

Potential Theory of Subordinate Brownian Motions Revisited

Panki Kim* Renming Song and Zoran Vondraček†

Abstract

The paper discusses and surveys some aspects of the potential theory of subordinate Brownian motion under the assumption that the Laplace exponent of the corresponding subordinator is comparable to a regularly varying function at infinity. This extends some results previously obtained under stronger conditions.

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1 Introduction

An \mathbb{R}^d -valued process $X = (X_t : t \geq 0)$ is called a Lévy process in \mathbb{R}^d if it is a right continuous process with left limits and if, for every $s, t \geq 0$, $X_{t+s} - X_s$ is independent of $\{X_r, r \in [0, s]\}$ and has the same distribution as $X_s - X_0$. A Lévy process is completely characterized by its Lévy exponent Φ via

$$\mathbb{E}[\exp\{i\langle \xi, X_t \rangle\}] = \exp\{-t\Phi(\xi)\}, \quad t \geq 0, \xi \in \mathbb{R}^d.$$

The Lévy exponent Φ of a Lévy process is given by the Lévy-Khintchine formula

$$\Phi(\xi) = i\langle l, \xi \rangle + \frac{1}{2}\langle \xi, A\xi^T \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, x \rangle} - i\langle \xi, x \rangle \mathbf{1}_{\{|x| < 1\}}\right) \Pi(dx), \quad \xi \in \mathbb{R}^d,$$

where $l \in \mathbb{R}^d$, A is a nonnegative definite $d \times d$ matrix, and Π is a measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int (1 \wedge |x|^2) \Pi(dx) < \infty$. A is called the diffusion matrix, Π the Lévy measure, and (l, A, Π) the generating triplet of the process.

Nowadays Lévy processes are widely used in various fields, such as mathematical finance, actuarial mathematics and mathematical physics. However, general Lévy processes are not very easy to deal with.

A subordinate Brownian motion in \mathbb{R}^d is a Lévy process which can be obtained by replacing the time of Brownian motion in \mathbb{R}^d by an independent subordinator (i.e., an increasing Lévy process).

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More precisely, let $B = (B_t : t \geq 0)$ be a Brownian motion in \mathbb{R}^d and $S = (S_t : t \geq 0)$ be a subordinator independent of B . The process $X = (X_t : t \geq 0)$ defined by $X_t = B_{S_t}$ is a rotationally invariant Lévy process in \mathbb{R}^d and is called a subordinate Brownian motion.

The subordinator S used to define the subordinate Brownian motion X can be interpreted as “operational” time or “intrinsic” time. For this reason, subordinate Brownian motions have been used in mathematical finance and other applied fields. Subordinate Brownian motions form a very large class of Lévy processes. Nonetheless, compared with general Lévy processes, subordinate Brownian motions are much more tractable. If we take the Brownian motion B as given, then X is completely determined by the subordinator S . Hence, one can deduce properties of X from properties of the subordinator S . On the analytic level this translates to the following: Let ϕ denote the Laplace exponent of the subordinator S , that is, $\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}$, $\lambda > 0$. Then the characteristic exponent Φ of the subordinate Brownian motion X takes on the very simple form $\Phi(x) = \phi(|x|^2)$ (our Brownian motion B runs at twice the usual speed). Therefore, properties of X should follow from properties of the Laplace exponent ϕ .

The Laplace exponent ϕ of a subordinator S is a Bernstein function, hence it has a representation of the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \mu(dt)$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t) \mu(dt) < \infty$. If μ has a completely monotone density, the function ϕ is called a complete Bernstein function. The purpose of this work is to study the potential theory of subordinate Brownian motion under the assumption that the Laplace exponent ϕ of the subordinator is a complete Bernstein function comparable to a regularly varying functions at infinity. More precisely, we will assume that there exist $\alpha \in (0, 2)$ and a function ℓ slowly varying at infinity such that

$$\phi(\lambda) \asymp \lambda^{\alpha/2} \ell(\lambda), \quad \lambda \rightarrow \infty.$$

Here and later, for two functions f and g we write $f(\lambda) \asymp g(\lambda)$ as $\lambda \rightarrow \infty$ if the quotient $f(\lambda)/g(\lambda)$ stays bounded between two positive constants as $\lambda \rightarrow \infty$.

A lot of progress has been made in recent years in the study of the potential theory of subordinate Brownian motions, see, for instance [13, 14, 25, 26, 27, 31, 35] and [6, Chapter 5]. In particular, an extensive survey of results obtained before 2007 is given in [6, Chapter 5]. At that time, the focus was on the potential theory of the process X in the whole of \mathbb{R}^d , the results for the (killed) subordinate Brownian motion in an open subset still being out of reach. In the last few years significant progress has been made in studying the potential theory of subordinate Brownian motion killed upon exiting an open subset of \mathbb{R}^d . The main results include the boundary Harnack principle and sharp Green function estimates. For the processes having a continuous component see [26] (for the one-dimensional case) and [12, 13, 14] (for multi-dimensional case). For purely discontinuous processes, the boundary Harnack principle was obtained in [25] and sharp Green function estimates were discussed in the recent preprint [27]. The main assumption in [13, 14, 25]

and [6, Chapter 5] is that the Laplace exponent of the subordinator is regularly varying at infinity. The results were established under different assumptions, some of which turned out to be too strong and some even redundant. Time is now ripe to put some of the recent progress under one unified setup and to give a survey of some of these results. The survey builds upon the work done in [6, Chapter 5] and [25]. The setup we are going to assume is more general than all these of the previous papers, so in this sense, most of the results contained in this paper are extensions of the existing ones.

In Section 2 we first recall some basic facts about subordinators, Bernstein functions and complete Bernstein functions. Then we establish asymptotic behaviors, near the origin, of the potential density and Lévy density of subordinators.

In Section 3 we establish the asymptotic behaviors, near the origin, of the Green function and the Lévy density of our subordinate Brownian motion. These results follow from the asymptotic behaviors, near the origin, of the potential density and Lévy density of the subordinator.

In Section 4 we prove that the Harnack inequality and the boundary Harnack principle hold for our subordinate Brownian motions.

The materials covered in this paper by no means include all that can be said about the potential theory of subordinate Brownian motions. One of the omissions is the sharp Green function estimates of (killed) subordinate Brownian motions in bounded $C^{1,1}$ open sets obtained in the recent preprint [27]. The present paper builds up the framework for [27] and can be regarded as a preparation for [27] in this sense. Another omission is the Dirichlet heat kernel estimates of subordinate Brownian motions in smooth open sets recently established in [8, 9, 10, 11]. One of the reasons we do not include these recent results in this paper is that all these heat kernel estimates are for particular subordinate Brownian motions only and are not yet established in the general case. A third notable omission is the spectral theory for killed subordinate Brownian motions developed in [17, 18, 19]. Some of these results have been summarized in [34, Section 12.3]. A fourth notable omission is the potential theory of subordinate killed Brownian motions developed in [21, 22, 37, 38, 40, 41]. Some of these results have been summarized in [6, Section 5.5] and [34, Chapter 13]. In this paper we concentrate on subordinate Brownian motions without diffusion components and therefore this paper does not include results from [13, 14, 26]. One of the reasons for this is that subordinate Brownian motions with diffusion components require a different treatment.

We end this introduction with few words on the notations. For functions f and g we write $f(t) \sim g(t)$ as $t \rightarrow 0+$ (resp. $t \rightarrow \infty$) if the quotient $f(t)/g(t)$ converges to 1 as $t \rightarrow 0+$ (resp. $t \rightarrow \infty$), and $f(t) \asymp g(t)$ as $t \rightarrow 0+$ (resp. $t \rightarrow \infty$) if the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \rightarrow 0+$ (resp. $t \rightarrow \infty$).

2 Subordinators

2.1 Subordinators and Bernstein functions

Let $S = (S_t : t \geq 0)$ be a subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. A subordinator S is completely characterized by its Laplace exponent ϕ via

$$\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)), \quad \lambda > 0.$$

The Laplace exponent ϕ can be written in the form (cf. [2, p. 72])

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu(dt).$$

Here $b \geq 0$, and μ is a σ -finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1) \mu(dt) < \infty.$$

The constant b is called the drift, and μ the Lévy measure of the subordinator S .

A C^∞ function $\phi : (0, \infty) \rightarrow [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leq 0$ for every positive integer n . Every Bernstein function has a representation (cf. [34, Theorem 3.2])

$$\phi(\lambda) = a + b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) \mu(dt)$$

where $a, b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge t) \mu(dt) < \infty$. a is called the killing coefficient, b the drift and μ the Lévy measure of the Bernstein function. Thus a nonnegative function ϕ on $(0, \infty)$ is the Laplace exponent of a subordinator if and only if it is a Bernstein function with $\phi(0+) = 0$.

A Bernstein function ϕ is called a complete Bernstein function if the Lévy measure μ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \geq 0$ for every non-negative integer n . Here and below, by abuse of notation we will denote the Lévy density by $\mu(t)$. Complete Bernstein functions form a large subclass of Bernstein functions. Most of the familiar Bernstein functions are complete Bernstein functions. See [34, Chapter 15] for an extensive table of complete Bernstein functions. Here are some examples of complete Bernstein functions:

- (i) $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0, 2]$;
- (ii) $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, $\alpha \in (0, 2)$, $m \geq 0$;
- (iii) $\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}$, $0 \leq \beta < \alpha \in (0, 2]$;
- (iv) $\phi(\lambda) = \lambda^{\alpha/2} (\log(1 + \lambda))^{\gamma/2}$, $\alpha \in (0, 2)$, $\gamma \in (0, 2 - \alpha)$;
- (v) $\phi(\lambda) = \lambda^{\alpha/2} (\log(1 + \lambda))^{-\beta/2}$, $0 \leq \beta < \alpha \in (0, 2]$.

An example of a Bernstein function which is not a complete Bernstein function is $1 - e^{-\lambda}$.

It is known (cf. [34, Proposition 7.1]) that ϕ is a complete Bernstein function if and only if the function $\lambda/\phi(\lambda)$ is a complete Bernstein function. For other properties of complete Bernstein functions we refer the readers to [34].

The following result, which will play an important role later, says that the Lévy density of a complete Bernstein function cannot decrease too fast in the following sense.

Lemma 2.1 ([27, Lemma 2.1]) *Suppose that ϕ is a complete Bernstein function with Lévy density μ . Then there exists $C_1 > 0$ such that $\mu(t) \leq C_1\mu(t+1)$ for every $t > 1$.*

Proof. Since μ is a completely monotone function, by Bernstein's theorem ([34, Theorem 1.4]), there exists a measure m on $[0, \infty)$ such that $\mu(t) = \int_{[0, \infty)} e^{-tx} m(dx)$. Choose $r > 0$ such that $\int_{[0, r]} e^{-x} m(dx) \geq \int_{(r, \infty)} e^{-x} m(dx)$. Then, for any $t > 1$, we have

$$\begin{aligned} \int_{[0, r]} e^{-tx} m(dx) &\geq e^{-(t-1)r} \int_{[0, r]} e^{-x} m(dx) \\ &\geq e^{-(t-1)r} \int_{(r, \infty)} e^{-x} m(dx) \geq \int_{(r, \infty)} e^{-tx} m(dx). \end{aligned}$$

Therefore, for any $t > 1$,

$$\mu(t+1) \geq \int_{[0, r]} e^{-(t+1)x} m(dx) \geq e^{-r} \int_{[0, r]} e^{-tx} m(dx) \geq \frac{1}{2} e^{-r} \int_{[0, \infty)} e^{-tx} m(dx) = \frac{1}{2} e^{-r} \mu(t).$$

□

The potential measure of the subordinator S is defined by

$$U(A) = \mathbb{E} \int_0^\infty \mathbf{1}_{\{S_t \in A\}} dt, \quad A \subset [0, \infty). \quad (2.1)$$

Note that $U(A)$ is the expected time the subordinator S spends in the set A . The Laplace transform of the measure U is given by

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t} dU(t) = \mathbb{E} \int_0^\infty \exp(-\lambda S_t) dt = \frac{1}{\phi(\lambda)}. \quad (2.2)$$

We call a subordinator S a complete subordinator if its Laplace exponent ϕ is a complete Bernstein function. The following characterization of complete subordinators is due to [41, Remark 2.2] (see also [6, Corollary 5.3]).

Proposition 2.2 *Let S be a subordinator with Laplace exponent ϕ and potential measure U . Then ϕ is a complete Bernstein function if and only if*

$$U(dt) = c\delta_0(dt) + u(t)dt$$

for some $c \geq 0$ and completely monotone function u .

In case the constant c in the proposition above is equal to zero, we will call u the potential density of the subordinator S .

An inspection of the argument, given in [6, Chapter 5] or [41] leading to the proposition above yields the following two results (cf. [6, Corollary 5.4 and Corollary 5.5] or [41, Corollary 2.3 and Corollary 2.4]).

Corollary 2.3 *Suppose that $S = (S_t : t \geq 0)$ is a subordinator whose Laplace exponent*

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu(dt)$$

is a complete Bernstein function with $b > 0$ or $\mu(0, \infty) = \infty$. Then the potential measure U of S has a completely monotone density u .

Proof. By [6, Corollary 5.4] or [41, Corollary 2.3], if the drift of complete subordinator S is zero or the Lévy measure μ has infinite mass, then the constant c in Proposition 2.2 above is equal to zero so the potential measure U of S has a density u . The completeness of the density follows directly from Proposition 2.2. \square

Corollary 2.4 *Let S be a complete subordinator with Laplace exponent ϕ . Suppose that the drift of S is zero and the Lévy measure μ has infinite mass. Then the potential measure of a subordinator with Laplace exponent $\psi(\lambda) := \lambda/\phi(\lambda)$ has a completely monotone density v given by*

$$v(t) = \mu(t, \infty).$$

Proof. Since the drift of S is zero and the Lévy measure μ has infinite mass, by [6, Corollary 5.5] or [41, Corollary 2.4], we have that

$$\psi(\lambda) = a + \int_0^\infty (1 - e^{-\lambda t}) \nu(dt)$$

where $a = (\int_0^\infty t\mu(t)dt)^{-1}$, the Lévy measure ν of ψ has infinite mass and the potential measure of a possibly killed (i.e., $a > 0$) subordinator with Laplace exponent ψ has a density v given by $v(t) = \mu(t, \infty)$. The completeness of the density follows from [6, Corollary 5.3], which works for killed subordinator. \square

2.2 Asymptotic behavior of the potential and Lévy densities

From now on we will always assume that S is a complete subordinator without drift and that the Laplace exponent ϕ of S satisfies $\lim_{\lambda \rightarrow \infty} \phi(\lambda) = \infty$ (or equivalently, the Lévy measure of S has infinite mass). Under this assumption, the potential measure U of S has a completely monotone density u (cf. Corollary 2.3). The main purpose of this subsection is to determine the asymptotic behaviors of u and μ near the origin. For this purpose, we will need the following result due to Zähle (cf. [46, Theorem 7]).

Proposition 2.5 *Suppose that w is a completely monotone function given by*

$$w(t) = \int_0^\infty e^{-st} f(s) ds,$$

where f is a nonnegative decreasing function. Then

$$f(s) \leq (1 - e^{-1})^{-1} s^{-1} w(s^{-1}), \quad s > 0.$$

If, furthermore, there exist $\delta \in (0, 1)$ and $a, s_0 > 0$ such that

$$w(\lambda t) \leq a \lambda^{-\delta} w(t), \quad \lambda \geq 1, t \geq 1/s_0, \quad (2.3)$$

then there exists $C_2 = C_2(w, f, a, s_0, \delta) > 0$ such that

$$f(s) \geq C_2 s^{-1} w(s^{-1}), \quad s \leq s_0.$$

Proof. Using the assumption that f is a nonnegative decreasing function, we get that, for any $r > 0$, we have

$$\begin{aligned} w(t) &= \frac{1}{t} \int_0^\infty e^{-s} f\left(\frac{s}{t}\right) ds \\ &\geq \frac{1}{t} \int_0^r e^{-s} f\left(\frac{s}{t}\right) ds \geq \frac{1}{t} f\left(\frac{r}{t}\right) (1 - e^{-r}). \end{aligned}$$

Thus

$$f\left(\frac{r}{t}\right) \leq \frac{tw(t)}{1 - e^{-r}}, \quad t > 0, r > 0.$$

In particular, we have

$$f(s) \leq (1 - e^{-1})^{-1} s^{-1} w(s^{-1}), \quad s > 0,$$

and

$$f\left(\frac{s}{t}\right) \leq (1 - e^{-1})^{-1} \frac{t}{s} w\left(\frac{t}{s}\right), \quad s > 0, t > 0. \quad (2.4)$$

On the other hand, for $r \in (0, 1]$, we have

$$\begin{aligned} tw(t) &= \int_0^r e^{-s} f\left(\frac{s}{t}\right) ds + \int_r^\infty e^{-s} f\left(\frac{s}{t}\right) ds \\ &\leq \int_0^r e^{-s} f\left(\frac{s}{t}\right) ds + f\left(\frac{r}{t}\right) e^{-r} \\ &\leq (1 - e^{-1})^{-1} t \int_0^r e^{-s} \frac{1}{s} w\left(\frac{t}{s}\right) ds + f\left(\frac{r}{t}\right) e^{-r}, \end{aligned}$$

where in the last line we used (2.4). Now we assume (2.3), then we get that

$$w\left(\frac{t}{s}\right) \leq a s^\delta w(t), \quad t \geq 1/s_0, s < r.$$

Thus, for $r \in (0, 1]$, we have,

$$tw(t) \leq a(1 - e^{-1})^{-1} tw(t) \int_0^r e^{-s} s^{\delta-1} ds + f\left(\frac{r}{t}\right) e^{-r}.$$

Choosing $r \in (0, 1]$ small enough so that

$$a(1 - e^{-1})^{-1} \int_0^r e^{-s} s^{\delta-1} ds \leq \frac{1}{2},$$

we conclude that for this choice of r , we have

$$f\left(\frac{r}{t}\right) \geq c_1 tw(t), \quad t \geq 1/s_0$$

for some constant $c_1 > 0$. Since w is decreasing, we have

$$f(s) \geq c_1 \frac{r}{s} w\left(\frac{r}{s}\right) \geq c_2 s^{-1} w(s^{-1}), \quad s \leq rs_0,$$

where $c_2 = c_1 r$. From this we immediately get that there exists $c_3 > 0$ such that

$$f(s) \geq c_3 s^{-1} w(s^{-1}), \quad s \leq s_0.$$

□

Corollary 2.6 *The potential density u of S satisfies*

$$u(t) \leq C_3 t^{-1} \phi(t^{-1})^{-1}, \quad t > 0. \quad (2.5)$$

Proof. Apply the first part of Proposition 2.5 to the function

$$w(t) := \int_0^\infty e^{-st} u(s) ds = \frac{1}{\phi(t)},$$

□

We introduce now the main assumption on our Laplace exponent ϕ of the complete subordinator S that we will use throughout the rest of the paper. Recall that a function $\ell : (0, \infty) \rightarrow (0, \infty)$ is slowly varying at infinity if

$$\lim_{t \rightarrow \infty} \frac{\ell(\lambda t)}{\ell(t)} = 1, \quad \text{for every } \lambda > 0.$$

Assumption (H): There exist $\alpha \in (0, 2)$ and a function $\ell : (0, \infty) \rightarrow (0, \infty)$ which is measurable, locally bounded and slowly varying at infinity such that

$$\phi(\lambda) \asymp \lambda^{\alpha/2} \ell(\lambda), \quad \lambda \rightarrow \infty. \quad (2.6)$$

Remark 2.7 The precise interpretation of (2.6) will be as follows: There exists a positive constant $c > 1$ such that

$$c^{-1} \leq \frac{\phi(\lambda)}{\lambda^{\alpha/2}\ell(\lambda)} \leq c \quad \text{for all } \lambda \in [1, \infty).$$

The choice of the interval $[1, \infty)$ is, of course, arbitrary. Any interval $[a, \infty)$ would do, but with a different constant. This follows from the assumption that ℓ is locally bounded. Moreover, by choosing $a > 0$ large enough, we could dispense with the local boundedness assumption. Indeed, by [3, Lemma 1.3.2], every slowly varying function at infinity is locally bounded on $[a, \infty)$ for a large enough.

Although the choice of interval $[1, \infty)$ is arbitrary, it will have as a consequence the fact that all relations of the type $f(t) \asymp g(t)$ as $t \rightarrow \infty$ (respectively $t \rightarrow 0+$) following from (2.6) will be interpreted as $\tilde{c}^{-1} \leq f(t)/g(t) \leq \tilde{c}$ for $t \geq 1$ (respectively $0 < t \leq 1$).

The assumption (2.6) is a very weak assumption on the asymptotic behavior of ϕ at infinity. All the examples (in (i), (iii) and (v), we need to take $\alpha < 2$) above Lemma 2.1 satisfy this assumption. In fact they satisfy the following stronger assumption

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda), \tag{2.7}$$

where ℓ is a function slowly varying at infinity. By inspecting the table in [34, Chapter 15], one can come up with a lot more examples of complete Bernstein functions satisfying this stronger assumption. In the next example we construct a complete Bernstein function satisfying (2.6), but not the stronger (2.7).

Example 2.8 Suppose that $\alpha \in (0, 2)$. Let F be a function on $[0, \infty)$ defined by $F(x) = 0$ on $0 \leq x < 1$ and

$$F(x) = 2^n, \quad 2^{2(n-1)/\alpha} \leq x < 2^{2n/\alpha}, \quad n = 1, 2, \dots$$

Then clearly F is non-decreasing and $x^{\alpha/2} \leq F(x) \leq 2x^{\alpha/2}$ for all $x \geq 1$. This implies that for all $t > 0$,

$$\frac{t^{\alpha/2}}{2} \leq \liminf_{x \rightarrow \infty} \frac{F(tx)}{F(x)} \leq \limsup_{x \rightarrow \infty} \frac{F(tx)}{F(x)} \leq 2t^{\alpha/2}.$$

If F were regularly varying, the above inequality would imply that the index was $\alpha/2$, hence the limit of $F(tx)/F(x)$ as $x \rightarrow \infty$ would be equal to $ct^{\alpha/2}$ for some positive constant c . But this does not happen because of the following. Take $t = 2^{2/\alpha}$ and a subsequence $x_n = 2^{2n/\alpha}$. Then $tx_n = 2^{2(n+1)/\alpha}$ and therefore

$$F(tx_n)/F(x_n) = 2^{n+2}/2^{n+1} = 2$$

which should be equal to $ct^{\alpha/2} = c(2^{2/\alpha})^{\alpha/2} = 2c$, implying $c = 1$. On the other hand, take any $t \in (1, 2^{2/\alpha})$ and the same subsequence $x_n = 2^{2n/\alpha}$. Then $tx_n \in [2^{2n/\alpha}, 2^{2(n+1)/\alpha})$ implying

$F(tx_n) = F(x_n)$. Thus the quotient $F(tx_n)/F(x_n) = 1$ which should be equal to $ct^\alpha = t^\alpha$ for all $t \in (1, 2^{1/\alpha})$. Clearly this is impossible, so F is not regularly varying. This also shows that $F(x)$ is not \sim to any $cx^{\alpha/2}$, as $x \rightarrow \infty$.

Let σ be the measure corresponding to the nondecreasing function F (in the sense that $\sigma(dt) = F(dt)$):

$$\sigma := \sum_{n=1}^{\infty} 2^n \delta_{2^{2n/\alpha}}.$$

Since $\int_{(0,\infty)} (1+t)^{-1} \sigma(dt) < \infty$, σ is a Stieltjes measure. Let

$$g(\lambda) := \int_{(0,\infty)} \frac{1}{\lambda+t} \sigma(dt) = \sum_{n=1}^{\infty} \frac{2^n}{\lambda + 2^{2n/\alpha}}$$

be the corresponding Stieltjes function. It follows from [3, Theorem 1.7.4] or [45, Lemma 6.2] that g is not regularly varying at infinity. Moreover, since $F(x) \asymp x^{\alpha/2}$, $x \rightarrow \infty$, it follows from [45, Lemma 6.3] that $g(\lambda) \asymp \lambda^{\alpha/2-1}$, $\lambda \rightarrow \infty$. Therefore, the function $f(\lambda) := 1/g(\lambda)$ is a complete Bernstein function which is not regularly varying at infinity, but satisfies $f(\lambda) \asymp \lambda^{1-\alpha/2}$, $\lambda \rightarrow \infty$.

Now we are going to establish the asymptotic behaviors of u and μ under the assumption **(H)**.

First we claim that under the assumption (2.6), there exist $\delta \in (0, 1)$ and $a, s_0 > 0$ such that

$$\phi(\lambda t) \geq a\lambda^\delta \phi(t), \quad \lambda \geq 1, t \geq 1/s_0. \quad (2.8)$$

Indeed, by Potter's theorem (cf. [3, Theorem 1.5.6]), for $0 < \epsilon < \alpha/2$ there exists t_1 such that

$$\frac{\ell(t)}{\ell(\lambda t)} \leq 2 \max \left(\left(\frac{t}{\lambda t} \right)^\epsilon, \left(\frac{\lambda t}{t} \right)^\epsilon \right) = 2\lambda^\epsilon, \quad \lambda \geq 1, t \geq t_1.$$

Hence,

$$\phi(\lambda t) \geq c_2(\lambda t)^{\alpha/2} \ell(\lambda t) = c_2 t^{\alpha/2} \ell(t) \lambda^{\alpha/2} \frac{\ell(\lambda t)}{\ell(t)} \geq c_3 \phi(t) \lambda^{\alpha/2-\epsilon}, \quad \lambda \geq 1, t \geq t_1.$$

By taking $\delta := \alpha/2 - \epsilon \in (0, 1)$, $a = c_3$, and $s_0 = 1/t_1$ we arrive at (2.8).

Theorem 2.9 *Let S be a complete subordinator with Laplace exponent ϕ satisfying **(H)**. Then the potential density u of S satisfies*

$$u(t) \asymp t^{-1} \phi(t^{-1})^{-1} \asymp \frac{t^{\alpha/2-1}}{\ell(t^{-1})}, \quad t \rightarrow 0+. \quad (2.9)$$

Proof. Put

$$w(t) := \int_0^\infty e^{-st} u(s) ds = \frac{1}{\phi(t)},$$

then by (2.8) we have

$$w(\lambda t) \leq a^{-1} \lambda^{-\delta} w(t), \quad \lambda \geq 1, t \geq 1/s_0.$$

Applying the second part of Proposition 2.5 we see that there is a constant $c > 0$ such that

$$u(t) \geq ct^{-1}w(t^{-1}),$$

for small $t > 0$. Combining this inequality with (2.5) we arrive at (2.9). \square

Theorem 2.10 *Let S be a complete subordinator with Laplace exponent ϕ satisfying **(H)**. Then the Lévy density μ of S satisfies*

$$\mu(t) \asymp t^{-1}\phi(t^{-1}) \asymp t^{-\alpha/2-1}\ell(t^{-1}), \quad t \rightarrow 0+ . \quad (2.10)$$

Proof. Since ϕ is a complete Bernstein function, the function $\psi(\lambda) := \lambda/\phi(\lambda)$ is also a complete Bernstein function and satisfies

$$\psi(\lambda) \asymp \frac{\lambda^{1-\alpha/2}}{\ell(\lambda)}, \quad \lambda \rightarrow \infty,$$

where $\alpha \in (0, 2)$ and ℓ are the same as in (2.6). It follows from Corollary 2.4 that the potential measure of a subordinator with Laplace exponent ψ has a complete monotone density v given by

$$v(t) = \mu(t, \infty) = \int_t^\infty \mu(s) ds.$$

Applying Theorem 2.9 to ψ and v we get

$$\mu(t, \infty) = v(t) \asymp t^{-1}\psi(t^{-1})^{-1} = \phi(t^{-1}), \quad t \rightarrow 0. \quad (2.11)$$

By using the elementary inequality $1 - e^{-cy} \leq c(1 - e^{-y})$ valid for all $c \geq 1$ and all $y > 0$, we get that $\phi(c\lambda) \leq c\phi(\lambda)$ for all $c \geq 1$ and all $\lambda > 0$. Hence $\phi(s^{-1}) = \phi(2(2s)^{-1}) \leq 2\phi((2s)^{-1})$ for all $s > 0$. Therefore, by (2.11), for all $s \in (0, 1/2)$

$$v(s) \leq c_1\phi(s^{-1}) \leq 2c_1\phi((2s)^{-1}) \leq c_2v(2s)$$

for some constants $c_1, c_2 > 0$. Since

$$v(t/2) \geq v(t/2) - v(t) = \int_{t/2}^t \mu(s) ds \geq (t/2)\mu(t),$$

we have for all $t \in (0, 1)$,

$$\mu(t) \leq 2t^{-1}v(t/2) \leq c_2t^{-1}v(t) \leq c_3t^{-1}\phi(t^{-1}),$$

for some constant $c_3 > 0$.

Using (2.8) we get that for every $\lambda \geq 1$

$$\phi(s^{-1}) = \phi(\lambda(\lambda s)^{-1}) \geq a\lambda^\delta\phi((\lambda s)^{-1}), \quad s \leq \frac{s_0}{\lambda}.$$

It follows from (2.11) that there exists a constant $c_4 > 0$ such that

$$c_4^{-1}\phi(s^{-1}) \leq v(s) \leq c_4\phi(s^{-1}), \quad s < 1.$$

Fix $\lambda := 2^{1/\delta}((c_4^2 a^{-1}) \vee 1)^{1/\delta} \geq 1$. Then for $s \leq (s_0 \wedge 1)/\lambda$,

$$v(\lambda s) \leq c_4\phi((\lambda s)^{-1}) \leq c_4 a^{-1} \lambda^{-\delta} \phi(s^{-1}) \leq c_4^2 a^{-1} \lambda^{-\delta} v(s) \leq \frac{1}{2}v(s)$$

by our choice of λ . Further,

$$(\lambda - 1)s\mu(s) \geq \int_s^{\lambda s} \mu(t) dt = v(s) - v(\lambda s) \geq v(s) - \frac{1}{2}v(s) = \frac{1}{2}v(s).$$

This implies that for all small t

$$\mu(t) \geq \frac{1}{2(\lambda - 1)} t^{-1} v(t) = c_5 t^{-1} v(t) \geq c_6 t^{-1} \phi(t^{-1})$$

for some constants $c_5, c_6 > 0$. The proof is now complete. \square

3 Subordinate Brownian motion

3.1 Definitions and technical lemma

Let $B = (B_t, \mathbb{P}_x)$ be a Brownian motion in \mathbb{R}^d with transition density $p(t, x, y) = p(t, y - x)$ given by

$$p(t, x) = (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad t > 0, x, y \in \mathbb{R}^d.$$

The semigroup $(P_t : t \geq 0)$ of B is defined by $P_t f(x) = \mathbb{E}_x[f(B_t)] = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$, where f is a nonnegative Borel function on \mathbb{R}^d . Recall that if $d \geq 3$, the Green function $G^{(2)}(x, y) = G^{(2)}(x - y)$, $x, y \in \mathbb{R}^d$, of B is well defined and is equal to

$$G^{(2)}(x) = \int_0^\infty p(t, x) dt = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}} |x|^{-d+2}.$$

Let $S = (S_t : t \geq 0)$ be a complete subordinator independent of B , with Laplace exponent $\phi(\lambda)$, Lévy measure μ and potential measure U . In the rest of the paper, we will always assume that S satisfies **(H)**. Hence $\lim_{\lambda \rightarrow \infty} \phi(\lambda) = \infty$, and thus S has a completely monotone potential density u . We define a new process $X = (X_t : t \geq 0)$ by $X_t := B(S_t)$. Then X is a Lévy process with characteristic exponent $\Phi(x) = \phi(|x|^2)$ (see e.g. [33, pp.197–198]) called a subordinate Brownian motion. The semigroup $(Q_t : t \geq 0)$ of the process X is given by

$$Q_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(B(S_t))] = \int_0^\infty P_s f(x) \mathbb{P}(S_t \in ds).$$

The semigroup Q_t has a density $q(t, x, y) = q(t, x - y)$ given by $q(t, x) = \int_0^\infty p(s, x) \mathbb{P}(S_t \in ds)$.

We will always assume that the subordinate Brownian motion X is transient. According to the criterion of Chung-Fuchs type (see ([30] or [33, pp. 252–253]), X is transient if and only if for some small $r > 0$, $\int_{|x|<r} \frac{1}{\Phi(x)} dx < \infty$. Since $\Phi(x) = \phi(|x|^2)$, it follows that X is transient if and only if

$$\int_{0+} \frac{\lambda^{d/2-1}}{\phi(\lambda)} d\lambda < \infty. \quad (3.1)$$

This is always true if $d \geq 3$, and, depending on the subordinator, may be true for $d = 1$ or $d = 2$. In the case $d \leq 2$, if there exists $\gamma \in [0, d/2)$ such that

$$\liminf_{\lambda \rightarrow 0} \frac{\phi(\lambda)}{\lambda^\gamma} > 0, \quad (3.2)$$

then (3.1) holds.

For $x \in \mathbb{R}^d$ and A Borel subset of \mathbb{R}^d , the potential measure of X is given by

$$\begin{aligned} G(x, A) &= \mathbb{E}_x \int_0^\infty \mathbf{1}_{\{X_t \in A\}} dt = \int_0^\infty Q_t \mathbf{1}_A(x) dt = \int_0^\infty \int_0^\infty P_s \mathbf{1}_A(x) \mathbb{P}(S_t \in ds) dt \\ &= \int_0^\infty P_s \mathbf{1}_A u(s) ds = \int_A \int_0^\infty p(s, x, y) u(s) ds dy, \end{aligned}$$

where the second line follows from (2.1). If A is bounded, then by the transience of X , $G(x, A) < \infty$ for every $x \in \mathbb{R}^d$. Let $G(x, y)$ denote the density of the potential measure $G(x, \cdot)$. Clearly, $G(x, y) = G(y - x)$ where

$$G(x) = \int_0^\infty p(t, x) U(dt) = \int_0^\infty p(t, x) u(t) dt. \quad (3.3)$$

The Lévy measure Π of X is given by (see e.g. [33, pp. 197–198])

$$\Pi(A) = \int_A \int_0^\infty p(t, x) \mu(dt) dx = \int_A J(x) dx, \quad A \subset \mathbb{R}^d,$$

where

$$J(x) := \int_0^\infty p(t, x) \mu(dt) = \int_0^\infty p(t, x) \mu(t) dt \quad (3.4)$$

is the Lévy density of X . Define the function $j : (0, \infty) \rightarrow (0, \infty)$ by

$$j(r) := \int_0^\infty (4\pi)^{-d/2} t^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(dt), \quad r > 0, \quad (3.5)$$

and note that by (3.4), $J(x) = j(|x|)$, $x \in \mathbb{R}^d \setminus \{0\}$.

Since $x \mapsto p(t, x)$ is continuous and radially decreasing, we conclude that both G and J are continuous on $\mathbb{R}^d \setminus \{0\}$ and radially decreasing.

The following technical lemma will play a key role in establishing the asymptotic behaviors of the Green function G and the jumping function J of the subordinate Brownian motion X in the next subsection.

Lemma 3.1 Suppose that $w : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function, $\ell : (0, \infty) \rightarrow (0, \infty)$ a measurable, locally bounded function which is slowly varying at ∞ , and $\beta \in [0, 2]$, $\beta > 1 - d/2$. If $d = 1$ or $d = 2$, we additionally assume that there exist constants $c > 0$ and $\gamma < d/2$ such that

$$w(t) \leq ct^{\gamma-1}, \quad \forall t \geq 1. \quad (3.6)$$

Let

$$I(x) := \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} w(t) dt.$$

(a) If

$$w(t) \asymp \frac{1}{t^\beta \ell(1/t)}, \quad t \rightarrow 0, \quad (3.7)$$

then

$$I(x) \asymp \frac{1}{|x|^{d+2\beta-2} \ell\left(\frac{1}{|x|^2}\right)} \asymp \frac{w(|x|^2)}{|x|^{d-2}}, \quad |x| \rightarrow 0.$$

(b) If

$$w(t) \sim \frac{1}{t^\beta \ell(1/t)}, \quad t \rightarrow 0, \quad (3.8)$$

then

$$I(x) \sim \frac{\Gamma(d/2 + \beta - 1)}{4^{1-\beta} \pi^{d/2}} \frac{1}{|x|^{d+2\beta-2} \ell\left(\frac{1}{|x|^2}\right)}, \quad |x| \rightarrow 0.$$

Proof. (a) Let us first note that the assumptions of the lemma guarantee that $I(x) < \infty$ for every $x \neq 0$. Now, let $\xi \geq 1/4$ to be chosen later. By a change of variable we get

$$\begin{aligned} & \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} w(t) dt \\ &= \frac{1}{4\pi^{d/2}} \left(|x|^{-d+2} \int_0^{\xi|x|^2} t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt + |x|^{-d+2} \int_{\xi|x|^2}^\infty t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt \right) \\ &=: \frac{1}{4\pi^{d/2}} \left(|x|^{-d+2} I_1(x) + |x|^{-d+2} I_2(x) \right). \end{aligned} \quad (3.9)$$

We first consider $I_1(x)$ for the case $d = 1$ or $d = 2$. It follows from the assumptions that there exists a positive constant c_1 such that $w(s) \leq c_1 s^{\gamma-1}$ for all $s \geq 1/(4\xi)$. Thus

$$I_1(x) \leq \int_0^{\xi|x|^2} t^{d/2-2} e^{-t} c_1 \left(\frac{|x|^2}{4t}\right)^{\gamma-1} dt \leq c_2 |x|^{2\gamma-2} \int_0^{\xi|x|^2} t^{d/2-\gamma-1} dt = c_3 |x|^{d-2}.$$

It follows that

$$\lim_{|x| \rightarrow 0} |x|^{-d+2} I_1(x) \left(|x|^{d-2+2\beta} \ell\left(\frac{1}{|x|^2}\right) \right) = 0. \quad (3.10)$$

In the case $d \geq 3$, we proceed similarly, using the bound $w(s) \leq w(1/(4\xi))$ for $s \geq 1/(4\xi)$.

Now we consider $I_2(x)$:

$$\begin{aligned} |x|^{-d+2}I_2(x) &= \frac{1}{|x|^{d-2}} \int_{\xi|x|^2}^{\infty} t^{d/2-2} e^{-t} w\left(\frac{|x|^2}{4t}\right) dt \\ &= \frac{4^\beta}{|x|^{d+2\beta-2} \ell\left(\frac{1}{|x|^2}\right)} \int_{\xi|x|^2}^{\infty} t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} dt. \end{aligned}$$

Using the assumption (3.7), we can see that there is a constant $c_1 > 1$ such that

$$c_1^{-1} \leq w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) < c_1,$$

for all t and x satisfying $|x|^2/(4t) \leq 1/(4\xi)$.

Now choose a $\delta \in (0, d/2 - 1 + \beta)$ (note that by assumption, $d/2 - 1 + \beta > 0$). By Potter's theorem (cf. [3, Theorem 1.5.6 (i)]), there exists $\rho = \rho(\delta) \geq 1$ such that

$$\frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \leq 2 \left(\left(\frac{1/|x|^2}{4t/|x|^2} \right)^\delta \vee \left(\frac{1/|x|^2}{4t/|x|^2} \right)^{-\delta} \right) = 2 \left((4t)^\delta \vee (4t)^{-\delta} \right) \leq c_2 (t^\delta \vee t^{-\delta}) \quad (3.11)$$

whenever $\frac{1}{|x|^2} > \rho$ and $\frac{4t}{|x|^2} > \rho$. By reversing the roles of $1/|x|^2$ and $4t/|x|^2$ we also get that

$$\frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \geq c_2^{-1} (t^\delta \wedge t^{-\delta}) \quad (3.12)$$

for $\frac{1}{|x|^2} > \rho$ and $\frac{4t}{|x|^2} > \rho$. Now we define $\xi := \frac{\rho}{4}$ so that for all $x \neq 0$ with $|x|^2 \leq \frac{1}{4\xi}$ and $t > \xi|x|^2$ we have that

$$\begin{aligned} c_1^{-1} c_2^{-1} t^{d/2-2+\beta} e^{-t} (t^\delta \wedge t^{-\delta}) &\leq t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \\ &\leq c_1 c_2 t^{d/2-2+\beta} e^{-t} (t^\delta \vee t^{-\delta}). \end{aligned} \quad (3.13)$$

Let

$$\begin{aligned} c_3 &:= c_1^{-1} c_2^{-1} \int_0^\infty t^{d/2-2+\beta} e^{-t} (t^\delta \wedge t^{-\delta}) dt < \infty, \\ c_4 &:= c_1 c_2 \int_0^\infty t^{d/2-2+\beta} e^{-t} (t^\delta \vee t^{-\delta}) dt < \infty. \end{aligned}$$

The integrals are finite because of assumption $d/2 - 2 + \beta - \delta > -1$. It follows from (3.13) that

$$\begin{aligned} c_3 &\leq \liminf_{|x| \rightarrow 0} \int_0^\infty t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \mathbf{1}_{(\xi|x|^2, \infty)}(t) dt \\ &\leq \limsup_{|x| \rightarrow 0} \int_0^\infty t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \mathbf{1}_{(\xi|x|^2, \infty)}(t) dt \leq c_4. \end{aligned}$$

This means that

$$\begin{aligned} & |x|^{-d+2} I_2(x) \left(|x|^{d-2\beta+2} \ell\left(\frac{1}{|x|^2}\right) \right) \\ &= 4^\beta \int_{\xi|x|^2}^{\infty} t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} dt \asymp 1. \end{aligned} \quad (3.14)$$

Combining (3.10) and (3.14) we have proved the first part of the lemma.

(b) The proof is almost the same with a small difference at the very end. Since ℓ is slowly varying at ∞ , we have that

$$\lim_{|x| \rightarrow 0} \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} = 1.$$

This implies that

$$\lim_{|x| \rightarrow 0} t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \mathbf{1}_{(\xi|x|^2, \infty)}(t) = t^{d/2-2+\beta} e^{-t} \mathbf{1}_{(0, \infty)}(t).$$

By the right-hand side inequality in (3.13), we can apply the dominated convergence theorem to conclude that

$$\begin{aligned} & \lim_{|x| \rightarrow 0} |x|^{-d+2} I_2(x) \left(|x|^{d-2\beta+2} \ell\left(\frac{1}{|x|^2}\right) \right) \\ &= \lim_{|x| \rightarrow 0} 4^\beta \int_0^{\infty} t^{d/2-2+\beta} e^{-t} w\left(\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t}\right)^\beta \ell\left(\frac{4t}{|x|^2}\right) \frac{\ell\left(\frac{1}{|x|^2}\right)}{\ell\left(\frac{4t}{|x|^2}\right)} \mathbf{1}_{(\xi|x|^2, \infty)}(t) dt \\ &= 4^\beta \Gamma(d/2 - 1 + \beta). \end{aligned}$$

Together with (3.9) and (3.10) this proves the second part of the lemma. \square

3.2 Asymptotic behavior of the Green function and Lévy density

The goal of this subsection is to establish the asymptotic behavior of the Green function $G(x)$ and Lévy density $J(x)$ of the subordinate process Y under certain assumptions on the Laplace exponent ϕ of the subordinator S . We start with the Green function.

Theorem 3.2 *Suppose that the Laplace exponent ϕ is a complete Bernstein function satisfying the assumption **(H)** and that $\alpha \in (0, 2 \wedge d)$. In the case $d \leq 2$, we further assume (3.2). Then*

$$G(x) \asymp \frac{1}{|x|^d \phi(|x|^{-2})} \asymp \frac{1}{|x|^{d-\alpha} \ell(|x|^{-2})}, \quad |x| \rightarrow 0.$$

Proof. It follows from Theorem 2.9 that the potential density u of S satisfies

$$u(t) \asymp t^{-1} \phi(t^{-1})^{-1} \asymp \frac{t^{\alpha/2-1}}{\ell(t^{-1})}, \quad t \rightarrow 0+.$$

Using (2.5) and (3.2) we conclude that in case $d \leq 2$ there exists $c > 0$ such that

$$u(t) \leq ct^{\gamma-1}, \quad t \geq 1.$$

We can now apply Lemma 3.1 with $w(t) = u(t)$, $\beta = 1 - \alpha/2$ to obtain the required asymptotic behavior. \square

Remark 3.3 (i) Since α is always assumed to be in $(0, 2)$, the assumption $\alpha \in (0, 2 \wedge d)$ in the theorem above makes a difference only in the case $d = 1$.

(ii) In case $d \geq 3$, the conclusion of the theorem above is proved in [46, Theorem 1 (ii)–(iii)] under weaker assumptions. The statement of [46, Theorem 1 (ii)] in case $d \leq 2$ is incorrect and the proof has an error.

The asymptotic behavior near the origin of $J(x)$ is contained in the following result.

Theorem 3.4 *Suppose that the Laplace exponent ϕ is a complete Bernstein function satisfying the assumption (H). Then*

$$J(x) \asymp \frac{\phi(|x|^{-2})}{|x|^d} \asymp \frac{\ell(|x|^{-2})}{|x|^{d+\alpha}}, \quad |x| \rightarrow 0.$$

Proof. It follows from Theorem 2.10 that the potential density u of S satisfies

$$\mu(t) \asymp t^{-1}\phi(t^{-1}) \asymp t^{-\alpha/2-1}\ell(t^{-1}), \quad t \rightarrow 0+.$$

Since $\mu(t)$ is decreasing and integrable at infinity, there exists $c > 0$ such that

$$\mu(t) \leq ct^{-1}, \quad t \geq 1.$$

We can now apply Lemma 3.1 with $w(t) = \mu(t)$, $\beta = 1 + \alpha/2$ and $\gamma = 0$ to obtain the required asymptotic behavior. \square

Proposition 3.5 *Suppose that the Laplace exponent ϕ is a complete Bernstein function satisfying the assumption (H). Then the following assertions hold.*

(a) *For any $K > 0$, there exists $C_4 = C_4(K) > 1$ such that*

$$j(r) \leq C_4 j(2r), \quad \forall r \in (0, K). \quad (3.15)$$

(b) *There exists $C_5 > 1$ such that*

$$j(r) \leq C_5 j(r+1), \quad \forall r > 1. \quad (3.16)$$

Proof. For simplicity we redefine in this proof the function j by dropping the factor $(4\pi)^{-d/2}$ from its definition. This does not effect (3.15) and (3.16). It follows from Lemma 2.1 and Theorem 2.10 that

(a) For any $K > 0$, there exists $c_1 = c_1(K) > 1$ such that

$$\mu(r) \leq c_1 \mu(2r), \quad \forall r \in (0, K). \quad (3.17)$$

(b) There exists $c_2 > 1$ such that

$$\mu(r) \leq c_2 \mu(r + 1), \quad \forall r > 1. \quad (3.18)$$

Let $0 < r < K$. We have

$$\begin{aligned} j(2r) &= \int_0^\infty t^{-d/2} \exp(-r^2/t) \mu(t) dt \\ &\geq \frac{1}{2} \left(\int_{K/2}^\infty t^{-d/2} \exp(-r^2/t) \mu(t) dt + \int_0^K t^{-d/2} \exp(-r^2/t) \mu(t) dt \right) \\ &= \frac{1}{2} (I_1 + I_2). \end{aligned}$$

Now,

$$\begin{aligned} I_1 &= \int_{K/2}^\infty t^{-d/2} \exp(-\frac{r^2}{t}) \mu(t) dt = \int_{K/2}^\infty t^{-d/2} \exp(-\frac{r^2}{4t}) \exp(-\frac{3r^2}{4t}) \mu(t) dt \\ &\geq \int_{K/2}^\infty t^{-d/2} \exp(-\frac{r^2}{4t}) \exp(-\frac{3r^2}{2K}) \mu(t) dt \geq e^{-3K/2} \int_{K/2}^\infty t^{-d/2} \exp(-\frac{r^2}{4t}) \mu(t) dt, \\ I_2 &= \int_0^K t^{-d/2} \exp(-\frac{r^2}{t}) \mu(t) dt = 4^{-d/2+1} \int_0^{K/2} s^{-d/2} \exp(-\frac{r^2}{4s}) \mu(4s) ds \\ &\geq c_1^{-2} 4^{-d/2+1} \int_0^{K/2} s^{-d/2} \exp(-\frac{r^2}{4s}) \mu(s) ds. \end{aligned}$$

Combining the three displays above we get that $j(2r) \geq c_3 j(r)$ for all $r \in (0, K)$.

To prove (3.16) we first note that for all $t \geq 2$ and all $r \geq 1$ it holds that

$$\frac{(r+1)^2}{t} - \frac{r^2}{t-1} \leq 1.$$

This implies that

$$\exp\left(-\frac{(r+1)^2}{4t}\right) \geq e^{-1/4} \exp\left(-\frac{r^2}{4(t-1)}\right), \quad \text{for all } r > 1, t > 2. \quad (3.19)$$

Now we have

$$\begin{aligned}
j(r+1) &= \int_0^\infty t^{-d/2} \exp\left(-\frac{(r+1)^2}{4t}\right) \mu(t) dt \\
&\geq \frac{1}{2} \left(\int_0^8 t^{-d/2} \exp\left(-\frac{(r+1)^2}{4t}\right) \mu(t) dt + \int_3^\infty t^{-d/2} \exp\left(-\frac{(r+1)^2}{4t}\right) \mu(t) dt \right) \\
&= \frac{1}{2} (I_3 + I_4).
\end{aligned}$$

For I_3 note that $(r+1)^2 \leq 4r^2$ for all $r > 1$. Thus

$$\begin{aligned}
I_3 &= \int_0^8 t^{-d/2} \exp\left(-\frac{(r+1)^2}{4t}\right) \mu(t) dt \geq \int_0^8 t^{-d/2} \exp(-r^2/t) \mu(t) dt \\
&= 4^{-d/2+1} \int_0^2 s^{-d/2} \exp\left(-\frac{r^2}{4s}\right) \mu(4s) ds \geq c_1^{-2} 4^{-d/2+1} \int_0^2 s^{-d/2} \exp\left(-\frac{r^2}{4s}\right) \mu(s) ds, \\
I_4 &= \int_3^\infty t^{-d/2} \exp\left(-\frac{(r+1)^2}{4t}\right) \mu(t) dt \geq \int_3^\infty t^{-d/2} \exp\{-1/4\} \exp\left(-\frac{r^2}{4(t-1)}\right) \mu(t) dt \\
&= e^{-1/4} \int_2^\infty (s-1)^{-d/2} \exp\left(-\frac{r^2}{4s}\right) \mu(s+1) ds \geq c_1^{-1} e^{-1/4} \int_2^\infty s^{-d/2} \exp\left(-\frac{r^2}{4s}\right) \mu(s) ds.
\end{aligned}$$

Combining the three displays above we get that $j(r+1) \geq c_4 j(r)$ for all $r > 1$. \square

3.3 Some results on subordinate Brownian motion in \mathbb{R}

In this subsection we assume $d = 1$. We will consider subordinate Brownian motions in \mathbb{R} . Let $B = (B_t : t \geq 0)$ be a Brownian motion in \mathbb{R} , independent of S , with

$$\mathbb{E} \left[e^{i\theta(B_t - B_0)} \right] = e^{-t\theta^2}, \quad \forall \theta \in \mathbb{R}, t > 0.$$

The subordinate Brownian motion $X = (X_t : t \geq 0)$ in \mathbb{R} defined by $X_t = B_{S_t}$ is a symmetric Lévy process with the characteristic exponent $\Phi(\theta) = \phi(\theta^2)$ for all $\theta \in \mathbb{R}$. In the first part of this subsection, up to Corollary 3.8, we do not need to assume that ϕ satisfies the assumption **(H)**.

Let $\bar{X}_t := \sup\{0 \vee X_s : 0 \leq s \leq t\}$ and let $L = (L_t : t \geq 0)$ be a local time of $\bar{X} - X$ at 0. L is also called a local time of the process X reflected at the supremum. Then the right continuous inverse L_t^{-1} of L is a subordinator and is called the ladder time process of X . The process $\bar{X}_{L_t^{-1}}$ is also a subordinator and is called the ladder height process of X . (For the basic properties of the ladder time and ladder height processes, we refer our readers to [2, Chapter 6].) Let χ be the Laplace exponent of the ladder height process of X . It follows from [20, Corollary 9.7] that

$$\chi(\lambda) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\log(\Phi(\lambda\theta))}{1+\theta^2} d\theta \right) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2} d\theta \right), \quad \forall \lambda > 0. \quad (3.20)$$

The next result, first proved independently in [27] and [28], tells us that χ is also complete Bernstein function. The proof presented below is taken from [27].

Proposition 3.6 *Suppose ϕ , the Laplace exponent of the subordinator S , is a complete Bernstein function. Then the Laplace exponent χ of the ladder height process of the subordinate Brownian motion $X_t = B_{S_t}$ is also a complete Bernstein function.*

Proof. It follows from Theorem [34, Theorem 6.10] that ϕ has the following representation:

$$\log \phi(\lambda) = \gamma + \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\lambda+t} \right) \eta(t) dt, \quad (3.21)$$

where η is a function such that $0 \leq \eta(t) \leq 1$ for all $t > 0$. By (3.21) and (3.20), we have

$$\log \chi(\lambda) = \frac{\gamma}{2} + \frac{1}{\pi} \int_0^\infty \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\lambda^2 \theta^2 + t} \right) \eta(t) dt \frac{d\theta}{1+\theta^2}.$$

By using $0 \leq \eta(t) \leq 1$, we have

$$\begin{aligned} \eta(t) \left| \frac{t}{1+t^2} - \frac{1}{\lambda^2 \theta^2 + t} \right| \frac{1}{1+\theta^2} &\leq \frac{1}{1+t^2} \frac{1}{1+\theta^2} \left(\frac{1}{\lambda^2 \theta^2 + t} + \frac{\lambda^2 \theta^2 t}{\lambda^2 \theta^2 + t} \right) \\ &\leq \frac{1}{1+t^2} \left(\frac{1}{\lambda^2 \theta^2 + t} + \frac{\lambda^2 t}{\lambda^2 \theta^2 + t} \right). \end{aligned}$$

Since

$$\int_0^\infty \frac{1}{\lambda^2 \theta^2 + t} d\theta = \frac{1}{t} \int_0^\infty \frac{1}{\frac{\lambda^2 \theta^2}{t} + 1} d\theta = \frac{1}{t} \frac{\sqrt{t}}{\lambda} \int_0^\infty \frac{1}{\gamma^2 + 1} d\gamma = \frac{\pi}{2\lambda\sqrt{t}},$$

we can use Fubini's theorem to get

$$\begin{aligned} \log \chi(\lambda) &= \frac{\gamma}{2} + \int_0^\infty \left(\frac{t}{2(1+t^2)} - \frac{1}{2\sqrt{t}(\lambda + \sqrt{t})} \right) \eta(t) dt \\ &= \frac{\gamma}{2} + \int_0^\infty \left(\frac{t}{2(1+t^2)} - \frac{1}{2(1+t)} \right) \eta(t) dt + \int_0^\infty \left(\frac{1}{2(1+t)} - \frac{1}{2\sqrt{t}(\lambda + \sqrt{t})} \right) \eta(t) dt \\ &= \gamma_1 + \int_0^\infty \left(\frac{s}{1+s^2} - \frac{1}{\lambda+s} \right) \eta(s^2) ds. \end{aligned} \quad (3.22)$$

Applying [34, Theorem 6.10] we get that χ is a complete Bernstein function. \square

The potential measure of the ladder height process of X is denoted by V and its density by v . We will also use V to denote the renewal function of X : $V(t) := V((0, t)) = \int_0^t v(s) ds$.

The following result is first proved in [27].

Proposition 3.7 *χ is related to ϕ by the following relation*

$$e^{-\pi/2} \sqrt{\phi(\lambda^2)} \leq \chi(\lambda) \leq e^{\pi/2} \sqrt{\phi(\lambda^2)}, \quad \text{for all } \lambda > 0.$$

Proof. According to (3.22), we have

$$\log \chi(\lambda) = \frac{\gamma}{2} + \frac{1}{2} \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\sqrt{t}(\lambda + \sqrt{t})} \right) \eta(t) dt.$$

Together with representation (3.21) we get that for all $\lambda > 0$

$$\begin{aligned} \left| \log \chi(\lambda) - \frac{1}{2} \log \phi(\lambda^2) \right| &= \frac{1}{2} \left| \int_0^\infty \left(\left(\frac{t}{1+t^2} - \frac{1}{\sqrt{t}(\lambda + \sqrt{t})} \right) - \left(\frac{t}{1+t^2} - \frac{1}{\lambda^2 + t} \right) \right) \eta(t) dt \right| \\ &\leq \frac{1}{2} \int_0^\infty \frac{\lambda(\sqrt{t} + \lambda)}{(\lambda^2 + t)\sqrt{t}(\lambda + \sqrt{t})} dt = \frac{1}{2} \int_0^\infty \frac{\lambda}{(\lambda^2 + t)\sqrt{t}} dt = \frac{\pi}{2}. \end{aligned}$$

This implies that

$$-\pi/2 \leq \log \chi(\lambda) - \frac{1}{2} \log \phi(\lambda^2) \leq \pi/2, \quad \text{for all } \lambda > 0,$$

i.e.,

$$e^{-\pi/2} \leq \chi(\lambda)\phi(\lambda^2)^{-1/2} \leq e^{\pi/2}, \quad \text{for all } \lambda > 0.$$

□

Combining the above two propositions with Corollary 2.3, we obtain

Corollary 3.8 *Suppose ϕ , the Laplace exponent of the subordinator S , is a complete Bernstein function satisfying $\lim_{\lambda \rightarrow \infty} \phi(\lambda) = \infty$. Then the potential measure of the ladder height process of the subordinate Brownian motion $X_t = B_{S_t}$ has a completely monotone density v . In particular, v and the renewal function V are C^∞ functions.*

In the remainder of this paper we will always assume that ϕ satisfies the assumption **(H)**. We will not explicitly mention this assumption anymore.

Since $\phi(\lambda) \asymp \lambda^{\alpha/2} \ell(\lambda)$ as $\lambda \rightarrow \infty$, Lemma 3.7 implies that

$$\chi(\lambda) \asymp \lambda^{\alpha/2} (\ell(\lambda^2))^{1/2}, \quad t \rightarrow \infty. \quad (3.23)$$

It follows from (3.23) that $\lim_{\lambda \rightarrow \infty} \chi(\lambda)/\lambda = 0$, hence the ladder height process does not have a drift. Recall that $V(t) = V((0, t)) = \int_0^t v(s) ds$ is the renewal function of the ladder height process of X . In light of (3.23), we have, as a consequence of Theorem 2.9, the following result.

Proposition 3.9 *As $t \rightarrow 0$, we have*

$$V(t) \asymp \phi(t^{-2})^{-1/2} \asymp \frac{t^{\alpha/2}}{(\ell(t^{-2}))^{1/2}}$$

and

$$v(t) \asymp t^{-1} \phi(t^{-2})^{-1/2} \asymp \frac{t^{\alpha/2-1}}{(\ell(t^{-2}))^{1/2}}.$$

Remark 3.10 It follows immediately from the proposition above that there exists a positive constant $c > 0$ such that $V(2t) \leq cV(t)$ for all $t \in (0, 2)$.

It follows from (3.23) above and [29, Lemma 7.10] that the process X does not creep upwards. Since X is symmetric, we know that X also does not creep downwards. Thus if, for any $a \in \mathbb{R}$, we define

$$\tau_a = \inf\{t > 0 : X_t < a\}, \quad \sigma_a = \inf\{t > 0 : X_t \leq a\},$$

then we have

$$\mathbb{P}_x(\tau_a = \sigma_a) = 1, \quad x > a. \quad (3.24)$$

Let $G^{(0,\infty)}(x, y)$ be the Green function of X in $(0, \infty)$. Then we have the following result.

Proposition 3.11 *For any $x, y > 0$ we have*

$$G^{(0,\infty)}(x, y) = \begin{cases} \int_0^x v(z)v(y+z-x)dz, & x \leq y, \\ \int_{x-y}^x v(z)v(y+z-x)dz, & x > y. \end{cases}$$

Proof. Let $X^{(0,\infty)}$ be the process obtained by killing X upon exiting from $(0, \infty)$. By using (3.24) above and [2, Theorem 20, p. 176] we get that for any nonnegative function f on $(0, \infty)$,

$$\mathbb{E}_x \left[\int_0^\infty f(X_t^{(0,\infty)}) dt \right] = k \int_0^\infty \int_0^x v(z)f(x+z-y)v(y)dzdy, \quad (3.25)$$

where k is the constant depending on the normalization of the local time of the process X reflected at its supremum. We choose $k = 1$. Then

$$\begin{aligned} \mathbb{E}_x \left[\int_0^\infty f(X_t^{(0,\infty)}) dt \right] &= \int_0^\infty v(y) \int_0^x v(z)f(x+y-z)dzdy \\ &= \int_0^x v(z) \int_0^\infty v(y)f(x+y-z)dydz = \int_0^x v(z) \int_{x-z}^\infty v(w+z-x)f(w)dw dz \\ &= \int_0^x f(w) \int_{x-w}^x v(z)v(w+z-x)dzdw + \int_x^\infty f(w) \int_0^x v(z)v(w+z-x)dzdw. \end{aligned} \quad (3.26)$$

On the other hand,

$$\begin{aligned} \mathbb{E}_x \left[\int_0^\infty f(X_t^{(0,\infty)}) dt \right] &= \int_0^\infty G^{(0,\infty)}(x, w)f(w) dw \\ &= \int_0^x G^{(0,\infty)}(x, w)f(w) dw + \int_x^\infty G^{(0,\infty)}(x, w)f(w) dw. \end{aligned} \quad (3.27)$$

By comparing (3.26) and (3.27) we arrive at our desired conclusion. \square

For any $r > 0$, let $G^{(0,r)}$ be the Green function of X in $(0, r)$. Then we have the following result.

Proposition 3.12 For all $r > 0$ and all $x \in (0, r)$

$$\int_0^r G^{(0,r)}(x, y) dy \leq 2V(x)V(r).$$

In particular, for any $R > 0$, there exists $C_6 = C_6(R) > 0$ such that for all $r \in (0, R)$ and all $x \in (0, r)$,

$$\int_0^r G^{(0,r)}(x, y) dy \leq C_6(\phi(r^{-2})\phi(x^{-2}))^{-1/2} \asymp \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{x^{\alpha/2}}{(\ell(x^{-2}))^{-1/2}}.$$

Proof. For any $x \in (0, r)$, we have

$$\begin{aligned} \int_0^r G^{(0,r)}(x, y) dy &\leq \int_0^r G^{(0,\infty)}(x, y) dy \\ &= \int_0^x \int_{x-y}^x v(z)v(y+z-x) dz dy + \int_x^r \int_0^x v(z)v(y+z-x) dz dy \\ &= \int_0^x v(z) \int_{x-z}^x v(y+z-x) dy dz + \int_0^x v(z) \int_x^r v(y+z-x) dy dz \leq 2V(r)V(x). \end{aligned}$$

Now the desired conclusion follows easily from Proposition 3.9. \square

As a consequence of the result above, we immediately get the following.

Corollary 3.13 For all $r > 0$ and all $x \in (0, r)$

$$\int_0^r G^{(0,r)}(x, y) dy \leq 2V(r)(V(x) \wedge V(r-x)).$$

In particular, for any $R > 0$, there exists $C_7 = C_7(R) > 0$ such that for all $x \in (0, r)$, and $r \in (0, R)$,

$$\begin{aligned} \int_0^r G^{(0,r)}(x, y) dy &\leq C_7(\phi(r^{-2}))^{-1/2} \left((\phi(x^{-2}))^{-1/2} \wedge (\phi((r-x)^{-2}))^{-1/2} \right) \\ &\asymp \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \left(\frac{x^{\alpha/2}}{(\ell(x^{-2}))^{1/2}} \wedge \frac{(r-x)^{\alpha/2}}{(\ell((r-x)^{-2}))^{1/2}} \right). \end{aligned}$$

Proof. The first inequality is a consequence of the identity $\int_0^r G^{(0,r)}(x, y) dy = \int_0^r G^{(0,r)}(r-x, y) dy$ which is true by symmetry of the process X . The second one now follows exactly as in the proof of Proposition 3.12. \square

Remark 3.14 With self-explanatory notation, an immediate consequence of the above corollary is the following estimate

$$\int_{-r}^r G^{(-r,r)}(x, y) dy \leq 2V(2r)(V(r+x) \wedge V(r-x)). \quad (3.28)$$

4 Harnack inequality and Boundary Harnack principle

From now on we will always assume that X is a subordinate Brownian motion in \mathbb{R}^d . Recall that **(H)** is the standing assumptions on the Laplace exponent ϕ . We additionally assume that $\alpha \in (0, 2 \wedge d)$ and in the case $d \leq 2$, we further assume (3.2). The goal of this section is to show that the Harnack inequality and the boundary Harnack principle hold for X . The infinitesimal generator \mathbf{L} of the corresponding semigroup is given by

$$\mathbf{L}f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbf{1}_{\{|y| \leq 1\}}) J(y) dy \quad (4.1)$$

for $f \in C_b^2(\mathbb{R}^d)$. Moreover, for every $f \in C_b^2(\mathbb{R}^d)$

$$f(X_t) - f(X_0) - \int_0^t \mathbf{L}f(X_s) ds$$

is a \mathbb{P}_x -martingale for every $x \in \mathbb{R}^d$. We recall the Lévy system formula for X which describes the jumps of the process X : for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y) J(X_s, y) dy \right) ds \right]. \quad (4.2)$$

(See, for example, [15, Proof of Lemma 4.7] and [16, Appendix A].)

4.1 Harnack inequality

It follows from Theorem 3.4 and the 0-version of [3, Propositions 1.5.8 and 1.5.10] that

$$r^{-2} \int_0^r s^{d+1} j(s) ds \asymp \frac{\ell(r^{-2})}{r^\alpha} \asymp \phi(r^{-2}), \quad r \rightarrow 0 \quad (4.3)$$

and

$$\int_r^\infty s^{d-1} j(s) ds \asymp \frac{\ell(r^{-2})}{r^\alpha} \asymp \phi(r^{-2}), \quad r \rightarrow 0. \quad (4.4)$$

For any open set D , we use τ_D to denote the first exit time from D , i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$.

Lemma 4.1 *There exists a constant $C_8 > 0$ such that for every $r \in (0, 1)$ and every $t > 0$,*

$$\mathbb{P}_x \left(\sup_{s \leq t} |X_s - X_0| > r \right) \leq C_8 \phi(r^{-2}) t.$$

Proof. It suffices to prove the lemma for $x = 0$. Let $f \in C_b^2(\mathbb{R}^d)$, $0 \leq f \leq 1$, $f(0) = 0$, and $f(y) = 1$ for all $|y| \geq 1$. Let $c_1 = \sup_y \sum_{j,k} |(\partial^2/\partial y_j \partial y_k) f(y)|$. Then $|f(z+y) - f(z) - y \cdot \nabla f(z)| \leq \frac{c_1}{2} |y|^2$. For $r \in (0, 1)$, let $f_r(y) = f(y/r)$. Then the following estimate is valid:

$$\begin{aligned} |f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z) \mathbf{1}_{\{|y| \leq r\}}| &\leq \frac{c_1}{2} \frac{|y|^2}{r^2} \mathbf{1}_{\{|y| \leq r\}} + \mathbf{1}_{\{|y| \geq r\}} \\ &\leq c_2 (\mathbf{1}_{\{|y| \leq r\}} \frac{|y|^2}{r^2} + \mathbf{1}_{\{|y| \geq r\}}). \end{aligned}$$

By using (4.3) and (4.4), we get the following estimate:

$$\begin{aligned} |\mathbf{L}f_r(z)| &\leq \int_{\mathbb{R}^d} |f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z) \mathbf{1}_{\{|y| \leq r\}}| J(y) dy \\ &\leq c_2 \int_{\mathbb{R}^d} \left(\mathbf{1}_{\{|y| \leq r\}} \frac{|y|^2}{r^2} + \mathbf{1}_{\{|y| \geq r\}} \right) J(y) dy \\ &\leq c_3 \phi(r^{-2}), \end{aligned}$$

where the constant c_3 is independent of r . Further, by the martingale property,

$$\mathbb{E}_0 f_r(X(\tau_{B(0,r)} \wedge t)) - f_r(0) = \mathbb{E}_0 \int_0^{\tau_{B(0,r)} \wedge t} \mathbf{L}f_r(X_s) ds$$

implying the estimate

$$\mathbb{E}_0 f_r(X(\tau_{B(0,r)} \wedge t)) \leq c_3 \phi(r^{-2}) t.$$

If X exits $B(0, r)$ before time t , then $f_r(X(\tau_{B(0,r)} \wedge t)) = 1$, so the left hand side is larger than $\mathbb{P}_0(\tau_{B(0,r)} \leq t)$. \square

Lemma 4.2 For every $r \in (0, 1)$, and every $x \in \mathbb{R}^d$,

$$\inf_{z \in B(x, r/2)} \mathbb{E}_z [\tau_{B(x,r)}] \geq \frac{1}{4C_8 \phi((r/2)^{-2})},$$

where C_8 is the constant from Lemma 4.1.

Proof. Let $z \in B(x, r/2)$. Then

$$\mathbb{P}_z(\tau_{B(x,r)} \leq t) \leq \mathbb{P}_z(\tau_{B(z, r/2)} \leq t) \leq C_8 \phi((r/2)^{-2}) t.$$

Therefore,

$$\mathbb{E}_z [\tau_{B(x,r)}] \geq t \mathbb{P}_z(\tau_{B(x,r)} \geq t) \geq t(1 - C_8 \phi((r/2)^{-2}) t).$$

Choose $t = 1/(2C_8 \phi((r/2)^{-2}))$ so that $1 - C_8 \phi((r/2)^{-2}) t = 1/2$. Then

$$\mathbb{E}_z [\tau_{B(x,r)}] \geq \frac{1}{2C_8 \phi((r/2)^{-2})} \frac{1}{2} = \frac{1}{4C_8 \phi((r/2)^{-2})}.$$

\square

Lemma 4.3 *There exists a constant $C_9 > 0$ such that for every $r \in (0, 1)$ and every $x \in \mathbb{R}^d$,*

$$\sup_{z \in B(x, r)} \mathbb{E}_z [\tau_{B(x, r)}] \leq \frac{C_9}{\phi(r^{-2})}.$$

Proof. This lemma follows immediately from Theorem 3.2 and the 0-version of [3, Proposition 1.5.8]. An improved version will be given in Proposition 4.9 later on. Here we present a different proof that does not need the transience of X .

Let $r \in (0, 1)$, and let $x \in \mathbb{R}^d$. Using the Lévy system formula (4.2), we get

$$\begin{aligned} 1 &\geq \mathbb{P}_z(|X(\tau_{B(x, r)}) - x| > r) \\ &= \int_{B(x, r)} G_{B(x, r)}(z, y) \int_{\overline{B(x, r)}^c} j(|u - y|) du dy, \end{aligned}$$

where $G_{B(x, r)}$ denotes the Green function of the process X in $B(x, r)$. Now we estimate the inner integral. Let $y \in B(x, r)$, $u \in \overline{B(x, r)}^c$. If $u \in B(x, 2)$, then $|u - y| \leq 2|u - x|$, while for $u \notin B(x, 2)$ we use $|u - y| \leq |u - x| + 1$. Then

$$\begin{aligned} \int_{\overline{B(x, r)}^c} j(|u - y|) du &= \int_{\overline{B(x, r)}^c \cap B(x, 2)} j(|u - y|) du + \int_{\overline{B(x, r)}^c \cap B(x, 2)^c} j(|u - y|) du \\ &\geq \int_{\overline{B(x, r)}^c \cap B(x, 2)} j(2|u - x|) du + \int_{\overline{B(x, r)}^c \cap B(x, 2)^c} j(|u - x| + 1) du \\ &\geq \int_{\overline{B(x, r)}^c \cap B(x, 2)} c^{-1} j(|u - x|) du + \int_{\overline{B(x, r)}^c \cap B(x, 2)^c} c^{-1} j(|u - x|) du \\ &= \int_{\overline{B(x, r)}^c} c^{-1} j(|u - x|) du, \end{aligned}$$

where in the next to last line we used (3.15) and (3.16). Now, It follows from (4.4) that

$$\begin{aligned} 1 &\geq \int_{B(x, r)} G_{B(x, r)}(z, y) dy \int_{\overline{B(x, r)}^c} c^{-1} j(|u - x|) du \\ &= \mathbb{E}_z [\tau_{B(x, r)}] c^{-1} c_1 \int_r^\infty v^{d-1} j(v) dv \\ &= c_2 \phi(r^{-2}) \mathbb{E}_z [\tau_{B(x, r)}] \end{aligned}$$

which implies the lemma. □

Lemma 4.4 *There exists a constant $C_{10} > 0$ such that for every $r \in (0, 1)$, every $x \in \mathbb{R}^d$, and any $A \subset B(x, r)$*

$$\mathbb{P}_y (T_A < \tau_{B(x, 3r)}) \geq C_{10} \frac{|A|}{|B(x, r)|}, \quad \text{for all } y \in B(x, 2r).$$

Proof. Without loss of generality assume that $\mathbb{P}_y(T_A < \tau_{B(x,3r)}) < 1/4$. Set $\tau = \tau_{B(x,3r)}$. By Lemma 4.1, $\mathbb{P}_y(\tau \leq t) \leq \mathbb{P}_y(\tau_{B(y,r)} \leq t) \leq c_1 \phi(r^{-2})t$. Choose $t_0 = 1/(4c_1 \phi(r^{-2}))$, so that $\mathbb{P}_y(\tau \leq t_0) \leq 1/4$. Further, if $z \in B(x, 3r)$ and $u \in A \subset B(x, r)$, then $|u - z| \leq 4r$. Since j is decreasing, $j(|u - z|) \geq j(4r)$. Thus,

$$\begin{aligned} \mathbb{P}_y(T_A < \tau) &\geq \mathbb{E}_y \sum_{s \leq T_A \wedge \tau \wedge t_0} \mathbf{1}_{\{X_{s-} \neq X_s, X_s \in A\}} \\ &= \mathbb{E}_y \int_0^{T_A \wedge \tau \wedge t_0} \int_A j(|u - X_s|) du ds \\ &\geq \mathbb{E}_y \int_0^{T_A \wedge \tau \wedge t_0} \int_A j(4r) du ds \\ &= j(4r)|A|\mathbb{E}_y[T_A \wedge \tau \wedge t_0], \end{aligned}$$

where in the second line we used properties of the Lévy system. Next,

$$\begin{aligned} \mathbb{E}_y[T_A \wedge \tau \wedge t_0] &\geq \mathbb{E}_y[t_0; T_A \geq \tau \geq t_0] \\ &= t_0 \mathbb{P}_y(T_A \geq \tau \geq t_0) \\ &\geq t_0 [1 - \mathbb{P}_y(T_A < \tau) - \mathbb{P}_y(\tau < t_0)] \\ &\geq \frac{t_0}{2} = \frac{1}{8c_1 \phi(r^{-2})}. \end{aligned}$$

The last two displays give that

$$\mathbb{P}_y(T_A < \tau) \geq j(4r)|A| \frac{1}{8c_1 \phi(r^{-2})} = \frac{1}{8c_1} |A| \frac{j(4r)}{\phi(r^{-2})}.$$

The claim now follows immediately from (2.6) and Theorem 3.4. \square

Lemma 4.5 *There exist positive constant C_{11} and C_{12} , such that if $r \in (0, 1)$, $x \in \mathbb{R}^d$, $z \in B(x, r)$, and H is a bounded nonnegative function with support in $B(x, 2r)^c$, then*

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \leq C_{11} \mathbb{E}_z[\tau_{B(x,r)}] \int H(y) j(|y - x|) dy,$$

and

$$\mathbb{E}_z H(X(\tau_{B(x,r)})) \geq C_{12} \mathbb{E}_z[\tau_{B(x,r)}] \int H(y) j(|y - x|) dy.$$

Proof. Let $y \in B(x, r)$ and $u \in B(x, 2r)^c$. If $u \in B(x, 2)$ we use the estimates

$$2^{-1}|u - x| \leq |u - y| \leq 2|u - x|, \tag{4.5}$$

while if $u \notin B(x, 2)$ we use

$$|u - x| - 1 \leq |u - y| \leq |u - x| + 1. \tag{4.6}$$

Let $B \subset B(x, 2r)^c$. Then using the Lévy system we get

$$\mathbb{E}_z \left[\mathbf{1}_B(X_{\tau_{B(x,r)}}) \right] = \mathbb{E}_z \int_0^{\tau_{B(x,r)}} \int_B j(|u - X_s|) du ds.$$

By use of (3.15), (3.16), (4.5), and (4.6), the inner integral is estimated as follows:

$$\begin{aligned} \int_B j(|u - X_s|) du &= \int_{B \cap B(x,2)} j(|u - X_s|) du + \int_{B \cap B(x,2)^c} j(|u - X_s|) du \\ &\leq \int_{B \cap B(x,2)} j(2^{-1}|u - x|) du + \int_{B \cap B(x,2)^c} j(|u - x| - 1) du \\ &\leq \int_{B \cap B(x,2)} cj(|u - x|) du + \int_{B \cap B(x,2)^c} cj(|u - x|) du \\ &= c \int_B j(|u - x|) du \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}_z \left[\mathbf{1}_B(X_{\tau_{B(x,r)}}) \right] &\leq \mathbb{E}_z \int_0^{\tau_{B(x,r)}} c \int_B j(|u - x|) du \\ &= c \mathbb{E}_z(\tau_{B(x,r)}) \int \mathbf{1}_B(u) j(|u - x|) du. \end{aligned}$$

Using linearity we get the above inequality when $\mathbf{1}_B$ is replaced by a simple function. Approximating H by simple functions and taking limits we have the first inequality in the statement of the lemma.

The second inequality is proved in the same way. \square

Definition 4.6 Let D be an open subset of \mathbb{R}^d . A function u defined on \mathbb{R}^d is said to be

(1) *harmonic in D with respect to X if*

$$\mathbb{E}_x [|u(X_{\tau_B})|] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x [u(X_{\tau_B})], \quad x \in B,$$

for every open set B whose closure is a compact subset of D ;

(2) *regular harmonic in D with respect to X if it is harmonic in D with respect to X and for each $x \in D$, $u(x) = \mathbb{E}_x [u(X_{\tau_D})]$.*

Now we give the proof of Harnack inequality. The proof below is basically the proof given in [39] which is an adaptation of the proof given in [1]. However, the proof below corrects some typos in the proof given in [39].

Theorem 4.7 *There exists $C_{13} > 0$ such that, for any $r \in (0, 1/4)$, $x_0 \in \mathbb{R}^d$, and any function u which is nonnegative on \mathbb{R}^d and harmonic with respect to X in $B(x_0, 16r)$, we have*

$$u(x) \leq C_{13}u(y), \quad \text{for all } x, y \in B(x_0, r).$$

Proof. Without loss of generality we may assume that u is strictly positive in $B(x_0, 16r)$. Indeed, if $u(x) = 0$ for some $x \in B(x_0, 16r)$, then by harmonicity $0 = u(x) = \mathbb{E}_x[u(X_{\tau_B})]$. For $x \in B = B(x, \epsilon) \subset B(x_0, 16r)$. This and the fact that the Levy measure of X is supported on all of \mathbb{R}^d and has a density imply that $u = 0$ a.e. with respect to Lebesgue measure. Moreover, by the harmonicity, for every $y \in B(x_0, 16r)$, $u(y) = \mathbb{E}_y[u(X_{\tau_B})] = 0$ where $B = B(y, \delta) \subset B(x_0, 16r)$. Therefore, if $u(x) = 0$ for some x , then u is identically zero in $B(x_0, 16r)$ and there is nothing to prove.

We first assume u is bounded on \mathbb{R}^d . Using the harmonicity of u and Lemma 4.4, one can show that u is bounded from below on $B(x_0, r)$ by a positive number. To see this, let $\epsilon > 0$ be such that $F = \{x \in B(x_0, 3r) \setminus B(x_0, 2r) : u(x) > \epsilon\}$ has positive Lebesgue measure. Take a compact subset K of F so that it has positive Lebesgue measure. Then by Lemma 4.4, for $x \in B(x_0, r)$, we have

$$u(x) = \mathbb{E}_x \left[u(X(T_K \wedge \tau_{B(x_0, 3r)})) \mathbf{1}_{\{T_K \wedge \tau_{B(x_0, 3r)} < \infty\}} \right] > c \epsilon \frac{|K|}{|B(x_0, 3r)|},$$

for some $c > 0$. By taking a constant multiple of u we may assume that $\inf_{B(x_0, r)} u = 1/2$. Choose $z_0 \in B(x_0, r)$ such that $u(z_0) \leq 1$. We want to show that u is bounded above in $B(x_0, r)$ by a positive constant independent of u and $r \in (0, 1/4)$. We will establish this by contradiction: If there exists a point $x \in B(x_0, r)$ with $h(u) = K$ where K is too large, we can obtain a sequence of points in $B(x_0, 2r)$ along which u is unbounded.

Using Lemmas 4.2, 4.3 and 4.5, one can see that there exists $c_1 > 0$ such that if $x \in \mathbb{R}^d$, $s \in (0, 1)$ and H is nonnegative bounded function with support in $B(x, 2s)^c$, then for any $y, z \in B(x, s/2)$,

$$\mathbb{E}_z H(X(\tau_{B(x, s)})) \leq c_1 \mathbb{E}_y H(X(\tau_{B(x, s)})). \quad (4.7)$$

By Lemma 4.4, there exists $c_2 > 0$ such that if $A \subset B(x_0, 4r)$ then

$$\mathbb{P}_y (T_A < \tau_{B(x_0, 16r)}) \geq c_2 \frac{|A|}{|B(x_0, 4r)|}, \quad \forall y \in B(x_0, 8r). \quad (4.8)$$

Again by Lemma 4.4, there exists $c_3 > 0$ such that if $x \in \mathbb{R}^d$, $s \in (0, 1)$ and $F \subset B(x, s/3)$ with $|F|/|B(x, s/3)| \geq 1/3$, then

$$\mathbb{P}_x (T_F < \tau_{B(x, s)}) \geq c_3. \quad (4.9)$$

Let

$$\eta = \frac{c_3}{3}, \quad \zeta = \left(\frac{1}{3} \wedge \frac{1}{c_1}\right) \eta. \quad (4.10)$$

Now suppose there exists $x \in B(x_0, r)$ with $u(x) = K$ for $K > K_0 := \frac{2|B(x_0, 1)|}{c_2 \zeta} \vee \frac{2(12)^d}{c_2 \zeta}$. Let s be chosen so that

$$|B(x, \frac{s}{3})| = \frac{2|B(x_0, 4r)|}{c_2 \zeta K} < 1. \quad (4.11)$$

Note that this implies

$$s = 12 \left(\frac{2}{c_2 \zeta}\right)^{1/d} r K^{-1/d} < r. \quad (4.12)$$

Let us write B_s for $B(x, s)$, τ_s for $\tau_{B(x,s)}$, and similarly for B_{2s} and τ_{2s} . Let A be a compact subset of

$$A' = \{y \in B(x, \frac{s}{3}) : u(y) \geq \zeta K\}.$$

It is well known that $u(X_t)$ is right continuous in $[0, \tau_{B(x_0, 16r)})$. Since $z_0 \in B(x_0, r)$ and $A' \subset B(x, \frac{s}{3}) \subset B(x_0, 2r)$, we can apply (4.8) to get

$$\begin{aligned} 1 &\geq u(z_0) \geq \mathbb{E}_{z_0}[u(X(T_A \wedge \tau_{B(x_0, 16r)})) \mathbf{1}_{\{T_A < \tau_{B(x_0, 16r)}\}}] \\ &\geq \zeta K \mathbb{P}_{z_0}(T_A < \tau_{B(x_0, 16r)}) \\ &\geq c_2 \zeta K \frac{|A|}{|B(x_0, 4r)|}. \end{aligned}$$

Hence

$$\frac{|A|}{|B(x, \frac{s}{3})|} \leq \frac{|B(x_0, 4r)|}{c_2 \zeta K |B(x, \frac{s}{3})|} = \frac{1}{2}.$$

This implies that $|A'|/|B(x, s/3)| \leq 1/2$. Let F be a compact subset of $B(x, s/3) \setminus A'$ such that

$$\frac{|F|}{|B(x, \frac{s}{3})|} \geq \frac{1}{3}. \quad (4.13)$$

Let $H = u \cdot \mathbf{1}_{B_{2s}^c}$. We claim that

$$\mathbb{E}_x[u(X(\tau_s)); X(\tau_s) \notin B_{2s}] \leq \eta K.$$

If not, $\mathbb{E}_x H(X(\tau_s)) > \eta K$, and by (4.7), for all $y \in B(x, s/3)$, we have

$$\begin{aligned} u(y) &= \mathbb{E}_y u(X(\tau_s)) \geq \mathbb{E}_y[u(X(\tau_s)); X(\tau_s) \notin B_{2s}] \\ &\geq c_1^{-1} \mathbb{E}_x H(X(\tau_s)) \geq c_1^{-1} \eta K \geq \zeta K, \end{aligned}$$

contradicting (4.13) and the definition of A' .

Let $M = \sup_{B_{2s}} u$. We then have

$$\begin{aligned} K &= u(x) = \mathbb{E}_x[u(X(\tau_s \wedge T_F))] \\ &= \mathbb{E}_x[u(X(T_F)); T_F < \tau_s] + \mathbb{E}_x[u(X(\tau_s)); \tau_s < T_F, X(\tau_s) \in B_{2s}] \\ &\quad + \mathbb{E}_x[u(X(\tau_s)); \tau_s < T_F, X(\tau_s) \notin B_{2s}] \\ &\leq \zeta K \mathbb{P}_x(T_F < \tau_s) + M \mathbb{P}_x(\tau_s < T_F) + \eta K \\ &= \zeta K \mathbb{P}_x(T_F < \tau_s) + M(1 - \mathbb{P}_x(T_F < \tau_s)) + \eta K, \end{aligned}$$

or equivalently

$$\frac{M}{K} \geq \frac{1 - \eta - \zeta}{1 - \mathbb{P}_x(T_F < \tau_s)} + \zeta.$$

Using (4.9) and (4.10) we see that there exists $\beta > 0$ such that $M \geq K(1 + 2\beta)$. Therefore there exists $x' \in B(x, 2s)$ with $u(x') \geq K(1 + \beta)$.

Now suppose there exists $x_1 \in B(x_0, r)$ with $u(x_1) = K_1 > K_0$. Define s_1 in terms of K_1 analogously to (4.11). Using the above argument (with x_1 replacing x and x_2 replacing x'), there exists $x_2 \in B(x_1, 2s_1)$ with $u(x_2) = K_2 \geq (1 + \beta)K_1$. We continue and obtain s_2 and then x_3, K_3, s_3 , etc. Note that $x_{i+1} \in B(x_i, 2s_i)$ and $K_i \geq (1 + \beta)^{i-1}K_1$. In view of (4.12),

$$\begin{aligned} \sum_{i=0}^{\infty} |x_{i+1} - x_i| &\leq r + 24 \left(\frac{2}{c_2 \zeta} \right)^{1/d} r \sum_{i=1}^{\infty} K_i^{-1/d} \leq r \left(1 + 24 \left(\frac{2}{c_2 \zeta} \right)^{1/d} K_1^{-1/d} \sum_{i=1}^{\infty} (1 + \beta)^{-(i-1)/d} \right) \\ &\leq r \left(\left(\frac{2}{c_2 \zeta} \right)^{1/d} K_1^{-1/d} + 24 \left(\frac{2}{c_2 \zeta} \right)^{1/d} K_1^{-1/d} \sum_{i=0}^{\infty} (1 + \beta)^{-i/d} \right) = c_4 r K_1^{-1/d} \end{aligned}$$

where $c_4 := 25 \left(\frac{2}{c_2 \zeta} \right)^{1/d} \sum_{i=0}^{\infty} (1 + \beta)^{-i/d}$. So if $K_1 > c_4^d \vee K_0$ then we have a sequence x_1, x_2, \dots contained in $B(x_0, 2r)$ with $u(x_i) \geq (1 + \beta)^{i-1}K_1 \rightarrow \infty$, a contradiction to u being bounded. Therefore we can not take K_1 larger than $c_4^d \vee K_0$, and thus $\sup_{y \in B(x_0, r)} u(y) \leq c_4^d \vee K_0$, which is what we set out to prove.

In the case that u is unbounded, one can follow the simple limit argument in the proof of [39, Theorem 2.4] to finish the proof. \square

By using the standard chain argument one can derive the following form of Harnack inequality.

Corollary 4.8 *For every $a \in (0, 1)$, there exists $C_{14} = C_{14}(a) > 0$ such that for every $r \in (0, 1/4)$, $x_0 \in \mathbb{R}^d$, and any function u which is nonnegative on \mathbb{R}^d and harmonic with respect to X in $B(x_0, r)$, we have*

$$u(x) \leq C_{14} u(y), \quad \text{for all } x, y \in B(x_0, ar).$$

4.2 Some estimates for the Poisson kernel

Recall that for any open set D in \mathbb{R}^d , τ_D is the first exit time of X from D . We will use $G_D(x, y)$ to denote the Green function of X in D . Using the continuity and the radial decreasing property of G , we can easily check that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$. We will frequently use the well-known fact that $G_D(\cdot, y)$ is harmonic in $D \setminus \{y\}$, and regular harmonic in $D \setminus \overline{B(y, \varepsilon)}$ for every $\varepsilon > 0$.

Using the Lévy system for X , we know that for every bounded open subset D , every $f \geq 0$ and all $x \in D$,

$$\mathbb{E}_x [f(X_{\tau_D}); X_{\tau_D-} \neq X_{\tau_D}] = \int_{\overline{D}^c} \int_D G_D(x, z) J(z - y) dz f(y) dy. \quad (4.14)$$

For notational convenience, we define

$$K_D(x, y) := \int_D G_D(x, z) J(z - y) dz, \quad (x, y) \in D \times \overline{D}^c. \quad (4.15)$$

Thus (4.14) can be simply written as

$$\mathbb{E}_x [f(X_{\tau_D}); X_{\tau_D-} \neq X_{\tau_D}] = \int_{\overline{D}^c} K_D(x, y) f(y) dy, \quad (4.16)$$

revealing $K_D(x, y)$ as a density of the exit distribution of X from D . The function $K_D(x, y)$ is called the Poisson kernel of X . Using the continuity of G_D and J , one can easily check that K_D is continuous on $D \times \overline{D}^c$.

The following proposition is an improvement of Lemma 4.3. The idea of the proof comes from [44].

Proposition 4.9 *For all $r > 0$ and all $x_0 \in \mathbb{R}^d$,*

$$\mathbb{E}_x[\tau_{B(x_0, r)}] \leq 2V(2r)V(r - |x - x_0|), \quad x \in B(x_0, r).$$

In particular, for any $R > 0$, $r \in (0, R)$ and $x_0 \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}_x[\tau_{B(x_0, r)}] &\leq C_7 (\phi(r^{-2})\phi((r - |x - x_0|)^{-2}))^{-1/2} \\ &\asymp \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{(r - |x - x_0|)^{\alpha/2}}{(\ell((r - |x - x_0|)^{-2}))^{1/2}}, \quad x \in B(x_0, r), \end{aligned}$$

where $C_7 = C_7(R)$ is the constant from Proposition 3.13.

Proof. Without loss of generality, we may assume that $x_0 = 0$. For $x \neq 0$, put $Z_t = \frac{X_t \cdot x}{|x|}$. Then Z_t is a Lévy process on \mathbb{R} with

$$\mathbb{E}(e^{i\theta Z_t}) = \mathbb{E}(e^{i\theta \frac{x}{|x|} \cdot X_t}) = e^{-t\phi(|\theta \frac{x}{|x|}|^2)} = e^{-t\phi(\theta^2)} \quad \theta \in \mathbb{R}.$$

Thus Z_t is of the type of one-dimensional subordinate Brownian motion studied in Section 3.3. It is easy to see that, if $X_t \in B(0, r)$, then $|Z_t| < r$, hence

$$\mathbb{E}_x[\tau_{B(0, r)}] \leq \mathbb{E}_{|x|}[\tilde{\tau}],$$

where $\tilde{\tau} = \inf\{t > 0 : |Z_t| \geq r\}$. Now the desired conclusion follows easily from Proposition 3.13 (more precisely, from (3.28)). \square

As a consequence of Lemma 4.2, Proposition 4.9 and (4.15), we get the following result.

Proposition 4.10 *There exist $C_{15}, C_{16} > 0$ such that for every $r \in (0, 1)$ and $x_0 \in \mathbb{R}^d$,*

$$\begin{aligned} K_{B(x_0, r)}(x, y) &\leq C_{15} j(|y - x_0| - r) (\phi(r^{-2})\phi((r - |x - x_0|)^{-2}))^{-1/2} \\ &\asymp j(|y - x_0| - r) \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{(r - |x - x_0|)^{\alpha/2}}{(\ell((r - |x - x_0|)^{-2}))^{1/2}}, \end{aligned} \quad (4.17)$$

for all $(x, y) \in B(x_0, r) \times \overline{B(x_0, r)}^c$ and

$$K_{B(x_0, r)}(x_0, y) \geq C_{16} \frac{j(|y - x_0|)}{\phi((r/2)^{-2})} \asymp j(|y - x_0|) \frac{r^\alpha}{\ell(r^{-2})} \quad (4.18)$$

for all $y \in \overline{B(x_0, r)}^c$.

Proof. Without loss of generality, we assume $x_0 = 0$. For $z \in B(0, r)$ and $r < |y| < 2$

$$|y| - r \leq |y| - |z| \leq |z - y| \leq |z| + |y| \leq r + |y| \leq 2|y|,$$

and for $z \in B(0, r)$ and $y \in B(0, 2)^c$,

$$|y| - r \leq |y| - |z| \leq |z - y| \leq |z| + |y| \leq r + |y| \leq |y| + 1.$$

Thus by the monotonicity of j , (3.15) and (3.16), there exists a constant $c > 0$ such that

$$cj(|y|) \leq j(|z - y|) \leq j(|y| - r), \quad (z, y) \in B(0, r) \times \overline{B(0, r)}^c.$$

Applying the above inequality, Lemma 4.2 and Proposition 4.9 to (4.15), we have proved the proposition. \square

Proposition 4.11 *For every $a \in (0, 1)$, $r \in (0, 1/4)$, $x_0 \in \mathbb{R}^d$ and $x_1, x_2 \in B(x_0, ar)$,*

$$K_{B(x_0, r)}(x_1, y) \leq C_{14}K_{B(x_0, r)}(x_2, y), \quad y \in \overline{B(x_0, r)}^c,$$

where $C_{14} = C_{14}(a)$ is the constant from Corollary 4.8.

Proof. Let $a \in (0, 1)$, $r \in (0, 1/4)$ and $x_0 \in \mathbb{R}^d$ be fixed. For every Borel set $A \subset \overline{B(x_0, r)}^c$, the function $x \mapsto \mathbb{P}_x(X_{\tau_{B(x_0, r)}} \in A)$ is harmonic in $B(x_0, r)$. By Corollary 4.8 and (4.16), we have for all $x_1, x_2 \in B(x_0, ar)$,

$$\begin{aligned} \int_A K_{B(x_0, r)}(x_1, y) dy &= \mathbb{P}_{x_1}(X_{\tau_{B(x_0, r)}} \in A) \\ &\leq C_{14}\mathbb{P}_{x_2}(X_{\tau_{B(x_0, r)}} \in A) = \int_A K_{B(x_0, r)}(x_2, y) dy. \end{aligned}$$

This implies that $K_{B(x_0, r)}(x_1, y) \leq C_{14}K_{B(x_0, r)}(x_2, y)$ for a.e. $y \in \overline{B(x_0, r)}^c$, and hence by continuity of $K_{B(x_0, r)}(x, \cdot)$ for every $y \in \overline{B(x_0, r)}^c$. \square

The next inequalities will be used several times in the remainder of this paper.

Lemma 4.12 *There exists $C > 0$ such that*

$$\frac{s^{\alpha/2}}{(\ell(s^{-2}))^{1/2}} \leq C \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}}, \quad 0 < s < r \leq 4, \quad (4.19)$$

$$\frac{s^{1-\alpha/2}}{(\ell(s^{-2}))^{1/2}} \leq C \frac{r^{1-\alpha/2}}{(\ell(r^{-2}))^{1/2}}, \quad 0 < s < r \leq 4, \quad (4.20)$$

$$s^{1-\alpha/2} (\ell(s^{-2}))^{1/2} \leq C r^{1-\alpha/2} (\ell(r^{-2}))^{1/2}, \quad 0 < s < r \leq 4, \quad (4.21)$$

$$\int_r^\infty \frac{(\ell(s^{-2}))^{1/2}}{s^{1+\alpha/2}} ds \leq C \frac{(\ell(r^{-2}))^{1/2}}{r^{\alpha/2}}, \quad 0 < r \leq 4, \quad (4.22)$$

$$\int_0^r \frac{(\ell(s^{-2}))^{1/2}}{s^{\alpha/2}} ds \leq C \frac{(\ell(r^{-2}))^{1/2}}{r^{\alpha/2-1}}, \quad 0 < r \leq 4, \quad (4.23)$$

$$\int_r^\infty \frac{\ell(s^{-2})}{s^{1+\alpha}} ds \leq C \frac{\ell(r^{-2})}{r^\alpha}, \quad 0 < r \leq 4, \quad (4.24)$$

$$\int_0^r \frac{\ell(s^{-2})}{s^{\alpha-1}} ds \leq C \frac{\ell(r^{-2})}{r^{\alpha-2}}, \quad 0 < r \leq 4, \quad (4.25)$$

and

$$\int_0^r \frac{s^{\alpha-1}}{\ell(s^{-2})} ds \leq C \frac{r^\alpha}{\ell(r^{-2})}, \quad 0 < r \leq 4. \quad (4.26)$$

Proof. The first three inequalities follow easily from [3, Theorem 1.5.3], while the last five from the 0-version of [3, 1.5.11]. \square

Proposition 4.13 *For every $a \in (0, 1)$, there exists $C_{17} = C_{17}(a) > 0$ such that for every $r \in (0, 1)$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0, r)}(x, y) \leq C_{17} \frac{r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \frac{(\ell(|y-x_0|-r)^{-2})^{1/2}}{(|y-x_0|-r)^{\alpha/2}}, \quad \forall x \in B(x_0, ar), y \in \{r < |x_0-y| \leq 2r\}.$$

Proof. By Proposition 4.11,

$$K_{B(x_0, r)}(x, y) \leq \frac{c_1}{r^d} \int_{B(x_0, ar)} K_{B(x_0, r)}(w, y) dw$$

for some constant $c_1 = c_1(a) > 0$. Thus from Proposition 4.9, (4.17) and Remark 3.10 we have that

$$\begin{aligned} K_{B(x_0, r)}(x, y) &\leq \frac{c_1}{r^d} \int_{B(x_0, r)} \int_{B(x_0, r)} G_{B(x_0, r)}(w, z) J(z-y) dz dw \\ &= \frac{c_1}{r^d} \int_{B(x_0, r)} \mathbb{E}_z[\tau_{B(x_0, r)}] J(z-y) dz \\ &\leq \frac{c_2}{r^d} \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \int_{B(x_0, r)} \frac{(r-|z-x_0|)^{\alpha/2}}{(\ell((r-|z-x_0|)^{-2}))^{1/2}} J(z-y) dz \end{aligned}$$

for some constant $c_2 = c_2(a) > 0$. Now applying Theorem 3.4, we get

$$K_{B(x_0, r)}(x, y) \leq \frac{c_3 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{B(x_0, r)} \frac{(r-|z-x_0|)^{\alpha/2}}{(\ell((r-|z-x_0|)^{-2}))^{1/2}} \frac{\ell(|z-y|^{-2})}{|z-y|^{d+\alpha}} dz$$

for some constant $c_3 = c_3(a) > 0$. Since $r-|z-x_0| \leq |y-z| \leq 3r \leq 3$, from (4.19) we see that

$$\frac{(r-|z-x_0|)^{\alpha/2}}{(\ell((r-|z-x_0|)^{-2}))^{1/2}} \leq c_4 \frac{(|y-z|)^{\alpha/2}}{(\ell(|y-z|^{-2}))^{1/2}}$$

for some constant $c_4 > 0$. Thus we have

$$\begin{aligned}
K_{B(x_0,r)}(x,y) &\leq \frac{c_5 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{B(x_0,r)} \frac{(\ell(|z-y|^{-2}))^{1/2}}{|z-y|^{d+\alpha/2}} dz \\
&\leq \frac{c_5 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{B(y,|y-x_0|-r)^c} \frac{(\ell(|z-y|^{-2}))^{1/2}}{|z-y|^{d+\alpha/2}} dz \\
&\leq \frac{c_6 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \int_{|y-x_0|-r}^{\infty} \frac{(\ell(s^{-2}))^{1/2}}{s^{1+\alpha/2}} ds
\end{aligned}$$

for some constants $c_5 = c_5(a) > 0$ and $c_6 = c_6(a) > 0$. Using (4.22) in the above equation, we conclude that

$$K_{B(x_0,r)}(x,y) \leq \frac{c_7 r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}} \frac{(\ell((|y-x_0|-r)^{-2}))^{1/2}}{(|y-x_0|-r)^{\alpha/2}}$$

for some constant $c_7 = c_7(a) > 0$. □

Remark 4.14 Note that the right-hand side of the estimate can be replaced by $\frac{V(r)}{r^d V(|y-x_0|-r)}$.

4.3 Boundary Harnack principle

The proof of the boundary Harnack principle is basically the proof given in [25], which is adapted from [4, 42]. The following result is a generalization of [42, Lemma 3.3].

Lemma 4.15 *There exists a positive constant $C_{19} > 0$ such that for any $r \in (0, 1)$ and any open set D with $D \subset B(0, r)$ we have*

$$\mathbb{P}_x(X_{\tau_D} \in B(0, r)^c) \leq C_{19} r^{-\alpha} \ell(r^{-2}) \int_D G_D(x, y) dy, \quad x \in D \cap B(0, r/2).$$

Proof. We will use $C_c^\infty(\mathbb{R}^d)$ to denote the space of infinitely differentiable functions with compact supports. Recall that \mathbf{L} is the L_2 -generator of X in (4.1) and that $G(x, y)$ and $G_D(x, y)$ are the Green functions of X in \mathbb{R}^d and D respectively. We have $\mathbf{L}G(x, y) = -\delta_x(y)$ in the weak sense. Since $G_D(x, y) = G(x, y) - \mathbb{E}_x[G(X_{\tau_D}, y)]$, we have, by the symmetry of \mathbf{L} , for any $x \in D$ and any nonnegative $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_D G_D(x, y) \mathbf{L}\phi(y) dy = \int_{\mathbb{R}^d} G_D(x, y) \mathbf{L}\phi(y) dy \\
&= \int_{\mathbb{R}^d} G(x, y) \mathbf{L}\phi(y) dy - \int_{\mathbb{R}^d} \mathbb{E}_x[G(X_{\tau_D}, y)] \mathbf{L}\phi(y) dy \\
&= \int_{\mathbb{R}^d} G(x, y) \mathbf{L}\phi(y) dy - \int_0^\infty \int_{D^c} \int_{\mathbb{R}^d} G(z, y) \mathbf{L}\phi(y) dy \mathbb{P}_x(X_{\tau_D} \in dz) \\
&= -\phi(x) + \int_0^\infty \int_{D^c} \phi(z) \mathbb{P}_x(X_{\tau_D} \in dz) = -\phi(x) + \mathbb{E}_x[\phi(X_{\tau_D})].
\end{aligned}$$

In particular, if $\phi(x) = 0$ for $x \in D$, we have

$$\mathbb{E}_x [\phi(X_{\tau_D})] = \int_D G_D(x, y) \mathbf{L}\phi(y) dy. \quad (4.27)$$

For fixed $a \in (0, 1)$, take a sequence of radial functions ϕ_m in $C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \phi_m \leq 1$,

$$\phi_m(y) = \begin{cases} 0, & |y| < a \\ 1, & 1 \leq |y| \leq m+1 \\ 0, & |y| > m+2, \end{cases}$$

and that $\sum_{i,j} |\frac{\partial^2}{\partial y_i \partial y_j} \phi_m|$ is uniformly bounded. Define $\phi_{m,r}(y) = \phi_m(\frac{y}{r})$ so that $0 \leq \phi_{m,r} \leq 1$,

$$\phi_{m,r}(y) = \begin{cases} 0, & |y| < ar \\ 1, & r \leq |y| \leq r(m+1) \\ 0, & |y| > r(m+2), \end{cases} \quad \text{and} \quad \sup_{y \in \mathbb{R}^d} \sum_{i,j} \left| \frac{\partial^2}{\partial y_i \partial y_j} \phi_{m,r}(y) \right| < c_1 r^{-2}. \quad (4.28)$$

We claim that there exists a constant $c_1 = c_1(a) > 0$ such that for all $r \in (0, 1)$,

$$\sup_{m \geq 1} \sup_{y \in \mathbb{R}^d} |\mathbf{L}\phi_{m,r}(y)| \leq c_1 r^{-\alpha} \ell(r^{-2}). \quad (4.29)$$

In fact, by Proposition 3.4 we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} (\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y) 1_{B(0,r)}(y)) J(y) dy \right| \\ & \leq \left| \int_{\{|y| \leq r\}} (\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y) 1_{B(0,r)}(y)) J(y) dy \right| + 2 \int_{\{r < |y|\}} J(y) dy \\ & \leq \frac{c_2}{r^2} \int_{\{|y| \leq r\}} |y|^2 J(y) dy + \int_{\{r < |y|\}} J(y) dy \\ & \leq \frac{c_3}{r^2} \int_{\{|y| \leq r\}} \frac{1}{|y|^{d+\alpha-2}} \ell(|y|^{-2}) dy + c_3 \int_{\{r < |y|\}} \frac{1}{|y|^{d+\alpha}} \ell(|y|^{-2}) dy \\ & \leq \frac{c_4}{r^2} \int_0^r \frac{\ell(s^{-2})}{s^{\alpha-1}} ds + c_4 \int_r^\infty \frac{\ell(s^{-2})}{s^{1+\alpha}} ds. \end{aligned}$$

Applying (4.24)-(4.25) to the above equation, we get

$$\left| \int_{\mathbb{R}^d} (\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y) 1_{B(0,r)}(y)) J(y) dy \right| \leq c_5 r^{-\alpha} \ell(r^{-2}),$$

for some constant $c_4 = c_4(d, \alpha, \ell) > 0$. So the claim follows. Let $A(x, a, b) := \{y \in \mathbb{R}^d : a \leq |y-x| < b\}$. When $D \subset B(0, r)$ for some $r \in (0, 1)$, we get, by combining (4.27) and (4.29), that for any $x \in D \cap B(0, ar)$,

$$\mathbb{P}_x(X_{\tau_D} \in B(0, r)^c) = \lim_{m \rightarrow \infty} \mathbb{P}_x(X_{\tau_D} \in A(0, r, (m+1)r)) \leq C r^{-\alpha} \ell(r^{-2}) \int_D G_D(x, y) dy.$$

□

Lemma 4.16 *There exists $C_{20} > 0$ such that for any open set D with $B(A, \kappa r) \subset D \subset B(0, r)$ for some $r \in (0, 1)$ and $\kappa \in (0, 1)$, we have that for every $x \in D \setminus B(A, \kappa r)$,*

$$\int_D G_D(x, y) dy \leq C_{20} r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})} \right) \mathbb{P}_x \left(X_{\tau_{D \setminus B(A, \kappa r)}} \in B(A, \kappa r) \right).$$

Proof. Fix a point $x \in D \setminus B(A, \kappa r)$ and let $B := B(A, \frac{\kappa r}{2})$. Note that, by the harmonicity of $G_D(x, \cdot)$ in $D \setminus \{x\}$ with respect to X , we have

$$G_D(x, A) \geq \int_{D \cap \overline{B}^c} K_B(A, y) G_D(x, y) dy \geq \int_{D \cap B(A, \frac{3\kappa r}{4})^c} K_B(A, y) G_D(x, y) dy.$$

Since $\frac{3\kappa r}{4} \leq |y - A| \leq 2r$ for $y \in B(A, \frac{3\kappa r}{4})^c \cap D$ and j is a decreasing function, it follows from (4.18) in Proposition 4.10 and Theorem 3.4 that

$$\begin{aligned} G_D(x, A) &\geq c_1 \frac{(\frac{\kappa r}{2})^\alpha}{\ell((\frac{\kappa r}{2})^{-2})} \int_{D \cap B(A, \frac{3\kappa r}{4})^c} G_D(x, y) J(y - A) dy \\ &\geq c_1 j(2r) \frac{(\frac{\kappa r}{2})^\alpha}{\ell((\frac{\kappa r}{2})^{-2})} \int_{D \cap B(A, \frac{3\kappa r}{4})^c} G_D(x, y) dy \\ &\geq c_2 \kappa^\alpha r^{-d} \frac{\ell((2r)^{-2})}{\ell((\frac{\kappa r}{2})^{-2})} \int_{D \cap B(A, \frac{3\kappa r}{4})^c} G_D(x, y) dy, \end{aligned}$$

for some positive constants c_1 and c_2 . On the other hand, applying Theorem 4.7 we get

$$\int_{B(A, \frac{3\kappa r}{4})} G_D(x, y) dy \leq c_3 \int_{B(A, \frac{3\kappa r}{4})} G_D(x, A) dy \leq c_4 r^d \kappa^d G_D(x, A),$$

for some positive constants c_3 and c_4 . Combining these two estimates we get that

$$\int_D G_D(x, y) dy \leq c_5 \left(r^d \kappa^d + r^d \kappa^{-\alpha} \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})} \right) G_D(x, A) \quad (4.30)$$

for some constant $c_5 > 0$.

Let $\Omega = D \setminus B(A, \frac{\kappa r}{2})$. Note that for any $z \in B(A, \frac{\kappa r}{4})$ and $y \in \Omega$, $2^{-1}|y - z| \leq |y - A| \leq 2|y - z|$. Thus we get from (4.15) and (3.15) that for $z \in B(A, \frac{\kappa r}{4})$,

$$c_6^{-1} K_\Omega(x, A) \leq K_\Omega(x, z) \leq c_6 K_\Omega(x, A) \quad (4.31)$$

for some $c_6 > 1$. Using the harmonicity of $G_D(\cdot, A)$ in $D \setminus \{A\}$ with respect to X , we can split $G_D(\cdot, A)$ into two parts:

$$\begin{aligned} G_D(x, A) &= \mathbb{E}_x [G_D(X_{\tau_\Omega}, A)] \\ &= \mathbb{E}_x \left[G_D(X_{\tau_\Omega}, A) : X_{\tau_\Omega} \in B(A, \frac{\kappa r}{4}) \right] + \mathbb{E}_x \left[G_D(X_{\tau_\Omega}, A) : X_{\tau_\Omega} \in \left\{ \frac{\kappa r}{4} \leq |y - A| \leq \frac{\kappa r}{2} \right\} \right] \\ &:= I_1 + I_2. \end{aligned}$$

Since $G_D(y, A) \leq G(y, A)$, by using (4.31) and Theorem 3.2, we have

$$I_1 \leq c_6 K_\Omega(x, A) \int_{B(A, \frac{\kappa r}{4})} G_D(y, A) dy \leq c_7 K_\Omega(x, A) \int_{B(A, \frac{\kappa r}{4})} \frac{1}{|y - A|^{d-\alpha} \ell(|y - A|^{-2})} dy,$$

for some constant $c_7 > 0$. Since $|y - A| \leq 4r \leq 4$, by (4.19),

$$\frac{|y - A|^{\alpha/2}}{\ell(|y - A|^{-2})} \leq c_8 \frac{(4r)^{\alpha/2}}{\ell((4r)^{-2})} \quad (4.32)$$

for some constant $c_8 > 0$. Thus

$$I_1 \leq c_7 c_8 K_\Omega(x, A) \int_{B(A, \frac{\kappa r}{4})} \frac{1}{|y - A|^{d-\alpha/2}} \frac{(4r)^{\alpha/2}}{\ell((4r)^{-2})} dy \leq c_9 \kappa^{\alpha/2} r^\alpha \frac{1}{\ell((4r)^{-2})} K_\Omega(x, A)$$

for some constant $c_9 > 0$. Now using (4.31) again, we get

$$\begin{aligned} I_1 &\leq c_{10} \kappa^{\alpha/2-d} r^{\alpha-d} \frac{1}{\ell((4r)^{-2})} \int_{B(A, \frac{\kappa r}{4})} K_\Omega(x, z) dz \\ &= c_{10} \kappa^{\alpha/2-d} r^{\alpha-d} \frac{1}{\ell((4r)^{-2})} \mathbb{P}_x \left(X_{\tau_\Omega} \in B(A, \frac{\kappa r}{4}) \right) \end{aligned}$$

for some constant $c_{10} > 0$. On the other hand, again by Theorem 3.2 and (4.32),

$$\begin{aligned} I_2 &= \int_{\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}} G_D(y, A) \mathbb{P}_x(X_{\tau_\Omega} \in dy) \\ &\leq c_{11} \int_{\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}} \frac{1}{|y - A|^{d-\alpha}} \frac{1}{\ell(|y - A|^{-2})} \mathbb{P}_x(X_{\tau_\Omega} \in dy) \\ &\leq c_{12} \kappa^{\alpha/2-d} r^{\alpha-d} \frac{1}{\ell((4r)^{-2})} \mathbb{P}_x \left(X_{\tau_\Omega} \in \left\{ \frac{\kappa r}{4} \leq |y - A| \leq \frac{\kappa r}{2} \right\} \right), \end{aligned}$$

for some constants $c_{11} > 0$ and $c_{12} > 0$.

Therefore

$$G_D(x, A) \leq c_{13} \kappa^{\alpha/2-d} r^{\alpha-d} \frac{1}{\ell((4r)^{-2})} \mathbb{P}_x \left(X_{\tau_\Omega} \in B(A, \frac{\kappa r}{2}) \right).$$

for some constant $c_{13} > 0$. Combining the above with (4.30), we get

$$\int_D G_D(x, y) dy \leq c_{14} r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})} \right) \mathbb{P}_x \left(X_{\tau_D \setminus B(A, \frac{\kappa r}{2})} \in B(A, \frac{\kappa r}{2}) \right),$$

for some constant $c_{14} > 0$. It follows immediately that

$$\int_D G_D(x, y) dy \leq c_{14} r^\alpha \kappa^{-d-\alpha/2} \frac{1}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})} \right) \mathbb{P}_x \left(X_{\tau_D \setminus B(A, \kappa r)} \in B(A, \kappa r) \right).$$

□

Combining Lemmas 4.15-4.16 and using the translation invariant property, we have the following

Lemma 4.17 *There exists $C_{21} > 0$ such that for any open set D with $B(A, \kappa r) \subset D \subset B(Q, r)$ for some $r \in (0, 1)$ and $\kappa \in (0, 1)$, we have that for every $x \in D \cap B(Q, \frac{r}{2})$,*

$$\mathbb{P}_x (X_{\tau_D} \in B(Q, r)^c) \leq C_{21} \kappa^{-d-\alpha/2} \frac{\ell(r^{-2})}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((2r)^{-2})} \right) \mathbb{P}_x \left(X_{\tau_{D \setminus B(A, \kappa r)}} \in B(A, \kappa r) \right).$$

Let $A(x, a, b) := \{y \in \mathbb{R}^d : a \leq |y - x| < b\}$.

Lemma 4.18 *Let D be an open set and $r \in (0, 1/2)$. For every $Q \in \mathbb{R}^d$ and any positive function u vanishing on $D^c \cap B(Q, \frac{11}{6}r)$, there is a $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$ such that for any $x \in D \cap B(Q, \frac{3}{2}r)$,*

$$\mathbb{E}_x \left[u(X_{\tau_{D \cap B(Q, \sigma)}}); X_{\tau_{D \cap B(Q, \sigma)}} \in B(Q, \sigma)^c \right] \leq C_{22} \frac{r^\alpha}{\ell((2r)^{-2})} \int_{B(Q, \frac{10}{6}r)^c} J(y - Q) u(y) dy \quad (4.33)$$

for some constant $C_{22} > 0$ independent of Q and u .

Proof. Without loss of generality, we may assume that $Q = 0$. Note that by (4.23)

$$\begin{aligned} & \int_{\frac{10}{6}r}^{\frac{11}{6}r} \int_{A(0, \sigma, 2r)} \ell((|y| - \sigma)^{-2})^{1/2} (|y| - \sigma)^{-\alpha/2} u(y) dy d\sigma \\ &= \int_{A(0, \frac{10}{6}r, 2r)} \int_{\frac{10}{6}r}^{|y| \wedge \frac{11}{6}r} \ell((|y| - \sigma)^{-2})^{1/2} (|y| - \sigma)^{-\alpha/2} d\sigma u(y) dy \\ &\leq \int_{A(0, \frac{10}{6}r, 2r)} \left(\int_0^{|y| - \frac{10}{6}r} \ell(s^{-2})^{1/2} s^{-\alpha/2} ds \right) u(y) dy \\ &\leq c_1 \int_{A(0, \frac{10}{6}r, 2r)} \ell \left(\left(|y| - \frac{10r}{6} \right)^{-2} \right)^{1/2} \left(|y| - \frac{10r}{6} \right)^{1-\alpha/2} u(y) dy \end{aligned}$$

for some positive constant c_1 . Using (4.20) and (4.21), we get that there are constants $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{aligned} & \int_{A(0, \frac{10}{6}r, 2r)} \ell \left(\left(|y| - \frac{10r}{6} \right)^{-2} \right)^{1/2} \left(|y| - \frac{10r}{6} \right)^{1-\alpha/2} u(y) dy \\ &\leq c_3 \int_{A(0, \frac{10}{6}r, 2r)} \ell(|y|^{-2})^{1/2} |y|^{1-\alpha/2} u(y) dy \\ &\leq c_3 \frac{r^{1-\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0, \frac{10}{6}r, 2r)} \ell(|y|^{-2}) u(y) dy. \end{aligned}$$

Thus, by taking $c_4 > 6c_1c_3$, we can conclude that there is a $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$ such that

$$\int_{A(0, \sigma, 2r)} \ell((|y| - \sigma)^{-2})^{1/2} (|y| - \sigma)^{-\alpha/2} u(y) dy \leq c_4 \frac{r^{-\alpha/2}}{\ell((2r)^{-2})^{1/2}} \int_{A(0, \frac{10}{6}r, 2r)} \ell(|y|^{-2}) u(y) dy. \quad (4.34)$$

Let $x \in D \cap B(0, \frac{3}{2}r)$. Note that, since X satisfies the hypothesis **H** in [43], by Theorem 1 in [43] we have

$$\begin{aligned} & \mathbb{E}_x \left[u(X_{\tau_{D \cap B(0, \sigma)}}); X_{\tau_{D \cap B(0, \sigma)}} \in B(0, \sigma)^c \right] \\ &= \mathbb{E}_x \left[u(X_{\tau_{D \cap B(0, \sigma)}}); X_{\tau_{D \cap B(0, \sigma)}} \in B(0, \sigma)^c, \tau_{D \cap B(0, \sigma)} = \tau_{B(0, \sigma)} \right] \\ &= \mathbb{E}_x \left[u(X_{\tau_{B(0, \sigma)}}); X_{\tau_{B(0, \sigma)}} \in B(0, \sigma)^c, \tau_{D \cap B(0, \sigma)} = \tau_{B(0, \sigma)} \right] \\ &\leq \mathbb{E}_x \left[u(X_{\tau_{B(0, \sigma)}}); X_{\tau_{B(0, \sigma)}} \in B(0, \sigma)^c \right] = \int_{B(0, \sigma)^c} K_{B(0, \sigma)}(x, y) u(y) dy. \end{aligned}$$

In the first equality above we have used the fact that u vanishes on $D^c \cap B(0, \sigma)$. Since $\sigma < 2r < 1$, from (4.17) in Proposition 4.10, Proposition 4.13 we have

$$\begin{aligned} & \mathbb{E}_x \left[u(X_{\tau_{D \cap B(0, \sigma)}}); X_{\tau_{D \cap B(0, \sigma)}} \in B(0, \sigma)^c \right] \leq \int_{B(0, \sigma)^c} K_{B(0, \sigma)}(x, y) u(y) dy \\ &\leq c_5 \int_{A(0, \sigma, 2r)} \frac{\sigma^{\alpha/2-d}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\ell((|y| - \sigma)^{-2}))^{1/2}}{(|y| - \sigma)^{\alpha/2}} u(y) dy \\ &\quad + c_5 \int_{B(0, 2r)^c} j(|y| - \sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma - |x|)^{\alpha/2}}{(\ell((\sigma - |x|)^{-2}))^{1/2}} u(y) dy \end{aligned}$$

for some constant $c_5 > 0$.

When $y \in A(0, 2r, 4)$ we have $\frac{1}{12}|y| \leq |y| - \sigma$, while when $|y| \geq 4$ we have $|y| - \sigma \geq |y| - 1$. Since $\sigma - |x| \leq \sigma \leq 2r$, we have by (4.19) and the monotonicity of j ,

$$j(|y| - \sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma - |x|)^{\alpha/2}}{(\ell((\sigma - |x|)^{-2}))^{1/2}} \leq c_6 j \left(\frac{|y|}{12} \right) \frac{r^\alpha}{\ell((2r)^{-2})}, \quad y \in A(0, 2r, 4)$$

and

$$j(|y| - \sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma - |x|)^{\alpha/2}}{(\ell((\sigma - |x|)^{-2}))^{1/2}} \leq c_6 j(|y| - 1) \frac{r^\alpha}{\ell((2r)^{-2})}, \quad |y| \geq 4$$

for some constant $c_6 > 0$. Thus by applying (3.15) and (3.16), we get

$$j(|y| - \sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma - |x|)^{\alpha/2}}{(\ell((\sigma - |x|)^{-2}))^{1/2}} \leq c_7 j(|y|) \frac{r^\alpha}{\ell((2r)^{-2})}$$

for some constant $c_7 > 0$. Therefore,

$$\int_{B(0, 2r)^c} j(|y| - \sigma) \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\sigma - |x|)^{\alpha/2}}{(\ell((\sigma - |x|)^{-2}))^{1/2}} u(y) dy \leq c_5 c_7 \frac{r^\alpha}{\ell((2r)^{-2})} \int_{B(0, 2r)^c} J(y) u(y) dy.$$

On the other hand, by (4.19), (4.34) and Theorem 3.4, we have that

$$\begin{aligned}
& \int_{A(0,\sigma,2r)} \frac{\sigma^{\alpha/2-d}}{(\ell(\sigma^{-2}))^{1/2}} \frac{(\ell(|y|-\sigma)^{-2})^{1/2}}{(|y|-\sigma)^{\alpha/2}} u(y) dy \\
& \leq \left(\frac{10r}{6}\right)^{-d} \frac{\sigma^{\alpha/2}}{(\ell(\sigma^{-2}))^{1/2}} \int_{A(0,\sigma,2r)} \frac{(\ell(|y|-\sigma)^{-2})^{1/2}}{(|y|-\sigma)^{\alpha/2}} u(y) dy \\
& \leq c_8 r^{-d} \frac{(2r)^{\alpha/2}}{(\ell((2r)^{-2}))^{1/2}} \frac{r^{-\alpha/2}}{(\ell((2r)^{-2}))^{1/2}} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2}) u(y) dy \\
& \leq c_9 \frac{r^\alpha}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},2r)} \ell(|y|^{-2}) |y|^{-d-\alpha} u(y) dy \\
& \leq c_{10} \frac{r^\alpha}{\ell((2r)^{-2})} \int_{A(0,\frac{10r}{6},2r)} J(y) u(y) dy
\end{aligned}$$

for some positive constants c_8 , c_9 and c_{10} . Hence, by combining the last two displays we arrive at

$$\mathbb{E}_x \left[u(X_{\tau_{D \cap B(0,\sigma)}}); X_{\tau_{D \cap B(0,\sigma)}} \in B(0,\sigma)^c \right] \leq c_{11} \frac{r^\alpha}{\ell((2r)^{-2})} \int_{B(0,\frac{10r}{6})^c} J(y) u(y) dy$$

for some constant $c_{11} > 0$. \square

Lemma 4.19 *Let D be an open set and $r \in (0, 1/2)$. Assume that $B(A, \kappa r) \subset D \cap B(Q, r)$ for $\kappa \in (0, 1/2]$. Suppose that $u \geq 0$ is regular harmonic in $D \cap B(Q, 2r)$ with respect to X and $u = 0$ in $D^c \cap B(Q, 2r)$. If w is a regular harmonic function with respect to X in $D \cap B(Q, r)$ such that*

$$w(x) = \begin{cases} u(x), & x \in B(Q, \frac{3r}{2})^c \cup (D^c \cap B(Q, r)), \\ 0, & x \in A(Q, r, \frac{3r}{2}), \end{cases}$$

then

$$u(A) \geq w(A) \geq C_{23} \kappa^\alpha \frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})} u(x), \quad x \in D \cap B(Q, \frac{3}{2}r)$$

for some constant $C_{23} > 0$.

Proof. Without loss of generality we may assume $Q = 0$. Let $x \in D \cap B(0, \frac{3}{2}r)$. The left hand side inequality in the conclusion of the lemma is clear from the fact that u dominates w on $(D \cap B(0, r))^c$ and both functions are regular harmonic in $D \cap B(0, r)$. Thus we only need to prove the right hand side inequality. By Lemma 4.18 there exists $\sigma \in (\frac{10r}{6}, \frac{11r}{6})$ such that (4.33) holds. Since u is regular harmonic in $D \cap B(0, 2r)$ with respect to X and equal to zero on $D^c \cap B(0, 2r)$, it follows that

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_{D \cap B(0,\sigma)}}); X_{\tau_{D \cap B(0,\sigma)}} \in B(0,\sigma)^c \right] \leq c_1 \frac{r^\alpha}{\ell((2r)^{-2})} \int_{B(0,\frac{10r}{6})^c} J(y) u(y) dy \quad (4.35)$$

for some constant $c_1 > 0$. On the other hand, by (4.18) in Proposition 4.10, we have that

$$\begin{aligned}
w(A) &= \int_{B(0,\frac{3r}{2})^c} K_{D \cap B(0,r)}(A, y) u(y) dy \geq \int_{B(0,\frac{3r}{2})^c} K_{B(A,\kappa r)}(A, y) u(y) dy \\
&\geq c_2 \int_{B(0,\frac{3r}{2})^c} J(A-y) \frac{(\kappa r)^\alpha}{\ell((\kappa r)^{-2})} u(y) dy
\end{aligned}$$

for some constant $c_2 > 0$. Note that $|y - A| \leq 2|y|$ in $A(0, \frac{3r}{2}, 4)$ and that $|y - A| \leq |y| + 1$ for $|y| \geq 4$. Hence by the monotonicity of j , (3.15) and (3.16),

$$w(A) \geq c_3 \frac{(\kappa r)^\alpha}{\ell((\kappa r)^{-2})} \int_{B(0, \frac{3r}{2})^c} J(y)u(y)dy$$

for some constant $c_3 > 0$. Therefore, by (4.35)

$$w(A) \geq c_4 c_1^{-1} \kappa^\alpha \frac{\ell((2r)^{-2})}{\ell((\kappa r)^{-2})} u(x).$$

□

Definition 4.20 Let $\kappa \in (0, 1/2]$. We say that an open set D in \mathbb{R}^d is κ -fat if there exists $R > 0$ such that for each $Q \in \partial D$ and $r \in (0, R)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (R, κ) is called the characteristics of the κ -fat open set D .

Note that all Lipschitz domain and all non-tangentially accessible domain (see [24] for the definition) are κ -fat. The boundary of a κ -fat open set can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred. Bounded κ -fat open set may be disconnected.

Since ℓ is slowly varying at ∞ , we get the Carleson's estimate from Lemma 4.19.

Corollary 4.21 Suppose that D is a κ -fat open set with the characteristics (R, κ) . There exists a constant C_{24} depending on the characteristics (R, κ) such that if $r \leq R \wedge \frac{1}{2}$, $Q \in \partial D$, $u \geq 0$ is regular harmonic in $D \cap B(Q, 2r)$ with respect to X and $u = 0$ in $D^c \cap B(Q, 2r)$, then

$$u(A_r(Q)) \geq C_{24} u(x), \quad \forall x \in D \cap B(Q, \frac{3}{2}r).$$

The next theorem is a boundary Harnack principle for (possibly unbounded) κ -fat open set and it is the main result of this subsection.

Theorem 4.22 Suppose that D is a κ -fat open set with the characteristics (R, κ) . There exists a constant $C_{25} > 1$ depending on the characteristics (R, κ) such that if $r \leq R \wedge \frac{1}{4}$ and $Q \in \partial D$, then for any nonnegative functions u, v in \mathbb{R}^d which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have

$$C_{25}^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \leq \frac{u(x)}{v(x)} \leq C_{25} \frac{u(A_r(Q))}{v(A_r(Q))}, \quad x \in D \cap B(Q, \frac{r}{2}).$$

Proof. Since ℓ is slowly varying at ∞ and locally bounded, there exists a constant $c > 0$ such that for every $r \in (0, 1/4)$,

$$\max \left(\frac{\ell(r^{-2})}{\ell((\kappa r)^{-2})}, \frac{\ell((2r)^{-2})}{\ell((4r)^{-2})}, \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})}, \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} \right) \leq c. \quad (4.36)$$

Fix $r \in (0, R \wedge \frac{1}{4})$ throughout this proof. Without loss of generality we may assume that $Q = 0$ and $u(A_r(0)) = v(A_r(0))$. For simplicity, we will write $A_r(0)$ as A in the remainder of this proof. Define u_1 and u_2 to be regular harmonic functions in $D \cap B(0, r)$ with respect to X such that

$$u_1(x) = \begin{cases} u(x), & x \in A(0, r, \frac{3r}{2}), \\ 0, & x \in B(0, \frac{3r}{2})^c \cup (D^c \cap B(0, r)) \end{cases}$$

and

$$u_2(x) = \begin{cases} 0, & x \in A(0, r, \frac{3r}{2}), \\ u(x), & x \in B(0, \frac{3r}{2})^c \cup (D^c \cap B(0, r)). \end{cases}$$

and note that $u = u_1 + u_2$. If $D \cap A(0, r, \frac{3r}{2}) = \emptyset$, then $u_1 = 0$ and the inequality (4.40) below holds trivially. So we assume that $D \cap A(0, r, \frac{3r}{2})$ is not empty. Then by Lemma 4.19,

$$u(y) \leq c_1 \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} u(A), \quad y \in D \cap B(0, \frac{3r}{2}),$$

for some constant $c_1 > 0$. For $x \in D \cap B(0, \frac{r}{2})$, we have

$$\begin{aligned} u_1(x) &= \mathbb{E}_x \left[u(X_{\tau_{D \cap B(0, r)}}) : X_{\tau_{D \cap B(0, r)}} \in D \cap A(0, r, \frac{3r}{2}) \right] \\ &\leq \left(\sup_{D \cap A(0, r, \frac{3r}{2})} u(y) \right) \mathbb{P}_x \left(X_{\tau_{D \cap B(0, r)}} \in D \cap A(0, r, \frac{3r}{2}) \right) \\ &\leq \left(\sup_{D \cap A(0, r, \frac{3r}{2})} u(y) \right) \mathbb{P}_x \left(X_{\tau_{D \cap B(0, r)}} \in B(0, r)^c \right) \\ &\leq c_1 \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} u(A) \mathbb{P}_x \left(X_{\tau_{D \cap B(0, r)}} \in B(0, r)^c \right). \end{aligned}$$

Now using Lemma 4.17 (with D replaced by $D \cap B(0, r)$) and (4.36), we have that for $x \in D \cap B(0, \frac{r}{2})$,

$$u_1(x) \tag{4.37}$$

$$\begin{aligned} &\leq c_2 \kappa^{-d - \frac{3}{2}\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} \frac{\ell(r^{-2})}{\ell((4r)^{-2})} \left(1 + \frac{\ell((\frac{\kappa r}{2})^{-2})}{\ell((4r)^{-2})} \right) u(A) \mathbb{P}_x \left(X_{\tau_{(D \cap B(0, r)) \setminus B(A, \frac{\kappa r}{2})}} \in B(A, \frac{\kappa r}{2}) \right) \\ &\leq c_3 u(A) \mathbb{P}_x \left(X_{\tau_{(D \cap B(0, r)) \setminus B(A, \frac{\kappa r}{2})}} \in B(A, \frac{\kappa r}{2}) \right) \end{aligned} \tag{4.38}$$

for some positive constants c_2 and $c_3 = c_3(\kappa)$. Since $r < 1/4$, Theorem 4.7 implies that

$$u(y) \geq c_4 u(A), \quad y \in B(A, \frac{\kappa r}{2})$$

for some constant $c_4 > 0$. Therefore for $x \in D \cap B(0, \frac{r}{2})$

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_{(D \cap B(0, r)) \setminus B(A, \frac{\kappa r}{2})}}) \right] \geq c_4 u(A) \mathbb{P}_x \left(X_{\tau_{(D \cap B(0, r)) \setminus B(A, \frac{\kappa r}{2})}} \in B(A, \frac{\kappa r}{2}) \right). \tag{4.39}$$

Using (4.38), the analogue of (4.39) for v , and the assumption that $u(A) = v(A)$, we get that for $x \in D \cap B(0, \frac{r}{2})$,

$$u_1(x) \leq c_3 v(A) \mathbb{P}_x \left(X_{\tau_{(D \cap B(0, r)) \setminus B(A, \frac{\kappa r}{2})}} \in B(A, \frac{\kappa r}{2}) \right) \leq c_5 v(x) \tag{4.40}$$

for some constant $c_5 = c_5(\kappa) > 0$. For $x \in D \cap B(0, r)$, we have

$$\begin{aligned} u_2(x) &= \int_{B(0, \frac{3r}{2})^c} K_{D \cap B(0, r)}(x, z) u(z) dz \\ &= \int_{B(0, \frac{3r}{2})^c} \int_{D \cap B(0, r)} G_{D \cap B(0, r)}(x, y) J(y - z) dy u(z) dz. \end{aligned}$$

Let

$$s(x) := \int_{D \cap B(0, r)} G_{D \cap B(0, r)}(x, y) dy.$$

Note that for every $y \in B(0, r)$ and $z \in B(0, \frac{3r}{2})^c$,

$$\frac{1}{3}|z| \leq |z| - r \leq |z| - |y| \leq |y - z| \leq |y| + |z| \leq r + |z| \leq 2|z|,$$

and that for every $y \in B(0, r)$ and $z \in B(0, 12)^c$,

$$|z| - 1 \leq |y - z| \leq |z| + 1.$$

So by the monotonicity of j , for every $y \in B(0, r)$ and $z \in A(0, \frac{3r}{2}, 12)$,

$$j(12|z|) \leq j(2|z|) \leq J(y - z) \leq j\left(\frac{|z|}{3}\right) \leq j\left(\frac{|z|}{12}\right),$$

and for every $y \in B(0, r)$ and every $z \in B(0, 12)^c$,

$$j(|z| - 1) \leq J(y - z) \leq j(|z| + 1).$$

Using (3.15) and (3.16), we have that, for every $y \in B(0, r)$ and $z \in B(0, \frac{3r}{2})^c$,

$$c_6^{-1} j(|z|) \leq J(y - z) \leq c_6 j(|z|)$$

for some constant $c_6 > 0$. Thus we have

$$c_7^{-1} \leq \left(\frac{u_2(x)}{u_2(A)}\right) \left(\frac{s(x)}{s(A)}\right)^{-1} \leq c_7, \quad (4.41)$$

for some constant $c_7 > 1$. Applying (4.41) to u , and v and Lemma 4.19 to v and v_2 , we obtain for $x \in D \cap B(0, \frac{r}{2})$,

$$u_2(x) \leq c_7 u_2(A) \frac{s(x)}{s(A)} \leq c_7^2 \frac{u_2(A)}{v_2(A)} v_2(x) \leq c_8 \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} \frac{u(A)}{v(A)} v_2(x) = c_8 \kappa^{-\alpha} \frac{\ell((\kappa r)^{-2})}{\ell((2r)^{-2})} v_2(x), \quad (4.42)$$

for some constant $c_8 > 0$. Combining (4.40) and (4.42) and applying (4.36), we have

$$u(x) \leq c_9 v(x), \quad x \in D \cap B(0, \frac{r}{2}),$$

for some constant $c_9 = c_9(\kappa) > 0$. □

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