Potential Theory of Subordinate Killed Brownian Motion in a Domain*

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Abstract

Subordination of a killed Brownian motion in a bounded domain $D \subset \mathbb{R}^d$ via an $\alpha/2$-stable subordinator gives a process Z_t whose infinitesimal generator is $-(-\Delta|_D)^{\alpha/2}$, the fractional power of the negative Dirichlet Laplacian. In this paper we study the properties of the process Z_t in a Lipschitz domain D by comparing the process with the rotationally invariant α-stable process killed upon exiting D. We show that these processes have comparable killing measures, prove the intrinsic ultracontractivity of the semigroup of Z_t, and, in the case when D is a bounded $C^{1,1}$ domain, obtain bounds on the Green function and the jumping kernel of Z_t.

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1 Introduction

Let X_t be a d-dimensional Brownian motion in $\mathbb{R}^d$ and let T_t be an $\alpha/2$-stable subordinator starting at zero, $0 < \alpha < 2$. It is well known that $Y_t = X_{T_t}$ is a rotationally invariant α-stable process whose generator is $-(-\Delta)^{\alpha/2}$, the fractional power of the negative Laplacian. The potential theory corresponding to the process Y is the Riesz potential theory of order α.

Suppose that D is a domain in $\mathbb{R}^d$, that is, an open connected subset of $\mathbb{R}^d$. We can kill the process Y upon exiting D. The killed process Y^D has been extensively studied in the last five years and various deep properties have been obtained. For instance, when D is a bounded $C^{1,1}$ domain, sharp estimates on the Green function of Y^D were established in [5] and [15], while the intrinsic ultracontractivity of the semigroup corresponding to Y^D was proved in [4], [6] and [16].

Let $\Delta|_D$ be the Dirichlet Laplacian in D. The fractional power $-(-\Delta|_D)^{\alpha/2}$ of the negative Dirichlet Laplacian is a very useful object in analysis and partial differential equations, see, for instance, [22] and [18]. There is a Markov process Z corresponding to $-(-\Delta|_D)^{\alpha/2}$ which can be obtained as follows: We first kill the Brownian motion X at τ_D, the first exit time of X from the domain D, and then we subordinate the killed Brownian motion using the $\alpha/2$-stable subordinator T_t. Note that in comparison with Y^D the order of killing and subordination has been reversed. The difference between the processes Y^D and Z can be explained as follows: Look at a path of the Brownian motion in $\mathbb{R}^d$, and put a mark on it at all the times given by the subordinator T_t. In this way we observe a trajectory of the process Y. The corresponding trajectory of Z is given by all the marks on the Brownian path prior to τ_D. There is the first mark on the Brownian path following the exit time τ_D. If this mark happens to be in D, the process Y has not been killed yet, and the mark corresponds to a point on the trajectory of Y^D, but not to a point on the trajectory of Z. If, on the other hand, the first mark on the Brownian path following the exit time τ_D happens to be in D^c, then trajectories of Z and Y^D are equal.

Despite its importance, the process Z has not been studied much. In [12], a relation between the harmonic functions of Z and the classical harmonic functions in D was established. In [14] (see also [10]) the domain of the Dirichlet form of Z was identified when D is a bounded smooth domain and $\alpha \neq 1$.

In this paper we study the process Z and some of its potential-theoretic properties. One way to understand the process Z is to describe its killing and jumping measures. It turns out that, at least when D is Lipschitz, the killing measure is comparable with the killing measure of the process Y^D. This fact, shown in Sections 2 and 3 of the paper, follows from an analysis of the lifetimes of processes Z and Y^D. In order to do that, we have to give a precise description of both processes in terms of the underlying Brownian motion X_t and the subordinator T_t. The process Z, being a symmetric Markov process, has an associated Dirichlet form. By using the comparability of killing measures of Z and Y^D, we show that the corresponding Dirichlet forms are also comparable. This fact is then used in Section 4 to prove the intrinsic ultracontractivity of the semigroup corresponding to Z. As a consequence of this result we derive a lower bound on the Green function of Z in terms of the first eigenfunction of the Dirichlet Laplacian $\Delta|_D$. In the last section we derive upper bounds on the Green function of Z for $C^{1,1}$ domains. These bounds may not be sharp, but they show that the behaviors of the Green functions of Z and Y^D are very different. In the same vein we obtain bounds for the jump kernel of Z which confirm that the jump kernel vanishes near the boundary of D.

2 Subordinate killed Brownian motion

Let $X^1 = (\Omega^1, \mathcal{F}^1, \mathcal{F}_t^1, X_t^1, \theta_t^1, \mathbb{P}_x^1)$ be a d-dimensional Brownian motion in $\mathbb{R}^d$, and let $T^2 = (\Omega^2, \mathcal{G}^2, T_t^2, \mathbb{P}^2)$ be an $\alpha/2$-stable subordinator starting at zero, $0 < \alpha < 2$. We will consider both processes on the product space $\Omega = \Omega^1 \times \Omega^2$. Thus we set $\mathcal{F} = \mathcal{F}^1 \times \mathcal{G}^2$, $\mathcal{F}_t = \mathcal{F}_t^1 \times \mathcal{G}^2$, and $\mathbb{P}_x = \mathbb{P}_x^1 \times \mathbb{P}^2$. Moreover, we define $X_t(\omega) = X_t^1(\omega^1)$, $T_t(\omega) = T_t^2(\omega^2)$, and $\theta_t(\omega) = \theta_t^1(\omega^1)$, where $\omega = (\omega^1, \omega^2) \in \Omega$. Then $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x)$ is a d-dimensional $\mathcal{F}_t$-Brownian motion, and $T = (\Omega, \mathcal{G}, T_t, \mathbb{P}_x)$ is an $\alpha/2$-stable subordinator starting at zero, independent of X for every $\mathbb{P}_x$. From now on, all processes and random variables will be defined on Ω.

Let $A_t = \inf\{s > 0 : T_s \ge t\}$ be the inverse of T. Since (T_t) is strictly increasing, (A_t) is continuous. Further, $A_{T_t} = t$ and $T_{A_s-} \le s \le T_{A_s}$.

We define a process Y subordinate to X by $Y_t = X_{T_t}$. It is well known that Y is a rotationally invariant α-stable process in $\mathbb{R}^d$. If $\mu_t^{\alpha/2}$ is the distribution of T_t (i.e., $(\mu_t^{\alpha/2}, t \ge 0)$ is one-sided $\alpha/2$-stable convolution semigroup), and $(P_t, t \ge 0)$ the semigroup corresponding to the Brownian motion X, then for any nonnegative Borel function f on $\mathbb{R}^d$, $\mathbb{E}_x(f(Y_t)) = \mathbb{E}_x(f(X_{T_t})) = \mathbb{E}_x(\int_0^\infty f(X_s)\, \mu_t^{\alpha/2}(ds)) = \int_0^\infty P_s f(x) \mu_t^{\alpha/2}(ds)$.

Let $D \subset \mathbb{R}^d$ be a bounded domain, and let $\tau_D^Y = \inf\{t > 0 : Y_t \notin D\}$ be the exit time of Y from D. The process Y killed upon exiting D is defined by

$$Y_t^D = \begin{cases} Y_t, & t < \tau_D^Y \\ \partial, & t \ge \tau_D^Y \end{cases} = \begin{cases} X_{T_t}, & t < \tau_D^Y \\ \partial, & t \ge \tau_D^Y \end{cases}$$

where ∂ is an isolated point serving as a cemetery.

Let $\tau_D = \inf\{t > 0 : X_t \notin D\}$ be the exit time of X from D. The Brownian motion killed upon exiting D is defined as

$$X_t^D = \begin{cases} X_t, & t < \tau_D \\ \partial, & t \geq \tau_D \end{cases}$$

We define now the subordinate killed Brownian motion as the process obtained by subordinating X^D via the $\alpha/2$-stable subordinator T_t. More precisely, let $Z_t = (X^D)_{T_t}$, $t \geq 0$. Then

$$Z_t = \begin{cases} X_{T_t}, & T_t < \tau_D \\ \partial, & T_t \geq \tau_D \end{cases} = \begin{cases} X_{T_t}, & t < A_{\tau_D} \\ \partial, & t \geq A_{\tau_D} \end{cases}$$

where the last equality follows from the fact $\{T_t < \tau_D\} = \{t < A_{\tau_D}\}$. Note that A_{τ_D} is the lifetime of the process Z. Moreover, it holds that $A_{\tau_D} \leq \tau_D^Y$. Indeed, if $s < A_{\tau_D}$, then $T_s < \tau_D$, implying that $Y_s = X_{T_s} \in D$. Hence, $s < \tau_D^Y$. Therefore, the lifetime of Z is less than or equal to the lifetime of Y^D.

Here is a very rough picture illustrating the differences between the processes Z and Y^D.

Figure 1: trajectories of Z and Y^D

In the picture above, the curve is a Brownian path, the points on the path marked by the little crosses, circles and squares represent a trajectory of Y, the points on the path marked by the little crosses, circles represent a trajectory of Y^D, and the points on the path marked by the little crosses represent a trajectory of Z.

We compare now the semigroups corresponding to Y^D and Z. For any nonnegative Borel function f on D, let

$$\begin{aligned} Q_t f(x) &= \mathbb{E}_x[f(Y_t^D)] = \mathbb{E}_x[f(Y_t), t < \tau_D^Y] = \mathbb{E}_x[f(X_{T_t}), t < \tau_D^Y] \\ R_t f(x) &= \mathbb{E}_x[f(Z_t)] = \mathbb{E}_x[f(X^D)_{T_t}] = \mathbb{E}_x[f(X_{T_t}), t < A_{\tau_D}] \end{aligned}$$

Since $A_{\tau_D} \le \tau_D^Y$, it follows that $R_t f(x) \le Q_t f(x)$ for all $t \ge 0$.

The following result will be needed in order to compare the killing functions of the processes Z and Y^D.

Proposition 2.1 *Suppose that there exists $C \in (0,1)$ such that $\mathbb{P}_x(X_t \in D) \le C$ for every $t > 0$ and every $x \in \partial D$. Then*

$$(1-C)(1-R_t 1(x)) \le 1 - Q_t 1(x) \le 1 - R_t 1(x) \tag{2.1}$$

for every $t > 0$ and every $x \in D$.

Proof. Let $\tau_D^1(\omega^1) = \inf\{t > 0,\ X_t^1(\omega^1) \notin D\}$, i.e., $\tau_D^1(\omega^1) = \tau_D(\omega)$. Then $\mathcal{F}^1_{\tau_D^1} \times \mathcal{G}^2 \subset \mathcal{F}_{\tau_D}$. Indeed, for $A_1 \in \mathcal{F}^1_{\tau_D^1}$ and $A_2 \in \mathcal{G}^2$, $(A_1 \times A_2) \cap \{\tau_D \le t\} = (A_1 \cap \{\tau_D^1 \le t\}) \times A_2 \in \mathcal{F}_t^1 \times \mathcal{G} = \mathcal{F}_t$. Thus, $A_1 \times A_2 \in \mathcal{F}_{\tau_D}$. Since such sets generate $\mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$, the claim follows.

We want to show that $T_{A_{\tau_D}}$ is $\mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$- measurable. Note first that τ_D and T_t, $t \ge 0$, are $\mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$-measurable. Therefore, $\{T_t < \tau_D\} \in \mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$. Since $\{A_{\tau_D} > t\} = \{T_t < \tau_D\}$, it follows that $\{A_{\tau_D} > t\}$ is $\mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$-measurable. Clearly, A_s is $\mathcal{F}_0 \times \mathcal{G}^2 \subset \mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$-measurable. Therefore, $\{T_{A_{\tau_D}} \ge s\} = \{A_s \ge A_{\tau_D}\} \in \mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$.

For any nonnegative Borel function f on $\mathbb{R}^d$, let $N_t(x,f) = \mathbb{E}_x(f(X_t))$. Since τ_D and $T_{A_{\tau_D}}$ are $\mathcal{F}_{\tau_D^1} \times \mathcal{G}^2$ -measurable, and $T_{A_{\tau_D}} = \tau_D + (T_{A_{\tau_D}} - \tau_D)$, by an extended version of the strong Markov property (see [2], pp. 43-44),

$$\mathbb{E}_x[1_D(X_{T_{A_{\tau_D}}})|\mathcal{F}_{\tau_D}] = \mathbb{E}_x[1_D(X_{\tau_D + (T_{A_{\tau_D}} - \tau_D)})|\mathcal{F}_{\tau_D}] = N_{T_{A_{\tau_D}} - \tau_D}(X_{\tau_D}, 1_D) \ \text{ a.s.} \tag{2.2}$$

By using the assumption of the proposition, we get $N_t(y, 1_D) = \mathbb{P}_y(X_t \in D) \le C$ for every $t > 0$ and every $y \in \partial D$. From (2.2) we obtain that $\mathbb{P}_x(X_{T_{A_{\tau_D}}} \in D|\mathcal{F}_{\tau_D}) \le C$ a.s. for every

$x \in D$. Since $\mathcal{F}^1_{\tau^1_D} \times \mathcal{G}^2 \subset \mathcal{F}_{\tau_D}$, it follows that $\mathbb{P}_x(X_{T_{A_{\tau_D}}} | \mathcal{F}^1_{\tau^1_D} \times \mathcal{G}^2) \le C$ a.s. Further,

$$\begin{aligned} \mathbb{P}_x(A_{\tau_D} \le t < \tau^Y_D) &\le \mathbb{P}_x(A_{\tau_D} \le t, A_{\tau_D} < \tau^Y_D) \\ &= \mathbb{P}_x(A_{\tau_D} \le t, X_{T_{A_{\tau_D}}} \in D) \\ &= \mathbb{P}_x[\mathbb{P}_x(A_{\tau_D} \le t, X_{T_{A_{\tau_D}}} \in D) | \mathcal{F}^1_{\tau^1_D} \times \mathcal{G}^2] \\ &= \mathbb{E}_x[1_{(A_{\tau_D} \le t)} \mathbb{P}_x(X_{T_{A_{\tau_D}}} \in D) | \mathcal{F}^1_{\tau^1_D} \times \mathcal{G}^2] \\ &\le C\mathbb{P}_x(A_{\tau_D} \le t)\,. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{P}_x(A_{\tau_D} \le t) &= \mathbb{P}_x(\tau^Y_D \le t) + \mathbb{P}_x(A_{\tau_D} \le t < \tau^Y_D) \\ &\le \mathbb{P}_x(\tau^Y_D \le t) + C\mathbb{P}_x(A_{\tau_D} \le t)\,, \end{aligned}$$

hence

$$\begin{aligned} \mathbb{P}_x(A_{\tau_D} \le t) &= \mathbb{P}_x(\tau^Y_D \le t) + \mathbb{P}_x(A_{\tau_D} \le t < \tau^Y_D) \\ &\le \mathbb{P}_x(\tau^Y_D \le t) + C\mathbb{P}_x(A_{\tau_D} \le t)\,. \end{aligned}$$

Since $\mathbb{P}_x(A_{\tau_D} \le t) = 1 - R_t 1(x)$ and $\mathbb{P}_x(\tau^Y_D \le t) = 1 - Q_t 1(x)$, (2.1) follows. □

A domain $D \subset \mathbb{R}^d$ is said to satisfy an exterior cone condition if there exist a cone K with vertex at the origin and a positive constant r_0, such that for each point $x \in \partial D$, there exist a translation and a rotation taking the cone K into a cone K_x with the vertex at x such that

$$K_x \cap B(x, r_0) \subset D^c \cap B(x, r_0)\,.$$

Here $B(x, r_0)$ denotes the ball of radius r_0 centered at x. We show now that the condition in Proposition 2.1 is true for a bounded domain $D \subset \mathbb{R}^d$ satisfying an exterior cone condition. Let $K_x(r_0) = K_x \cap B(x, r_0)$ and $K(r_0) = K \cap B(0, r_0)$. Then we have for each $x \in \partial D$,

$$\mathbb{P}_x(X_t \notin D) \ge \mathbb{P}_x(X_t \in K_x(r_0)) = \mathbb{P}_0(X_t \in K(r_0))\,.$$

By scaling,

$$\mathbb{P}_0(X_t \in K(r_0)) = \mathbb{P}_0(X_1 \in \frac{1}{\sqrt{t}} K(r_0)) \ge \mathbb{P}_0(X_1 \in K(r_0)) =: C_1 \in (0, 1)$$

for every $t \in (0, 1]$, where for any $\rho > 0$, $\rho K(r_0)$ is defined to be the set $\{\rho x : x \in K(r_0)\}$. The last two displays show that $\mathbb{P}_x(X_t \notin D) \ge C_1$, for every $t \in (0, 1]$ and every $x \in \partial D$. Since D is bounded, there exists $R > 0$ such that for every $x \in \partial D$, $D \subset B(x, R)$. Hence,

$$\mathbb{P}_x(X_t \notin D) \ge \mathbb{P}_0(|X_t| > R) \ge \mathbb{P}_0(|X_1| > R) =: C_2 \in (0, 1)$$

for every $t \geq 1$ and every $x \in \partial D$. Let $C = 1 - \min\{C_1, C_2\}$. Then $C \in (0,1)$ and $\mathbb{P}_x(X_t \in D) \leq C$ for every $t > 0$ and every $x \in \partial D$.

It is well known (see [4], for instance) that the transition semigroup Q_t corresponding to the killed stable process has a density with respect to the Lebesgue measure. Let $q(t,x,y)$ be this density. Let $r(t,x,y)$ be the density of R_t and let $p^D(t,x,y)$ be the transition density of the killed Brownian motion X^D. The density $r(t,x,y)$ is given by the formula

$$r(t,x,y) = \int_0^\infty p^D(s,x,y)\, \mu_t^{\alpha/2}(ds)\,, \tag{2.3}$$

where $(\mu_t^{\alpha/2}, t \geq 0)$ is the one-sided $\alpha/2$-stable convolution semigroup. Let $G_D(x,y)$ and $G_D^Y(x,y)$ denote Green functions of Z and Y^D respectively. The Green function of Z is given by

$$G_D(x,y) = \int_0^\infty r(t,x,y)\, dt = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p^D(t,x,y) t^{\alpha/2-1}\, dt\,. \tag{2.4}$$

Proposition 2.2 *Let D be a bounded domain in $\mathbb{R}^d$.*
(i) The transition density $r(t,x,y)$ of Z is jointly continuous in (x,y) for each fixed t. Further, $r(t,x,y) \leq q(t,x,y)$ for all $t > 0$ and all $(x,y) \in D \times D$.
(ii) When $d \geq 2$ or $\alpha \leq 1 = d$, the Green function $G_D(x,y)$ is finite and continuous on $D \times D \setminus \{(x,x), x \in D\}$. When $\alpha > 1 = d$, the Green function $G_D(x,y)$ is finite and continuous on $D \times D$. Further, $G_D(x,y) \leq G_D^Y(x,y)$ on $D \times D$.

Proof. (i) Note that $p^D(s,x,y) \leq (2\pi s)^{-d/2} \exp\{-|x-y|^2/2s\} \leq (2\pi s)^{-d/2}$ for all $x,y \in D$. It follows from the asymptotic behavior near zero of the density of $\mu_t^{\alpha/2}$ given in [20] that the integral $\int_0^\infty s^{-d/2} \mu_t^{\alpha/2}(ds)$ is finite. So the continuity of $r(t,\cdot,\cdot)$ follows from the dominated convergence theorem. Since $R_t f(x) \leq Q_t f(x)$ for every $x \in D$ and every nonnegative Borel function f, we get $r(t,x,y) \leq q(t,x,y)$ for all $y \in D \setminus N(x)$ with $N(x)$ having zero Lebesgue measure. By continuity, the inequality holds for all $x,y \in D$.

(ii) The fact that $G_D(x,y) \leq G_D^Y(x,y)$ follows immediately from $r(t,x,y) \leq q(t,x,y)$. We now prove the continuity of G_D by treating three cases separately. (a) The case when $d \geq 2$ or when $\alpha < 1 = d$. Let $x,y \in D$, $|x-y| > 2\eta > 0$. Let (x_n, y_n) be a sequence in $D \times D$ converging to (x,y) such that $|x_n - y_n| > \eta$. Note that,

$$p^D(t,x_n,y_n) t^{-1+\alpha/2} \leq (2\pi t)^{-d/2} \exp\{-|x_n - y_n|^2/2t\} t^{-1+\alpha/2} \leq c_1 t^{-d/2+\alpha/2-1} \exp\{-\eta^2/2t\}$$

which is integrable on $(0,\infty)$. The continuity now follows from the dominated convergence theorem. (b) The case when $\alpha = 1 = d$. Let $x,y \in D$, $|x-y| > 2\eta > 0$. Let (x_n, y_n)

be a sequence in $D \times D$ converging to (x, y) such that $|x_n - y_n| > \eta$. Using the intrinsic ultracontractivity of the killed Brownian semigroup on a bounded interval and Theorem 4.2.5 of [8], we know that there exists a $T > 0$ such that for any $t \geq T$,

$$p^D(t, x, y) \leq \frac{3}{2} e^{-\lambda_0 t} \phi_0(x) \phi_0(y), \quad x, y \in D,$$

where $-\lambda_0 < 0$ and ϕ_0 are the first eigenvalue and eigenfunction of the Dirichlet Laplacian in D respectively. Thus in this case, the functions $p^D(t, x_n, y_n) t^{\alpha/2-1} = p^D(t, x_n, y_n) t^{-1/2}$ is dominated by the function

$$g(t) = \begin{cases} c_1 t^{-1} \exp\{-\eta^2/2t\}, & t \leq T \\ c_2 t^{-1} e^{-\lambda_0 t}, & t \geq T \end{cases}$$

which is integrable on $(0, \infty)$. Now we can repeat the argument in the first case to arrive at the claimed continuity. (c) The case when $\alpha > 1 = d$. In this case, the family of functions $\{p^D(t, \cdot, \cdot) t^{\alpha/2-1} : x, y \in D\}$ is dominated by the function

$$h(t) = \begin{cases} c_1 t^{-3/2+\alpha/2}, & t \leq T \\ c_2 t^{-3/2+\alpha/2} e^{-\lambda_0 t}, & t \geq T \end{cases}$$

which is integrable on $(0, \infty)$. The continuity now follows from the dominated convergence theorem. $\square$

3 The Dirichlet form of the subordinate killed Brownian motion

Recall that Y is a rotationally invariant α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$. It is well known that the Dirichlet form $(\mathcal{E}^Y, \mathcal{F})$ associated with Y is given by

$$\mathcal{E}^Y(u, v) = \frac{1}{2} A(d, -\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \, dx dy$$
$$\mathcal{F} = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \, dx dy < \infty \right\},$$

where

$$A(d, -\alpha) = \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha} \pi^{d/2} \Gamma(1 - \frac{\alpha}{2})}.$$

It follows from Remark 4 in Section 2.5.1 of [21] that $\mathcal{F}$ is the same as the space $W^{\alpha/2,2}(\mathbb{R}^d)$.

Recall that, for any $s \in \mathbb{R}$, the classical Bessel potential space $H^s(\mathbb{R}^d)$ is defined to be

$$H^s(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty\},$$

where $S'(\mathbb{R}^d)$ stands for the space of tempered distributions on $\mathbb{R}^d$ and $\hat{u}$ stands for the Fourier transform of u. Using Fourier analysis, one can easily show (cf. Example 1.4.1 of [11]) that the spaces $W^{\alpha/2,2}(\mathbb{R}^d)$ and $H^{\alpha/2}(\mathbb{R}^d)$ are the same. Hence we have $\mathcal{F} = W^{\alpha/2,2}(\mathbb{R}^d) = H^{\alpha/2}(\mathbb{R}^d)$.

In this section we assume that D is a bounded domain in $\mathbb{R}^d$. The Dirichlet space on $L^2(D, dx)$ of the killed rotationally invariant α-stable process Y^D is $(\mathcal{E}^Y, H_0^{\alpha/2}(D))$ (cf. Theorem 4.4.3 of [11]), where

$$H_0^{\alpha/2}(D) = \{f \in H^{\alpha/2}(\mathbb{R}^d) : \, f = 0 \text{ q.e. on } D^c\}.$$

Here q.e. is the abbreviation for quasi-everywhere with respect to the Riesz capacity determined by $(\mathcal{E}^Y, W^{\alpha/2,2}(\mathbb{R}^d))$ (cf. [11]). The space $H_0^{\alpha/2}(D)$ can also be characterized as the $\mathcal{E}^Y$-closure of $C_0^\infty(D)$, the space of smooth functions with compact support in D. For $u \in H_0^{\alpha/2}(D)$,

$$\mathcal{E}^Y(u, v) = \int_D \int_D (u(x) - u(y))(v(x) - v(y)) J^Y(x, y) \, dxdy + \int_D u(x)v(x)\kappa^Y(x)dx,$$

where

$$J^Y(x, y) = \frac{1}{2} A(d, \, -\alpha)|x - y|^{-(d+\alpha)} \tag{3.1}$$

$$\kappa^Y(x) = A(d, \, -\alpha) \int_{D^c} \frac{1}{|x - y|^{d+\alpha}} dy \tag{3.2}$$

are the densities of the jumping and killing measures of Y^D.

Recall that Z is the process obtained by subordinating the killed Brownian motion on D with the one-sided $\alpha/2$-stable process. Z is a symmetric Markov process and so there is a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ associated with Z. It follows further from Theorem 1.18.10 of [21] that the domain $D(\mathcal{E})$ of $\mathcal{E}$ is the complex interpolation space $[L^2(D), H_0^1(D)]_{\alpha/2}$. It follows from Proposition 2.2 of [7] that, when D is a bounded Lipschitz domain, $[L^2(D), H_0^1(D)]_{\alpha/2} = H_0^{\alpha/2}(D)$. Recall that Hilbert spaces are identified if they coincide in the set theoretical sense and if they have equivalent norms. Therefore, there exists a constant C such that for any $u \in H_0^{\alpha/2}(D)$,

$$C^{-1}(\mathcal{E}^Y(u, u) + (u, u)) \leq \mathcal{E}(u, u) + (u, u) \leq C(\mathcal{E}^Y(u, u) + (u, u)).$$

One immediate consequence of the comparability above is that for a Borel subset A of D, A is polar for Z is equivalent to that A is polar for the killed rotationally invariant α-stable process Y^D, which in turn is equivalent to that A is polar for the rotationally invariant α-stable process Y.

Let P_t^D be the transition semigroup corresponding to the Brownian motion killed upon exiting D and recall that the corresponding transition density is denoted by $p^D(t,x,y)$. It follows from [3] and [17] (see also [13]) that the jumping measure $J(x,dy)$ and the killing measure $\kappa(dx)$ of the process Z have densities $J(x,y)$ and $\kappa(x)$ given by the following formulae respectively:

$$
\begin{aligned}
J(x,y) &= \int_0^\infty p^D(t,x,y)\,\nu(dt) &\quad (3.3)\\
\kappa(x) &= \int_0^\infty (1-P_t^D 1(x))\,\nu(dt) &\quad (3.4)
\end{aligned}
$$

Here

$$
\nu(dt) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} t^{-\alpha/2-1}\,dt
$$

is the Lévy measure of the $\alpha/2$-stable subordinator.

It is easy to see from (3.3) that $J(x,y) \le J^Y(x,y)$ for every $x,y \in D$. Now we are going to compare $\kappa(x)$ with $\kappa^Y(x)$. To do that we are going to use the following simple result.

Lemma 3.1 *Let $(X_t, \mathbb{P}_x)$ be a d-dimensional Brownian motion, and let τ_D be the exit time of X from D. Then*

$$
\kappa(x) = \frac{1}{\Gamma(1-\alpha/2)} \mathbb{E}_x({\tau_D}^{-\alpha/2}) \qquad (3.5)
$$

for every $x \in \mathbb{R}^d$.

Proof. Let F denote the $\mathbb{P}_x$-distribution function of τ_D. Note that $1-P_t^D 1(x) = \mathbb{P}_x(\tau_D \le t) = F(t)$. By using (3.4)

$$
\begin{aligned}
\kappa(x) &= \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty F(t) t^{-\alpha/2-1}\,dt \\
&= \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty \int_s^\infty t^{-\alpha/2-1}\,dt\,dF(s) \\
&= \frac{1}{\Gamma(1-\alpha/2)} \int_0^\infty s^{-\alpha/2}\,dF(s) = \frac{1}{\Gamma(1-\alpha/2)} \mathbb{E}_x({\tau_D}^{-\alpha/2})
\end{aligned}
$$

□

It was proved in [12] that $x \to \mathbb{E}_x({\tau_D}^{-\alpha/2})$ is continuous, hence κ is a continuous function. We will use this fact in the next result.

Proposition 3.2 *Suppose that there exists $C \in (0,1)$ such that $\mathbb{P}_x(X_t \in D) \le C$ for every $t > 0$ and every $x \in \partial D$. Then*

$$(1-C)\kappa(x) \le \kappa^Y(x) \le \kappa(x), \quad \textit{for every } x \in D\,. \tag{3.6}$$

Proof. By the proof of Lemma 4.5.2 in [11], there exists a sequence $t_n \downarrow 0$ such that

$$\begin{aligned}
\lim_{t_n\to 0} \frac{1}{t_n}\int_D f(x)(1-R_{t_n}1(x))\,dx &= \int_D f(x)\kappa(x)\,dx \\
\lim_{t_n\to 0} \frac{1}{t_n}\int_D f(x)(1-Q_{t_n}1(x))\,dx &= \int_D f(x)\kappa^Y(x)\,dx
\end{aligned}$$

for every $f \in C_0(D)$. By Proposition 2.1 this implies that

$$\int_D f(x)(1-C)\kappa(x)\,dx \le \int_D f(x)\kappa^Y(x)\,dx \le \int_D f(x)\kappa(x)\,dx\,,$$

for every nonnegative $f \in C_0(D)$. Since both κ and κ^Y are continuous, the last relation implies that

$$(1-C)\kappa(x) \le \kappa^Y(x) \le \kappa(x)\,, \quad x \in D\,.$$

□

Remark 3.3 Let $\delta(x)$ be the distance between x and ∂D. When D is a bounded Lipschitz domain, it follows easily from (3.2) that there exists a positive constant C_1 such that

$$C_1^{-1}(\delta(x))^{-\alpha} \le \kappa^Y(x) \le C_1(\delta(x))^{-\alpha}\,.$$

By using this and Proposition 3.2 it follows that there exists a constant C_2 such that

$$C_2^{-1}(\delta(x))^{-\alpha} \le \kappa(x) \le C_2(\delta(x))^{-\alpha}\,.$$

4 Intrinsic ultracontractivity

In this section we assume that D is a bounded Lipschitz domain and Z is the subordinate killed Brownian motion on D. The generator of Z is $-(-\Delta|_D)^{\alpha/2}$, where $\Delta|_D$ is the Dirichlet Laplacian in D. It is well known that if $\{-\lambda_k, k = 0,1,\dots\}$ are the eigenvalues of $\Delta|_D$ written in decreasing order and each repeated according to its multiplicity, and if $\{\phi_k, k = 0,1,\dots\}$ are the corresponding eigenfunctions, then $\{-(\lambda_k)^{\alpha/2}, k = 0,1,\dots\}$ are the eigenvalues of $-(-\Delta|_D)^{\alpha/2}$ written in decreasing order and each repeated according to its multiplicity, and $\{\phi_k, k = 0,1,\dots\}$ are the corresponding eigenfunctions.

Similar to Theorem 4.1 of [4], we have the following result.

Theorem 4.1 *For any $\eta > 0$ and $f \in H_0^{\alpha/2}(D) \cap L^\infty(D, dx)$, we have*

$$\int_D f^2 \log |f| dx \leq \eta \mathcal{E}(f, f) + \beta(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2,$$

with

$$\beta(\eta) = -\frac{d}{2\alpha} \log \eta + c$$

for some constant $c > 0$.

Proof. It follows from Proposition 2.2 that $r(t, x, y) \leq q(t, x, y)$, hence there exists a $c > 0$ such that $r(t, x, y) \leq ct^{-d/\alpha}$. Now we can repeat the proof of Theorem 4.1 of [4] to arrive at the conclusion. □

The following lemma appears on p.71 of [14]. The key ingredient in the proof there is an inequality (inequality (4.1) of [14]) proved in [19]. We include an elementary proof based on the behavior of the killing function of Z.

Lemma 4.2 *There exists a constant $C_1 > 0$ such that*

$$C_1^{-1} \mathcal{E}^Y(u, u) \leq \mathcal{E}(u, u) \leq C_1 \mathcal{E}^Y(u, u), \quad u \in H_0^{\alpha/2}(D).$$

Proof. Recall that the killing measures of Z and Y^D have densities κ and κ^Y respectively, which are both of the order $\delta(x)^{-\alpha}$. This implies that there is a constant c_1 such that

$$\int_D u^2(x)\, dx \leq c_1 \int_D u^2(x) \kappa(x)\, dx \tag{4.1}$$

From the last section we know that there exists a constant $c_2 > 0$ such that

$$c_2^{-1}(\mathcal{E}^Y(u, u) + (u, u)) \leq \mathcal{E}(u, u) + (u, u) \leq c_2(\mathcal{E}^Y(u, u) + (u, u)), \quad u \in H_0^{\alpha/2}(D).$$

Therefore the 1-norms are equivalent to the 0-norms for both forms. Thus there is a constant C_1 such that

$$C_1^{-1} \mathcal{E}^Y(u, u) \leq \mathcal{E}(u, u) \leq C_1 \mathcal{E}^Y(u, u), \quad u \in H_0^{\alpha/2}(D).$$

□

Recall that for any domain D in $\mathbb{R}^d$, the quasi–hyperbolic distance between any two points x_1 and x_2 in D is defined by

$$\rho_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta(x)}$$

where the infimum is taken over all rectifiable curves γ joining x_1 to x_2 in D and $\delta(x)$ is the Euclidean distance between x and ∂D. Fix a point $x_0 \in D$ which we call the center of D and we may assume without loss of generality that $\delta(x_0) = 1$.

Lemma 4.3 *There is a constant $C_2 = C_2(D) > 0$ such that for any $\beta > 0$*

$$\int_D (\rho_D(x_0, x))^\beta u^2(x)dx \le C_2 \mathcal{E}(u, u), \qquad u \in H_0^{\alpha/2}(D)$$

Proof. It follows from Lemma 3.2 of [6] that there is a constant $c = c(D) > 0$ such that for any $\beta > 0$

$$\int_D (\rho_D(x_0, x))^\beta u^2(x)dx \le c \mathcal{E}^Y(u, u), \qquad u \in H_0^{\alpha/2}(D).$$

Now the result follows from Lemma 4.2. □

Repeating the argument of Theorem 3.3 of [6](see also [1]), we get

Theorem 4.4 *For any $\varepsilon > 0$ and any $\sigma > 0$ we have*

$$\int_D f^2 \log \frac{1}{\phi_0} dx \le \varepsilon \mathcal{E}(f, f) + \beta(\varepsilon)\|f\|_2^2, \qquad f \in H_0^{\alpha/2}(D)$$

with

$$\beta(\varepsilon) = C_3 \varepsilon^{-\sigma} + C_4$$

for some positive constants C_3 and C_4.

Combining Theorems 4.1 and 4.4 we get

Theorem 4.5 *For any $\varepsilon > 0$ and any $\sigma > 0$ we have*

$$\int_D f^2 \log \frac{|f|}{\varphi_0} dx \le \eta \mathcal{E}(f, f) + \beta(\eta)\|f\|_2^2 + \|f\|_2^2 \log \|f\|_2$$

for all $f \in H_0^{\alpha/2}(D) \cap L^\infty(D, dx)$, with

$$\beta(\varepsilon) = -\frac{d}{2\alpha} \log \varepsilon + C_5 \varepsilon^{-\sigma} + C_6$$

for some positive constants C_5 and C_6.

Using this and Corollary 2.2.8 of [8] we immediately get

Theorem 4.6 *The semigroup corresponding to the subordinate killed Brownian motion Z is intrinsic ultracontractive.*

Here is an immediate corollary of the intrinsic ultracontractivity. Recall that the Green function of the process Z is given by the formula (2.4).

Corollary 4.7 *There exists a constant C_8 such that for all $x, y \in D$,*

$$\begin{aligned} G_D(x,y) &\geq C_8\phi_0(x)\phi_0(y), \\ J(x,y) &\geq C_8\phi_0(x)\phi_0(y). \end{aligned}$$

Proof. The first inequality follows immediately from the intrinsic ultracontractivity and Theorem 4.2.5 of [8]. Now we show the second inequality. Since the semigroup of the killed Brownian motion in D is intrinsic ultracontractive, Theorem 4.2.5 of [8] implies that there exists $T > 1$ such that for all $t \geq T$,

$$p^D(t,x,y) \geq \frac{1}{2}e^{-\lambda_0 t}\phi_0(x)\phi_0(y), \quad xy \in D.$$

Thus

$$\begin{aligned} J(x,y) &= c_1\int_0^\infty p^D(t,x,y)t^{-\alpha/2-1}dt \\ &\geq \frac{c_1}{2}\int_T^\infty e^{-\lambda_0 t}\phi_0(x)\phi_0(y)dt \\ &= c_2\phi_0(x)\phi_0(y). \end{aligned}$$

□

Note that these lower bound of G_D and J are of no use when x, y are away from the boundary. The next result gives lower bound when x and y are away from the boundary and it does not need the Lipschitz assumption.

Proposition 4.8 *For any bounded domain D in $\mathbb{R}^d$, there exists a constant $C_9 = C_9(\alpha, d)$ such that if $x, y \in D$ satisfy $|x-y| \leq \max\{\delta(x)/2, \delta(y)/2\}$, then*

$$G_D(x,y) \geq C_9|x-y|^{\alpha-d}, \tag{4.2}$$

$$J(x,y) \geq C_9|x-y|^{-\alpha-d}. \tag{4.3}$$

Proof. We prove the first inequality. The second is proved in the same way. Let $x, y \in D$ such that $|x-y| \le \max\{\delta(x)/2, \delta(y)/2\}$. By using (2.4) and the formula for the transition density of the killed Brownian motion X^D, we get that

$$G_D(x,y) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p(s,x,y) s^{\alpha/2-1}\, ds - \frac{1}{\Gamma(\alpha/2)} \mathbb{E}_x \int_{\tau_D}^\infty p(s, X_{\tau_D}, y) s^{\alpha/2-1}\, ds, , \quad (4.4)$$

where $p(s,x,y)$ denotes the transition density of the Brownian motion X. Since $|X_{\tau_D} - y| \ge \delta(y)$ for each $y \in D$, we obtain the estimate

$$\begin{aligned} \frac{1}{\Gamma(\alpha/2)} \mathbb{E}_x \int_{\tau_D}^\infty p(s, X_{\tau_D}, y) s^{\alpha/2-1}\, ds &\le \frac{(2\pi)^{-d/2}}{\Gamma(\alpha/2)} \int_0^\infty s^{-d/2+\alpha/2-1} \exp\{-\delta(y)^2)/2s\}\, ds \\ &\le c_1 \delta(y)^{\alpha-d} \\ &\le c_1\, 2^{\alpha-d} |x-y|^{\alpha-d}. \end{aligned}$$

The estimate (4.2) follows from (4.4) and the last display. □

5 Upper bounds on the Green function and the jumping kernel

For any bounded domain D in R^d, we have seen that

$$G_D(x,y) \le G_D^Y(x,y), \quad J(x,y) \le J^Y(x,y), \quad x, y \in D.$$

Recall that G_D^Y and J^Y are the Green function and jumping function of Y^D respectively. These estimates are not useful near the boundary of D. Now we are going to derive estimates that are useful near the boundary when D is a bounded $C^{1,1}$ domain.

Theorem 5.1 *Suppose that D is a bounded $C^{1,1}$ domain in R^d. Then there exists a constant C_1 such that for all $x, y \in D$,*

$$\begin{aligned} G_D(x,y) &\le C_1 \frac{\phi_0(x)\phi_0(y)}{|x-y|^{d+2-\alpha}}, \\ J(x,y) &\le C_1 \frac{\phi_0(x)\phi_0(y)}{|x-y|^{d+2+\alpha}}. \end{aligned}$$

Proof. The proof of these two inequalities are very similar. We only give the proof of the first. It is well known that when D is a bounded $C^{1,1}$ domain, there exists a constant c_1 such that

$$c_1^{-1}\delta(x) \le \phi_0(x) \le c_1 \delta(x), \quad x \in D.$$

Now we can repeat the proof of Theorem 4.6.9 of [8] to get that the density p^D of the killed Brownian motion on D satisfies the following estimate

$$p^D(t,x,y) \le c_2 t^{-(d+2)/2}\phi_0(x)\phi_0(y)e^{-\frac{|x-y|^2}{6t}}, \quad t>0,\ x,y\in D,$$

where c_2 is some constant independent of t, x, and y. Now using (2.4) we get that

$$\begin{aligned} G_D(x,y) &\le c_2\phi_0(x)\phi_0(y)\int_0^\infty t^{-(d+2)/2}e^{-\frac{|x-y|^2}{6t}}t^{\alpha/2-1}dt \\ &\le c_3\frac{\phi_0(x)\phi_0(y)}{|x-y|^{d+2-\alpha}}. \end{aligned}$$

□

Remark 5.2 *If we only assume that D is a bounded Lipschitz domain, then we can get a similar upper bound for G_D with $d+2$ replaced by some number $\mu \ge d$, where μ depends on the Lipschitz characteristics of D.*

Remark 5.3 *The estimates in the theorem above can also be written as*

$$\begin{aligned} G_D(x,y) &\le C_2\frac{\delta(x)\delta(y)}{|x-y|^{d+2-\alpha}}, \quad x,y\in D \\ J(x,y) &\le C_2\frac{\delta(x)\delta(y)}{|x-y|^{d+2+\alpha}}, \quad x,y\in D, \end{aligned}$$

for some positive constant C_2.

Summarizing our estimates on the Green function and the jumping kernel, we have the following:

Theorem 5.4 *Suppose that D is a bounded $C^{1,1}$ domain in R^d. Then there exist positive constants C_3 and C_4 such that for all $x,y\in D$,*

$$\begin{aligned} C_3\delta(x)\delta(y) &\le G_D(x,y) \le C_4\min(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta(x)\delta(y)}{|x-y|^{d+2-\alpha}}), \\ C_3\delta(x)\delta(y) &\le J(x,y) \le C_4\min(\frac{1}{|x-y|^{d+\alpha}}, \frac{\delta(x)\delta(y)}{|x-y|^{d+2+\alpha}}) \end{aligned}$$

Comparing the estimates on the Green function of Z with the estimates on the Green function of Y^D obtained in [5] and [15], we see that their boundary behaviors are different.

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