $y \in D_\delta$. Then $\mathbb{E}^x[u(X_{\tau_D},y)] \le k^{-1}c(d,\alpha)\mathbb{E}^x[|X_{\tau_D}-y|^{\alpha-d}] \le k^{-1}c(d,\alpha)\delta^{\alpha-d}$. Hence,

$$u_D(x,y) = u(x,y) - \mathbb{E}^x[u(X_{\tau_D},y)] \ge c(d,\alpha)(k|x-y|^{\alpha-d} - k^{-1}\delta^{\alpha-d})\,. \tag{4.6}$$

Let $\eta = \eta(\delta) = (k^2/2)^{1/(d-\alpha)}\delta$. For $x \in D$ such that $|x-y| \le \eta$, we have that

$$k|x-y|^{\alpha-d} - k^{-1}\delta^{\alpha-d} \ge \frac{k}{2}|x-y|^{\alpha-d}. \tag{4.7}$$

From (4.6) and (4.7) it follows that

$$u_D(x,y) \ge \frac{k}{2}c(d,\alpha)|x-y|^{\alpha-d}. \tag{4.8}$$

Since always $u_D(x,y) \le u(x,y)$, we have that for $y \in D_\delta$ and $x \in D$ such that $|x-y| \le \eta$

$$u_D(x,y) \le u(x,y) \le k^{-1}c(d,\alpha)|x-y|^{\alpha-d}. \tag{4.9}$$

By putting these estimates together, we get that there exists $c_{4.2} \in (0,\infty)$ depending on δ such that

$$c_{4.2}|x-y|^{\alpha-d} \le u_D(x,y) \le c_{4.2}^{-1}|x-y|^{\alpha-d} \tag{4.10}$$

for all $y \in D_\delta$, and all $x \in D$ such that $|x-y| \le \eta$.
The set $F_\delta = D_\delta \times D_\delta \setminus \{(x,y) : |x-y| \le \eta\}$ is a compact subset of $D \times D \setminus \{(x,x) : x \in D\}$. Since both u_D and $(x,y) \mapsto |x-y|^{\alpha-d}$ are strictly positive and continuous on F_δ, they are bounded and bounded away from zero on this set. Hence, (4.10) holds true on F_δ with a constant $c_{4.3}$ depending on δ. Therefore, (4.5) holds with $c_{4.1} = \min\{c_{4.2}, c_{4.3}\}$. ■

As a consequence of Theorem 4.4, we may conclude that there exists a positive constant $c_{4.4}$ depending on δ such that for all $x, y \in D_\delta$

$$c_{4.1}G_D(x,y) \le u_D(x,y) \le c_{4.1}^{-1}G_D(x,y). \tag{4.11}$$

5 Exit distributions and the Poisson kernel

In this section we study the exit distributions of the process X from a bounded domain D, show that in case D satisfies the uniform volume condition these exit distributions have a density, and under an additional assumption show that the density is continuous. A domain D is said to satisfy the *uniform volume condition* if there exists a constant $\rho > 0$ such that for every $x \in D$ the following holds:

$$\mathrm{vol}(D^c \cap B(x, 2\mathrm{dist}(x, \partial D))) \geq \rho\, \mathrm{dist}(x, \partial D)^d\,.$$

Note that the uniform exterior cone condition implies the uniform volume condition.

Let $X = (X_t, \mathbb{P}^x)$ be a strictly stable process satisfying assumption from Section 2., and let $\tau_D = \inf\{t > 0 : X_t \notin D\}$ be the first exit time from a bounded domain D. The following formula establishing connection between the Lévy measure of the process and the harmonic measure (i.e. the exit distribution) was proved in [14] (see also Theorem 3.1 in [10]): For $A \subset D^c$ and every $x \in D$

$$\mathbb{P}^x(X_{\tau_D} \in A, X_{\tau_D} \neq X_{\tau_D -}) = \int_D \nu(A - y)\, U_D(x, dy)\,. \tag{5.1}$$

In particular, if $\mathrm{dist}(A, D) > 0$, then $X_{\tau_D} \in A$ only if the process jumps out from D, implying

$$\mathbb{P}^x(X_{\tau_D} \in A) = \int_D \nu(A - y)\, U_D(x, dy), \quad A \subset D^c,\ \mathrm{dist}(A, D) > 0\,. \tag{5.2}$$

Under our assumptions, the Lévy measure ν has a density f, and the potential measure $U_D(x, \cdot)$ has a density $u_D(x, \cdot)$. Hence, the formula (5.2) takes the following form:

$$\mathbb{P}^x(X_{\tau_D} \in A) = \int_A \left(\int_D f(z - y) u_D(x, y)\, dy \right) dz, \quad A \subset D^c,\ \mathrm{dist}(A, D) > 0\,. \tag{5.3}$$

We would like to show that if D is a bounded domain satisfying the uniform volume condition, then $\mathbb{P}^x(X_{\tau_D} \neq X_{\tau_D -}) = 1$. For a symmetric α-stable process and domains satisfying the uniform exterior cone condition this is proved in Lemma 6 of [5]. Those arguments carry over to our situation. Some changes are needed to show the following lemma.

Lemma 5.1 *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform volume condition. There exists a constant $p > 0$ depending on κ, ρ, α, d, such that for every $x \in D$*

$$\mathbb{P}^x(X_{\tau_{B(x,r)}} \in D^c) \geq p\,, \tag{5.4}$$

where $r = r_x = dist(x, D^c)/3$.

Proof: By applying the formula (5.3) to $B(x,r)$ and D^c, we get

$$\begin{aligned}
\mathbb{P}^x(X_{\tau_{B(x,r)}} \in D^c) &= \int_{D^c}\left(\int_{B(x,r)} f(z-y)u_{B(x,r)}(x,y)\,dy\right)dz \\
&\geq \int_{D^c}\left(\int_{B(x,r)} \kappa|z-y|^{-(\alpha+d)}\,u_{B(x,r)}(x,y)\,dy\right)dz \\
&\geq \int_{D^c}\left(\int_{B(x,r)} \kappa 2^{-(\alpha+d)}|z-x|^{-(\alpha+d)}\,u_{B(x,r)}(x,y)\,dy\right)dz \\
&= \kappa 2^{-(\alpha+d)} U_{B(x,r)}(x, B(x,r)) \int_{D^c} |z-x|^{-(\alpha+d)}\,dz
\end{aligned}$$

where the first inequality follows from assumption (2.2), and the second from the fact that $|z-y| \leq 2|z-x|$ for $y \in B(x,r)$ and $z \in D^c$. Note that $U_{B(x,r)}(x, B(x,r)) = \mathbb{E}^x[\tau_{B(x,r)}] = c(\alpha)r^\alpha$ by the scaling property. Hence,

$$\mathbb{P}^x(X_{\tau_{B(x,r)}} \in D^c) \geq \kappa c(\alpha) 2^{-(\alpha+d)} r^\alpha \int_{D^c} |z-x|^{-(\alpha+d)}\,dz \tag{5.5}$$

The integral is estimated by means of the uniform volume condition:

$$\begin{aligned}
\int_{D^c} |z-x|^{-(\alpha+d)}\,dz &\geq \int_{D^c \cap B(x,6r)} |z-x|^{-(\alpha+d)}\,dz \\
&\geq (6r)^{-(\alpha+d)} \mathrm{vol}(D^c \cap B(x,6r)) \\
&\geq (6r)^{-(\alpha+d)} \rho (3r)^d = c(\rho,\alpha,d) r^{-\alpha}\,.
\end{aligned} \tag{5.6}$$

Now (5.2) follows from (5.5) and (5.6). ∎

The following result is proved exactly as in Lemma 6 of [5].

Proposition 5.2 *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform volume condition. Then for every $x \in D$,*

$$\mathbb{P}^x(X_{\tau_D} \neq X_{\tau_D -}) = 1\,.$$

We assume from now on that $D \subset \mathbb{R}^d$ satisfies the uniform volume condition. From (5.1) and Proposition 5.2 we get that for all $x \in D$

$$\mathbb{P}^x(X_{\tau_D} \in A) = \int_A \left(\int_D f(z-y) u_D(x,y) \right) dz, \quad A \subset \bar{D}^c, . \tag{5.7}$$

Let us define the Poisson kernel $P_D(x,z)$, $x \in D, z \in \bar{D}^c$ by

$$P_D(x,z) = \int_D f(z-y) u_D(x,y)\, dy\,. \tag{5.8}$$

The Poisson kernel $P_D(x,\cdot)$ is the density of the exit distribution of X under $\mathbb{P}^x$:

$$\mathbb{P}^x(X_{\tau_D} \in A) = \int_A P_D(x,z)\, dz\,, A \subset \bar{D}^c\,. \tag{5.9}$$

We will later need the following estimate of the Poisson kernel for points z away from the boundary ∂D.

Lemma 5.3 *Let $x \in \mathbb{R}^d$ and let $r > 0$. For all $z \in B(x,2r)^c$, and all $y \in B(x,r)$ we have that*

$$\kappa 2^{-(\alpha+d)} \frac{\mathbb{E}^y(\tau_{B(x,r)})}{|x-z|^{\alpha+d}} \le P_{B(x,r)}(y,z) \le \kappa^{-1} 2^{\alpha+d} \frac{\mathbb{E}^y(\tau_{B(x,r)})}{|x-z|^{\alpha+d}}\,. \tag{5.10}$$

Proof: Note that $2^{-1}|x-z| \le |w-z| \le 2|x-z|$ for any $w \in B(x,r)$ and $z \in B(x,2r)^c$. Then

$$\begin{aligned}
P_{B(x,r)}(y,z) &= \int_{B(x,r)} f(z-w) u_{B(x,r)}(y,w)\, dw \\
&\ge \int_{B(x,r)} \kappa |w-z|^{-(\alpha+d)} u_{B(x,r)}(y,w)\, dw \\
&\ge \int_{B(x,r)} \kappa 2^{-(\alpha+d)} |x-z|^{-(\alpha+d)} u_{B(x,r)}(y,w)\, dw \\
&= \kappa 2^{-(\alpha+d)} \mathbb{E}^y(\tau_{B(x,r)}) |x-z|^{-(\alpha+d)}
\end{aligned}$$

where the first inequality follows from (2.3). The other inequality is proved exactly in the same way. ∎

A similar estimate comes from [5] (Lemma 7). We give a simple proof.

Lemma 5.4 *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform volume condition, and let $\lambda > 0$. For all $z \in D^c$ satisfying $dist(z, D) \geq \lambda\, diam(D)$, and all $x \in D$,*

$$\kappa(1+\lambda^{-1})^{-(\alpha+d)} \frac{\mathbb{E}^x(\tau_D)}{dist(z,D)^{\alpha+d}} \leq P_D(x,z) \leq \kappa^{-1} \frac{\mathbb{E}^x(\tau_D)}{dist(z,D)^{\alpha+d}} \tag{5.11}$$

Proof: Let $z \in D^c$ be such that $\mathrm{dist}(z, D) \geq \lambda \mathrm{diam}(D)$. Then

$$\mathrm{dist}(z,D) \leq |z-y| \leq \mathrm{dist}(z,D) + \mathrm{diam}(D) \leq (1+1/\lambda)\mathrm{dist}(z,D).$$

Since $\kappa|z-y|^{-(\alpha+d)} \leq f(z-y) \leq \kappa^{-1}|z-y|^{-(\alpha+d)}$, we get that

$$\kappa(1+1/\lambda)^{-(\alpha+d)}\mathrm{dist}(z,D)^{-(\alpha+d)} \leq f(z-y) \leq \kappa^{-1}\mathrm{dist}(z,D)^{-(\alpha+d)}.$$

We integrate above inequalities against $u_D(x,y)dy$ to get (5.11). ■

Corollary 5.5 *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform exterior cone condition. Then for every $\zeta \in \partial D$*

$$\lim_{D \ni x \to \zeta} P_D(x,z) = 0\,,\ z \in \bar{D}^c\,.$$

Proof: The statement follows from the right hand side of (5.11) and Corollary 4.3. ■

In the remaining part of this section we will assume that the density f of the Lévy measure ν is a continuous function which satisfies (2.3). Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform volume condition. We will show that the Poisson kernel $P_D(\cdot,\cdot)$ is jointly continuous on $D \times \bar{D}^c$. Let us first prepare a lemma.

Lemma 5.6 *Let $x \in D$, $0 < \delta < dist(x, \partial D)$, and $\epsilon > 0$. There exists a constant $c_{5.1} > 0$ depending on ϵ, such that for all $w \in B(x, \delta/2)$ and all $v \in D^c$ satisfying $dist(v, D) \geq \epsilon$,*

$$\int_{B(x,\delta)} f(v-y) u_D(w,y)\, dy \leq c_{5.1}\delta^\alpha\,. \tag{5.12}$$

Proof: Since $|v-y| \geq \epsilon$, it follows that $f(v-y) \leq \kappa^{-1}\epsilon^{-(\alpha+d)}$. Thus,

$$\int_{B(x,\delta)} f(v-y)u_D(w,y)\,dy \leq \kappa^{-1}\epsilon^{-(\alpha+d)} \int_{B(x,\delta)} u_D(w,y)\,dy\,,$$

so it suffices to estimate $\int_{B(x,\delta)} u_D(w,y)\,dy$. This is done exactly as in the proof of Corollary 4.3. ■

Theorem 5.7 *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform volume condition, and let the Lévy measure of X have a continuous density $f : \mathbb{R}^d \setminus \{0\} \to (0,+\infty)$ satisfying (2.3). Then the Poisson kernel $P_D : D \times \bar{D}^c \to (0,+\infty)$ is a jointly continuous function.*

Proof: Let $x \in D$ and $z \in \bar{D}^c$ be fixed. Let $(x_n : n \geq 1)$ be a sequence of points in D such that $\lim_n x_n = x$, and let $(z_n : n \geq 1)$ be a sequence of points in $\bar{D}^c$ such that $\lim_n z_n = z$. Choose $\delta > 0$ and $\epsilon > 0$ such that $\mathrm{dist}(x, D^c) \geq 2\delta$ and $\mathrm{dist}(z, D) \geq 2\epsilon$. Then $|x - x_n| < \delta/2$ and $\mathrm{dist}(z_n, D) \geq \epsilon$ for all but finitely many n.
If $y \in D \setminus B(x,\delta)$, then $u_D(w,y) \leq k^{-1}(\delta/2)^{\alpha-d}$ for $w \in B(x,\delta/2)$. Also, $(|z_n - y| : n \geq 1)$ is bounded away from zero. By continuity of f and u_D,

$$f(z_n - y)u_D(x_n,y)1_{D\setminus B(x,\delta)}(y) \longrightarrow f(z-y)u_D(x,y)1_{D\setminus B(x,\delta)}(y)\,, \quad n \to \infty\,,$$

and convergence is bounded by a finite constant. Since $|D| < \infty$, dominated convergence theorem implies that

$$\int_{D\setminus B(x,\delta)} f(z_n - y)u_D(x_n,y)\,dy \longrightarrow \int_{D\setminus B(x,\delta)} f(z-y)u_D(x,y)\,dy\,, \quad n \to \infty\,. \tag{5.13}$$

Now

$$\begin{aligned} |P_D(x_n,z_n) - P_D(x,z)| \;\; &\leq \;\; \Big|\int_{D\setminus B(x,\delta)} f(z_n - y)u_D(x_n,y)\,dy \\ &\quad - \int_{D\setminus B(x,\delta)} f(z-y)u_D(x,y)\,dy\Big| \end{aligned}$$

$$
\begin{aligned}
&+ \int_{B(x,\delta)} f(z_n - y) u_D(x_n, y)\, dy \\
&+ \int_{B(x,\delta)} f(z - y) u_D(x, y)\, dy \\
&\leq \left| \int_{D\setminus B(x,\delta)} f(z_n - y) u_D(x_n, y)\, dy \right. \\
&- \left. \int_{D\setminus B(x,\delta)} f(z - y) u_D(x, y)\, dy \right| + 2c_{5.1}\delta^\alpha
\end{aligned}
$$

where the last inequality follows from Lemma 5.6. Let $n \to \infty$. By (5.13),

$$
\limsup_n |P_D(x_n, z_n) - P_D(x, z)| \leq 2c_{5.1}\delta^\alpha \, .
$$

Since δ can be chosen arbitrarily small, the claim follows. ∎

Remark Let us assume that the density f of the Lévy measure of X satisfies (2.3), but is not necessarily continuous. By letting $z_n = z$ for all $n \in \mathbb{N}$ in the above proof, we conclude that for every $z \in \bar{D}^c$, the function $x \mapsto P_D(x, z)$ is continuous in D.

6 Harmonic functions and Harnack inequality

Harnack inequality for a symmetric α-stable process is an easy consequence of the explicit formula for the Poisson kernel for the ball. The lack of such a formula for other strictly stable processes makes Harnack inequality more difficult task. In this section we would like to point out that the very recent proof of Harnack inequality for jump processes given in [2] carries over to our situation with only minor modifications. We begin this section by recalling definition of harmonic functions and collecting some known properties.

Let $X = (X_t, \mathbb{P}^x)$ be a strictly α-stable processes satisfying assumptions from Section 2. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a Borel function bounded from below. We say that h is *harmonic for the process X* (or simply harmonic) in an open, bounded set $D \subset \mathbb{R}^d$ if

$$h(x) = \mathbb{E}^x[h(X_{\tau_U})]\,, \quad x \in U\,, \tag{6.1}$$

for every open set U such that $\bar{U} \subset D$. A function h is *regular harmonic* in D, if

$$h(x) = \mathbb{E}^x[h(X_{\tau_D})]\,, \quad x \in D\,, \tag{6.2}$$

A function $h : \mathbb{R}^d \to \mathbb{R}$ harmonic in D is said to be *stochastically regular* if for every nondecreasing sequence of stopping times $(T_n : n \geq 1)$ such that $T_n \to \tau_D$, it holds that $h(X_{T_n}) \to h(X_{\tau_D})$, $\mathbb{P}^x$ a.s.

The analogous definitions of harmonic function for symmetric α-stable process appear in [5], while a similar definition for more general Lévy processes can be found in [18]. Stochastic regularity comes from [18].

Let $D \subset \mathbb{R}^d$ be a bounded open set. For a bounded Borel function $g : D^c \to \mathbb{R}$ define

$$H_D g(x) = \mathbb{E}^x[g(X_{\tau_D})]\,, \quad x \in \mathbb{R}^d\,. \tag{6.3}$$

In the next proposition we list several known properties of harmonic functions.

Proposition 6.1 *Let $h : \mathbb{R}^d \to \mathbb{R}$ be a Borel function bounded from below, and let $D \subset \mathbb{R}^d$ be open and bounded.*

(1) If h is a regular harmonic function in D, then it is harmonic in D.

(2) If h is bounded on $\mathbb{R}^d$ and if $(h(X_{t\wedge\tau_D}) : t \geq 0)$ is a $\mathbb{P}^x$-martingale for every $x \in D$, then h is regular harmonic in D. Conversely, if h is bounded on D^c and regular harmonic in D, then $(h(X_{t\wedge\tau_D}) : t \geq 0)$ is a $\mathbb{P}^x$-martingale for every $x \in D$.

(3) Let $g : D^c \to \mathbb{R}$ be a bounded Borel function. Then $H_D g$ is a stochastically regular harmonic function in D. Conversely, if h is a bounded (on $\mathbb{R}^d$) stochastically regular harmonic function on D, then $h = H_D h$, i.e., h is a regular harmonic function on D.

(4) Let $g : D^c \to \mathbb{R}$ be a bounded Borel function. Then $H_D g$ is a continuous function in D.

(5) Let $g : D^c \to \mathbb{R}$ be a bounded Borel function. If g is continuous at $z \in \partial D$, then

$$\lim_{D \ni x \to z} H_D g(x) = g(z) .$$

(6) If h is harmonic in D and continuous on $\bar{D}$, then h is regular harmonic in D.

(7) Assume, additionally, that D is a bounded domain satisfying the uniform volume condition. If h is bounded on D and harmonic in D, then h is regular harmonic in D.

Proof: Assertion (1) is a consequence of the strong Markov property, (3) is proved in [18], Section 24, (4) follows from the strong Feller property of X (see [18], Section 25), while (2) and (5) are standard facts. The assertion (7) is proved in the same way as Lemma 17 in [5], and (6) can be proved similarly by use of the bounded convergence theorem. ∎

Assume that D is a bounded domain satisfying the uniform volume condition. If h is a regular harmonic function in D, then $h(x) = \mathbb{E}^x[h(X_{\tau_D})]$. From (5.9) we get the following representation of h:

$$h(x) = \int_{\bar{D}^c} P_D(x,z) h(z)\, dz\,, \quad x \in D\,. \tag{6.4}$$

In the sequel we closely follow [2].

Proposition 6.2 *(1) There exists a constant $c_{6.1}$ not depending on x such that*

$$\mathbb{P}^x(\sup_{0 \le s \le t} |X_s - X_0| > 1) \le c_{6.1} t$$

(2) Let $\epsilon > 0$. There exists a constant $c_{6.2}$ depending only on ϵ such that if $x \in \mathbb{R}^d$ and $r > 0$, then

$$\inf_{y \in B(x,(1-\epsilon)r)} \mathbb{E}^y[\tau_{B(x,r)}] \ge c_{6.2} r^\alpha .$$

(3) There exists a constant $c_{6.3}$ such that $\sup_y \mathbb{E}^y[\tau_{B(x,r)}] \le c_{6.3} r^\alpha$.

(4) Let $A \subset B(x,1)$. There exists a constant $c_{6.4}$ not depending on x and A such that

$$\mathbb{P}^y(T_A < \tau_{B(x,3)}) \ge c_{6.4}|A| \quad y \in B(x,2)\,.$$

Proof: This is proved exactly as in [2].

Proposition 6.3 *There exist constants $c_{6.5}$ and $c_{6.6}$ such that if $x \in \mathbb{R}^d$, $r > 0$, $y \in B(x,r)$, and g is a bounded nonnegative function supported in $B(x,2r)^c$, then*

$$c_{6.5}\left(\mathbb{E}^y \tau_{B(x,r)}\right) \int \frac{g(z)}{|z-x|^{\alpha+d}}\,dz \le \mathbb{E}^y g(X_{\tau_{B(x,r)}}) \le c_{6.6}\left(\mathbb{E}^y \tau_{B(x,r)}\right) \int \frac{g(z)}{|z-x|^{\alpha+d}}\,dz\,.$$

Proof: Note that

$$\mathbb{E}^y g(X_{\tau_{B(x,r)}}) = \int_{B(x,r)^c} P_{B(x,r)}(y,z) g(z)\,dz = \int_{B(x,2r)^c} P_{B(x,r)}(y,z) g(z)\,dz\,,$$

where the last equality follows from the fact that g is supported in $B(x,2r)^c$. The result now follows from Lemma 5.3. ∎

Let $h : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative and bounded function (on $\mathbb{R}^d$), harmonic in a bounded domain D which satisfies the uniform volume condition. Let $K \subset D$ be compact. If $h(x) > 0$ for some $x \in D$, then $\inf_{y\in K} h(y) > 0$. Indeed, $h(x) = \int_{\bar{D}^c} P_D(x,z)h(z)\,dz$, for all $x \in D$, and by definition (5.6), $P_D(x,z) > 0$ for every $z \in D^c$. Hence, if $h(x) = 0$ for some $x \in D$, then $h(z) = 0$ for almost all $z \in D^c$, and thus $h = 0$ in D. Therefore, $h > 0$ in D. By Proposition 6.1, (4) and (7), h is continuous on D. Therefore, $\inf_{y\in K} h(y) > 0$. With this fact, the proof of the following result follows verbatim the proof of Theorem 3.6 in [2].

Theorem 6.4 *There exists a constant $c_{6.7}$ such that if $h : \mathbb{R}^d \to \mathbb{R}$ is nonnegative and bounded in $\mathbb{R}^d$ and harmonic in the ball $B(x_0, 16)$, then*

$$h(x) \le c_{6.7} h(y)\,, \quad \text{for all } x, y \in B(x_0,1)\,. \tag{6.5}$$

Corollary 6.5 *Let $D \subset \mathbb{R}^d$ be a bounded domain, and let $K \subset D$ be compact. There exists a constant $c_{6.8}$ such that if $h : \mathbb{R}^d \to \mathbb{R}$ is nonnegative and bounded in $\mathbb{R}^d$ and harmonic in D, then $h(x) \le c_{6.8} h(y)$ for all $x, y \in K$.*

Proof: This is proved by the standard chain argument. ■

The last result is the Harnack inequality stated for nonnegative harmonic functions which are bounded on $\mathbb{R}^d$. In the sequel we are going to remove the restriction on boundedness.

Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the uniform volume condition. For a Borel set $A \subset \bar{D}^c$, the harmonic measure of A at $x \in D$ is defined by

$$\omega^x(A) = \mathbb{P}^x(X_{\tau_D} \in A) = \mathbb{E}^x[1_A(X_{\tau_D})] .$$

By Proposition 6.1, $x \to \omega^x(A)$ is a bounded, regular harmonic function in D, and admits a representation

$$x \to \omega^x(A) = \int_A P_D(x,z)\, dz .$$

Let $K \subset D$ be compact. By Corollary 6.5 there exists a constant $c_{6.8}$ such that $\omega^x(A) \le c_{6.8}\, \omega^y(A)$ for all $x, y \in K$ and all $A \subset \bar{D}^c$. Thus

$$\int_A P_D(x,z)\, dz \le c_{6.8} \int_A P_D(y,z)\, dz ,$$

for all $x, y \in K$ and all $A \subset \bar{D}^c$. This implies that for $x, y \in K$,

$$P_D(x,z) \le c_{6.8} P_D(y,z) \quad \text{for a.e. } z \in \bar{D}^c . \tag{6.6}$$

Note that this inequality for a ball is the starting point for usual proofs of Harnack inequality.

Lemma 6.6 *Let $h : \mathbb{R}^d \to [0, \infty)$ be regular harmonic in D. Then h is continuous in D.*

Proof: Let $x \in D$ and let $(x_n : n \geq 1)$ be a sequence of points in D such that $x = \lim_{n\to\infty} x_n$. By the remark following the proof of Theorem 5.7, $\lim_{n\to\infty} P_D(x_n, z) = P_D(x, z)$, for every $z \in \bar{D}^c$. There is a compact set $K \subset D$ such that $x_n, x \in D$. By (6.6),

$$P_D(x_n, z) \leq c_{6.8} P_D(x, z) \quad \text{for every } n \geq 1, \text{ for a.e. } z \in \bar{D}^c \,. \tag{6.7}$$

Since $\int_{D^c} P_D(x, z) h(z) = h(x) < \infty$, the function $z \mapsto P_D(x, z)h(z)$ is integrable on D^c. By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{D^c} P_D(x_n, z) u(z)\, dz = \int_{D^c} P_D(x, z) u(z)\, dz$$

proving the lemma. ∎

Theorem 6.7 *Let $D \subset \mathbb{R}^d$ be a bounded domain, and let $K \subset D$ be compact. There exists a constant $c_{6.9}$ such that if h is a nonnegative harmonic function in D, then*

$$h(x) \leq c_{6.9} h(y) \quad x, y \in K \,. \tag{6.8}$$

Moreover, every nonnegative harmonic function in D is continuous in D.

Proof: There exists a C^∞ domain U such that $K \subset U \subset \bar{U} \subset D$ (e.g. Lemma 2.4 of [8]). Clearly, such U satisfies the uniform volume condition, so by (6.6), there exists a constant $c_{6.10}$ such that for all $x, y \in K$

$$P_U(x, z) \leq c_{6.10} P_U(y, z), \quad \text{for a.e. } z \in \bar{U}^c \,.$$

Let h be a nonnegative harmonic function in D. Then

$$h(x) = \mathbb{E}^x[h(X_{\tau_U})] = \int_{\bar{U}^c} P_U(x, z) h(z)\, dz \,,$$

for all $x \in U$. In particular, for $x, y \in K$,

$$h(x) = \int_{\bar{U}^c} P_U(x, z) h(z)\, dz \leq c_{6.10} \int_{\bar{U}^c} P_U(y, z) h(z)\, dz = c_{6.10} h(y) \,.$$

Since h is nonnegative and regular harmonic in U, Lemma 6.6 implies that h is continuous in U. Again by [8], there exists an increasing sequence of C^∞ domains $(U_n,\ n \geq 1)$ such that $\bar{U}_n \subset U_{n+1}$ for all $n \geq 1$, and $\cup_{n=1}^\infty U_n = D$. Therefore, h is continuous on D. ■

Having the full Harnack inequality, we can prove the following version originally due to Bogdan [5] for a symmetric α-stable process.

Lemma 6.8 *Let $x_1, x_2 \in \mathbb{R}^d$, $r > 0$, $k \in \mathbb{N}$, and let $|x_1 - x_2| < 2^k r$. There exits a constant $c_{6.11}$ such that for every nonnegative function h which is harmonic in $B(x_1, r) \cup B(x_2, r)$,*

$$c_{6.11} 2^{-k(\alpha+d)} h(x_2) \leq h(x_1) \leq c_{6.11}^{-1} 2^{k(\alpha+d)} \tag{6.9}$$

Proof: This is proved by following arguments from Lemma 2 in [5], and using estimate (5.11) for the Poisson kernel. ■

The last lemma makes it possible to prove the Harnack inequality for not necessarily connected open sets.

Corollary 6.9 *Let $D \subset \mathbb{R}^d$ be a bounded open set, and let $K \subset D$ be compact. There exists a constant $c_{6.12}$ such that for every nonnegative function h harmonic in D,*

$$h(x) \leq c_{6.12} h(y) \quad \textit{for all } x, y \in K\,.$$

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