PARABOLIC HARNACK INEQUALITY FOR THE MIXTURE OF BROWNIAN MOTION AND STABLE PROCESS

RENMING SONG* AND ZORAN VONDRAČEK**

Abstract. Let X be a mixture of independent Brownian motion and symmetric stable process. In this paper we establish sharp bounds for transition density of X, and prove a parabolic Harnack inequality for nonnegative parabolic functions of X.

1. Introduction

Let $W = (W_t : t \geq 0)$ be a Brownian motion in Euclidean d-space $\mathbb{R}^d$, and let $Y = (Y_t : t \geq 0)$ be a rotationally invariant α-stable process in $\mathbb{R}^d$, where $0 < \alpha < 2$. Suppose that W and Y are independent and define the process $X = (X_t : t \geq 0)$ by $X_t = W_t + Y_t$. The law of X started from $x \in \mathbb{R}^d$ will be denoted by $\mathbb{P}^x$. We will call the process X the mixture of the Brownian motion W and the stable process Y. Although X is a Lévy process with explicitly known generator and Lévy measure, until recently not much was known about the Green function and transition density of this process. The main difficulty in studying the process is the fact that it runs on two different scales. By realizing the process X as a subordinate Brownian motion and using Tauberian theorems, the asymptotic behaviors of the Green function of X near zero and infinity were established in [7]. These asymptotics were used in proving an elliptic Harnack inequality for the nonnegative harmonic functions of X. The study of elliptic Harnack inequality for purely discontinuous processes was initiated only recently by Bass and Levin in [1] whose approach was also used in [7].

Parabolic Harnack inequality for nonnegative parabolic functions of purely discontinuous symmetric Markov processes was established by Chen and Kumagai in [4] based on the ideas developed in [2]. The processes they studied have a scaling property that was essentially used in their argument. In a work in progress Chen and Kumagai were able to extend the parabolic Harnack inequality to a more general class of purely discontinuous symmetric Markov processes including sums of independent stable processes with different scales. Their work so far does not include the process X described in the paragraph above.

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The goal of this paper is to establish a parabolic Harnack inequality for the nonnegative parabolic functions of the process X. In order to do this, we first establish sharp upper and lower bounds for the transition density of X. Although our proof of these bounds is elementary and does not extend to general Markov processes which have both a continuous component and a discontinuous component, these bounds can serve as guidelines for the general case.

The content of this paper is organized as follows. The upper and lower bounds on the density of X are established in Section 2. In Section 3 we establish a lower bound for the transition density of the process X killed upon exiting a ball, and in Section 4 we prove the parabolic Harnack inequality.

2. Bounds for transition densities of the mixture

Let $p^{(2)}(t,x)$ be the transition density of W, and $p^{(\alpha)}(t,x)$ the transition density of Y. Then

$$p^{(2)}(t,x) = (4\pi t)^{-d/2} \exp(-\frac{|x|^2}{4t}),$$

while it follows from [3] that there are positive constants C_1, C_2 such that for all $t > 0$ and $x \in \mathbb{R}^d$,

$$C_1 \min(t^{-d/\alpha}, t|x|^{-d-\alpha}) \le p^{(\alpha)}(t,x) \le C_2 \min(t^{-d/\alpha}, t|x|^{-d-\alpha}). \tag{1}$$

The transition density $p(t,x)$ of X is given by

$$p(t,x) := \int_{\mathbb{R}^d} p^{(2)}(t, x-y) p^{(\alpha)}(t,y)\, dy.$$

The purpose of this section is to obtain sharp bounds on $p(t,x)$. In order to do this, we will need to compare $p^{(2)}(t,x)$ and $p^{(\alpha)}(t,x)$.

Lemma 2.1. *Let $\gamma > 0$.*

(i) *There exists a positive constant $c > 0$ such that for all $x \in \mathbb{R}^d$ and all $t > 0$ satisfying $|x| \le 1 \le t$, it holds that*

$$t^{-d/\alpha} \le c t^{-d/2} \exp(-\frac{|x|^2}{\gamma t}).$$

(ii) *For all $x \in \mathbb{R}^d$ and all $t \in (0,1)$, it holds that*

$$t^{-d/\alpha} \ge t^{-d/2} \exp(-\frac{|x|^2}{\gamma t}).$$

(iii) *There exists a positive constant $c > 0$ such that for all $t > 0$ and all $|x| \ge 1$, it holds that*

$$t^{-d/2} \exp(-\frac{|x|^2}{\gamma t}) \le c t |x|^{-d-\alpha}. \tag{2}$$

PROOF. We omit the easy proofs of (i) and (ii), and only give a proof of (iii).

For fixed $x \neq 0$, define $f : (0,\infty) \to (0,\infty)$ by

$$f(t) := t^{-d/2-1} \exp(-|x|^2/(\gamma t)) .$$

Then $f(0+) = f(+\infty) = 0$. Further,

$$f'(t) = f(t)t^{-2}(-(d/2+1)t + |x|^2/\gamma) .$$

This derivative is zero for

$$t_0 = \frac{|x|^2}{(d/2+1)\gamma} ,$$

positive for $t < t_0$, and negative for $t > t_0$. Thus f attains its maximum value at t_0, and

$$\begin{aligned} \max f = f(t_0) &= \left(\frac{|x|^2}{(d/2+1)\gamma}\right)^{-d/2-1} \exp\left(-\frac{|x|^2}{\gamma}\,\frac{(d/2+1)\gamma}{|x|^2}\right) \\ &= ((d/2+1)\gamma)^{d/2+1} \exp(-d/2-1)\, |x|^{-d-2} = c|x|^{-d-2} . \end{aligned}$$

It follows that for all $t > 0$

$$t^{-d/2} \exp(-|x|^2/(\gamma t)) \le t f(t) \le tc|x|^{-d-2} = ct|x|^{-d-\alpha}|x|^{\alpha-2} \le ct|x|^{-d-\alpha}$$

since $|x| \ge 1$. □

REMARK 2.2. Note that the proof of (iii) shows that for $|x| < 1$ there does not exist a positive constant c independent of x such that (2) holds. Clearly, the reverse inequality cannot be true either.

Now we will establish upper bounds for $p(t,x)$.

LEMMA 2.3. *There exists a positive constant c such that for $t \ge |x|^\alpha$,*

$$p(t,x) \le c p^{(\alpha)}(t,x) .$$

PROOF. For all $y \in \mathbb{R}^d$, $p^{(\alpha)}(t,y) \le C_2 t^{-d/\alpha}$. For $t > |x|^\alpha$, we have that $p^{(\alpha)}(t,x) \ge C_1 t^{-d/\alpha}$. Hence,

$$\begin{aligned} p(t,x) &= \int_{\mathbb{R}^d} p^{(2)}(t,x-y) p^{(\alpha)}(t,y)\, dy \\ &\le C_2 t^{-d/\alpha} \int_{\mathbb{R}^d} p^{(2)}(t,x-y)\, dy \le (C_2/C_1) p^{(\alpha)}(t,x) . \end{aligned}$$

□

LEMMA 2.4. *There exists a positive constant c such that for $t \ge |x|^2$,*

$$p(t,x) \le c p^{(2)}(t,x) .$$

PROOF.

$$
\begin{aligned}
p(t,x) &= \int_{\mathbb{R}^d} p^{(2)}(t,x-y)p^{(\alpha)}(t,y)\,dy \\
&= p^{(2)}(t,x)\int_{\mathbb{R}^d} \frac{p^{(2)}(t,x-y)}{p^{(2)}(t,x)}p^{(\alpha)}(t,y)\,dy \\
&= p^{(2)}(t,x)\int_{\mathbb{R}^d} \exp(\frac{1}{4t}(|x|^2-|x-y|^2))p^{(\alpha)}(t,y)\,dy \\
&\le p^{(2)}(t,x)\int_{\mathbb{R}^d} \exp(|x|^2/(4t))p^{(\alpha)}(t,y)\,dy \\
&= p^{(2)}(t,x)\exp(|x|^2/(4t)) \le e^{1/4}\,p^{(2)}(t,x)\,.
\end{aligned}
$$

□

LEMMA 2.5. *Let* $\tilde{p}^{(2)}(t,x) := (4\pi t)^{-d/2}\exp\{-|x|^2/(16t)\}$. *There exists a positive constant* c *such that for all* $|x|\le 1$ *and all* $t<|x|^2$,

$$
p(t,x) \le \max(\tilde{p}^{(2)}(t,x), p^{(\alpha)}(t,x))\,.
$$

PROOF.

$$
\begin{aligned}
p(t,x) &= \int_{\mathbb{R}^d} p^{(2)}(t,x-y)p^{(\alpha)}(t,y)\,dy \\
&= \int_{|y|<\frac{t^{1/\alpha}}{2}} + \int_{\frac{t^{1/\alpha}}{2}\le|y|<\frac{t^{1/2}}{2}} + \int_{|y|\ge\frac{t^{1/2}}{2},\,|y-x|>|x|/2} + \int_{|y|\ge\frac{t^{1/2}}{2},\,|y-x|\le|x|/2} \\
&=: I_1+I_2+I_3+I_4\,.
\end{aligned}
$$

(i) For $|y|<t^{1/\alpha}/2$ and $t^{1/2}<|x|\le 1$, it holds that $2|y|<t^{1/\alpha}<t^{1/2}<|x|$. Hence $|x-y|>|x|/2$, and so $\exp\{-|x-y|^2/(4t)\}\le\exp\{-|x|^2/(16t)\}$. Clearly, $p^{(\alpha)}(t,y)\le C_2t^{-d/\alpha}$. Therefore,

$$
\begin{aligned}
I_1 &\le C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))t^{-d/\alpha}\int_{|y|<\frac{t^{1/\alpha}}{2}} dy \\
&= c_1(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,.
\end{aligned}
$$

(ii) For $|y|<t^{1/2}/2$ we have that $2|y|<t^{1/2}<|x|$, and so again $|x-y|>|x|/2$. Clearly, $p^{(\alpha)}(t,y)\le C_2t|y|^{-d-\alpha}$. Therefore,

$$
\begin{aligned}
I_2 &\le C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\int_{\frac{t^{1/\alpha}}{2}\le|y|<\frac{t^{1/2}}{2}} ty^{-d-\alpha}\,dy \\
&= c_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,t\int_{\frac{t^{1/\alpha}}{2}}^{\frac{t^{1/2}}{2}} r^{d-1}r^{-d-\alpha}\,dr \\
&= c_3(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,t\,((t^{1/\alpha})^{-\alpha}-(t^{1/2})^{-\alpha}) \\
&= c_3(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,(1-t^{1-\alpha/2}) \\
&\le c_3(4\pi t)^{-d/2}\exp\{-|x|^2/(16t)\}\,.
\end{aligned}
$$

(iii) Similarly as in (ii),

$$\begin{aligned}
I_3 &\leq C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\int_{|y|\geq \frac{t^{1/2}}{2},\,|y-x|>|x|/2} ty^{-d-\alpha}\,dy\\
&\leq C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,t\int_{|y|\geq\frac{t^{1/2}}{2}}|y|^{-d-\alpha}\,dy\\
&= c_4(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,t\,(t^{1/2})^{-\alpha}\\
&\leq c_5(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\,,
\end{aligned}$$

since $t^{1-\alpha/2}\leq 1$ for $t\leq 1$.
(iv) For $|y-x|\leq |x|/2$, we have $|y|\geq |x|/2$, and hence $|y|^{-d-\alpha}\leq 2^{d+\alpha}|x|^{-d-\alpha}$. Further, since $t<|x|^2<|x|^\alpha$ for $|x|\leq 1$, it holds that $t|x|^{-d-\alpha}\leq (1/C_1)p^{(\alpha)}(t,x)$. Thus

$$\begin{aligned}
I_4 &\leq C_2\int_{|y|\geq\frac{t^{1/2}}{2},\,|y-x|\leq|x|/2} p^{(2)}(t,x-y)t|y|^{-d-\alpha}\,dy\\
&\leq c_6\int_{|y|\geq\frac{t^{1/2}}{2},\,|y-x|\leq|x|/2} p^{(2)}(t,x-y)t|x|^{-d-\alpha}\,dy\\
&\leq c_6 t|x|^{-d-\alpha}\int_{\mathbb{R}^d}p^{(2)}(t,x-y)\,dy\leq c_7p^{(\alpha)}(t,x)\,.
\end{aligned}$$

From the estimates above it follows that

$$p(t,x)\leq c_8\tilde{p}^{(2)}(t,x)+c_7p^{(\alpha)}(t,x)\leq c_9\max(\tilde{p}^{(2)}(t,x),p^{(\alpha)}(t,x))\,.$$

□

LEMMA 2.6. *There exists a positive constant c such that for all $t<|x|^\alpha$ and $|x|\geq 1$, it holds that*

$$p(t,x)\leq cp^{(\alpha)}(t,x)\,.$$

PROOF.

$$\begin{aligned}
p(t,x) &= \int_{\mathbb{R}^d}p^{(2)}(t,x-y)p^{(\alpha)}(t,y)\,dy\\
&= \int_{|x-y|\leq|x|/2}+\int_{|x-y|>|x|/2,|y|\geq t^{1/\alpha}}+\int_{|x-y|>|x|/2,|y|<t^{1/\alpha}}\\
&=: I_1+I_2+I_3\,.
\end{aligned}$$

(i) For $|x-y|\leq|x|/2$ we have $|y|\geq|x|/2$. Hence $p^{(\alpha)}(t,y)\leq C_2t|y|^{-d-\alpha}\leq C_22^{d+\alpha}t|x|^{-d-\alpha}=c_{10}t|x|^{-d-\alpha}$. Also, $p^{(\alpha)}(t,x)\geq C_1t|x|^{-d-\alpha}$. Therefore,

$$\begin{aligned}
I_1 &\leq c_{10}t|x|^{-d-\alpha}\int_{|x-y|\leq|x|/2}p^{(2)}(t,x-y)\,dy\\
&\leq c_{10}t|x|^{-d-\alpha}\leq c_{11}p^{(\alpha)}(t,x)\,.
\end{aligned}$$

(ii) For $|x-y| \geq |x|/2$ we have $\exp\{-|x-y|^2/(4t)\} \leq \exp(-|x|^2/(16t))$. Also, $p^{(\alpha)}(t,y) \leq C_2 t|y|^{-d-\alpha}$. Therefore,

$$\begin{aligned}
I_3 &\leq C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\int_{|x-y|\geq|x|/2,|y|\geq t^{1/\alpha}} t|y|^{-d-\alpha}\,dy \\
&\leq C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))t\int_{|y|\geq t^{1/\alpha}} |y|^{-d-\alpha}\,dy \\
&\leq c_{12}(4\pi t)^{-d/2}\exp(-|x|^2/(16t))t(t^{1/\alpha})^{-\alpha} \\
&= c_{13}(4\pi t)^{-d/2}\exp(-|x|^2/(16t)) \leq c_{14}p^{(\alpha)}(t,x),
\end{aligned}$$

where in the last inequality we used Lemma 2.1 (iii).
(iii) For $|x-y| \geq |x|/2$ we have $\exp(-|x-y|^2/(4t)) \leq \exp\{-|x|^2/(16t)\}$. Also, $p^{(\alpha)}(t,y) \leq C_2 t^{-d/\alpha}$. Therefore,

$$\begin{aligned}
I_3 &\leq C_2(4\pi t)^{-d/2}\exp(-|x|^2/(16t))\int_{|y|<t^{1/\alpha}} t^{-d/\alpha}dy \\
&\leq c_{15}(4\pi t)^{-d/2}\exp(-|x|^2/(16t)) \leq c_{16}p^{(\alpha)}(t,x)\,,
\end{aligned}$$

where in the last inequality we used Lemma 2.1 (iii).

From the estimates above it follows that $p(t,x) \leq c_{17}p^{(\alpha)}(t,x)$. □

REMARK 2.7. Suppose that $t < |x|^\alpha$ and $|x| \geq R$, where $0 < R < 1$. Then $p(t,x) \leq cR^{\alpha-2}p^{(\alpha)}(t,x)$, where the constant c does not depend on R. This can be proved by changing the estimates for I_2 and I_3, by using a modification of Lemma 2.1 (iii).

Next we establish lower bounds for $p(t,x)$.

LEMMA 2.8. *Let* $\hat{p}^{(2)}(t,x) := (4\pi t)^{-d/2}\exp(-|x|^2/t)$. *There exists a positive constant c such that for all $t \leq |x|^\alpha$,*

$$p(t,x) \geq c\hat{p}^{(2)}(t,x)\,.$$

PROOF. For $|y| \leq |x|$ we have that $|y-x| \leq 2|x|$, and hence $\exp(-|x-y|^2/(4t)) \geq \exp(-|x|^2/t)$. Therefore,

$$\begin{aligned}
p(t,x) &\geq \int_{B(0,|x|)} p^{(2)}(t,x-y)p^{(\alpha)}(t,y)\,dy \\
&\geq (4\pi t)^{-d/2}\exp(-|x|^2/t)\int_{B(0,|x|)} p^{(\alpha)}(t,y)\,dy \\
&= (4\pi t)^{-d/2}\exp(-|x|^2/t)\int_{B(0,|x|)} t^{-d/\alpha}p^{(\alpha)}(1,t^{-1/\alpha}y)\,dy \\
&= (4\pi t)^{-d/2}\exp(-|x|^2/t)\int_{B(0,t^{-1/\alpha}|x|)} p^{(\alpha)}(1,u)\,du \\
&\geq (4\pi t)^{-d/2}\exp(-|x|^2/t)\int_{B(0,1)} p^{(\alpha)}(1,u)\,du \\
&= c_1(4\pi t)^{-d/2}\exp(-|x|^2/t)\,.
\end{aligned}$$

□

LEMMA 2.9. *There exists a positive constant c such that for every $x \in \mathbb{R}^d$ and every $y \in B(x,|x|/2)$, it holds that*

$$\frac{p^{(\alpha)}(t,y)}{p^{(\alpha)}(t,x)} \geq c\,.$$

PROOF. This result can be easily proved by looking at the following four cases. Case 1: $t > |x|^\alpha, t > |y|^\alpha$; Case 2: $t \leq |x|^\alpha, t \leq |y|^\alpha$; Case 3: $t > |x|^\alpha, t \leq |y|^\alpha$ and Case 4: $t \leq |x|^\alpha, t > |y|^\alpha$. We omit the details. □

LEMMA 2.10. *There exists a positive constant c such that for all $t \leq |x|^2$,*

$$p(t,x) \geq cp^{(\alpha)}(t,x)\,.$$

Proof.

$$
\begin{aligned}
p(t,x) &= \int_{\mathbb{R}^d} p^{(\alpha)}(t,y)p^{(2)}(t,x-y)\,dy \\
&= p^{(\alpha)}(t,x)\int_{\mathbb{R}^d} \frac{p^{(\alpha)}(t,y)}{p^{(\alpha)}(t,x)} p^{(2)}(t,x-y)\,dy \\
&\geq c_1 p^{(\alpha)}(t,x)\int_{B(x,|x|/2)} p^{(2)}(t,x-y)\,dy \\
&= c_1 p^{(\alpha)}(t,x)\int_{B(0,|x|/2)} p^{(2)}(t,y)\,dy \\
&= \tilde{c} p^{(\alpha)}(t,x)\int_{B(0,t^{-1/2}|x|/2)} p^{(2)}(1,u)\,du \\
&\geq c_1 p^{(\alpha)}(t,x)\int_{B(0,1/2)} p^{(2)}(1,u)\,du = c_2 p^{(\alpha)}(t,x)\,,
\end{aligned}
$$

where the third line follows from Lemma 2.9. □

Lemma 2.11. *There exists a positive constant c such that for all $t \geq 1$ and all $|x|^\alpha < t$ we have*

$$p(t,x) \geq c p^{(\alpha)}(t,x)\,.$$

Proof. For $|x|^\alpha < t$, $p^{(\alpha)}(t,x) \leq C_2 t^{-d/\alpha}$. If $|y-x| \leq t^{1/\alpha}$, then $|y| \leq |x-y| + |x| \leq 2t^{1/\alpha}$. If $|y| \leq t^{1/\alpha}$, then $p^{(\alpha)}(t,y) \geq C_1 t^{-d/\alpha}$. If $t^{1/\alpha} < |y| \leq 2t^{1/\alpha}$, then $p^{(\alpha)}(t,y) \geq C_1 t|y|^{-d-\alpha} \geq C_1 2^{-d-\alpha} t^{-d/\alpha}$. Therefore,

$$
\begin{aligned}
p(t,x) &\geq \int_{|x-y|\leq t^{1/\alpha}} p^{(2)}(t,x-y)p^{(\alpha)}(t,y)\,dy \\
&\geq c_3 t^{-d/\alpha}\int_{|x-y|\leq t^{1/\alpha}} p^{(2)}(t,x-y)\,dy \\
&= c_3 t^{-d/\alpha}\int_{|y|\leq t^{1/\alpha}} p^{(2)}(t,y)\,dy = c_3 t^{-d/\alpha}\int_{|y|\leq t^{-1/2}t^{1/\alpha}} p^{(2)}(1,u)\,du \\
&\geq c_3 t^{-d/\alpha}\int_{|y|\leq 1} p^{(2)}(1,u)\,du = c_4 t^{-d/\alpha} \geq c_5 p^{(\alpha)}(t,x)\,.
\end{aligned}
$$

□

Lemma 2.12. *There exists a positive constant c such that for all $|x|^\alpha \leq t \leq 1$ it holds that*

$$p(t,x) \geq c p^{(2)}(t,x)\,.$$

Proof. If $|y| < t^{1/\alpha}$, then $p^{(\alpha)}(t,y) \ge C_1 t^{-d/\alpha}$. Also, $|x-y| \le |x|+|y| \le 2t^{1/\alpha}$ implying $\exp(-|x-y|^2/(4t)) \ge \exp(-4t^{2/\alpha}/(4t)) = \exp(-t^{2/\alpha-1}) \ge e^{-1}$, since $t^{2/\alpha-1} \le 1$ for $t \le 1$. Therefore,

$$\begin{aligned}
p(t,x) &\ge \int_{|y|<t^{1/\alpha}} p^{(2)}(t,x-y)p^{(\alpha)}(t,y)\,dy \\
&\ge C_1 t^{-d/\alpha} \int_{|y|<t^{1/\alpha}} p^{(2)}(t,x-y)\,dy \\
&\ge C_1 t^{-d/\alpha}(4\pi t)^{-d/2} e^{-1} \int_{|y|<t^{1/\alpha}} dy \\
&= c_6 (4\pi t)^{-d/2} \ge c_6 (4\pi t)^{-d/2} \exp(-|x|^2/(4t)) \\
&= c_6 p^{(2)}(t,x)\,.
\end{aligned}$$

□

By collecting the results from previous lemmas, we obtain the lower and upper bounds for the transition density $p(t,x)$. In order to briefly state the result, we define

$$q_1(t,x) = \begin{cases} \hat{p}^{(2)}(t,x), & |x|^2 < t < |x|^\alpha \le 1\,, \\ \max(\hat{p}^{(2)}(t,x), p^{(\alpha)}(t,x)), & t < |x|^2 \le 1\,, \\ p^{(2)}(t,x), & |x|^\alpha \le t \le 1\,, \\ p^{(\alpha)}(t,x), & t \ge 1 \text{ or } |x| \ge 1\,, \end{cases}$$

and

$$q_2(t,x) = \begin{cases} p^{(2)}(t,x), & |x|^2 < t < |x|^\alpha \le 1\,, \\ \max(\tilde{p}^{(2)}(t,x), p^{(\alpha)}(t,x)), & t < |x|^2 \le 1\,, \\ p^{(2)}(t,x), & |x|^\alpha \le t \le 1\,, \\ p^{(\alpha)}(t,x), & t \ge 1 \text{ or } |x| \ge 1\,. \end{cases}$$

Theorem 2.13. *There exists a positive constant C_3 such that*

$$C_3^{-1} q_1(t,x) \le p(t,x) \le C_3 q_2(t,x).$$

3. Lower bounds for transition densities of the killed process

In this section we will establish a lower bound for the transition density of the process X killed upon exiting a ball of radius R. Let $p(t,x,y) := p(t,y-x)$. We first need the following lemma.

Lemma 3.1. *There exists a constant $C_4 > 0$ such that for every $R > 0$, every $x \in \mathbb{R}^d$ and every $t > 0$,*

$$\mathbb{P}^x(\tau_{B(x,R)} \le t) \le \frac{C_4 t}{R^2 \wedge R^\alpha}\,, \tag{3}$$

where $\tau_{B(x,R)} = \inf\{s > 0;\ X_t \notin B(x,R)\}$.

This result for $R \in (0,1]$ appears as Lemma 2.1 in [8]. By a slight modification of the proof, the result follows for $R \geq 1$ as well.

Let $R > 0$, $B = B(0,R)$, and let τ_B denote the first exit time of X from B. Let X^B denote the process X killed upon exiting B. The transition density of X^B is given by

$$p^B(t,x,y) = p(t,x,y) - r^B(t,x,y)\,, \quad x,y \in B\,,$$

where

$$r^B(t,x,y) = \mathbb{E}^x[p(t-\tau_B, X_{\tau_B}, y)\, 1_{(t>\tau_B)}]\,.$$

Lemma 3.2. *There exist constants $C_5 > 0$ and $C_6 \in (0, 1/10)$ such that:*

(i) *For every $R > 0$, for all $x, y \in B(0,2R/3)$ and all $0 < t < 1$ satisfying $|x-y|^2 < t \leq C_6(R^2 \wedge R^\alpha)$ it holds that*

$$p^B(t,x,y) \geq C_5 t^{-d/2}\,.$$

(ii) *For every $R \geq 1$, for all $x, y \in B(0,2R/3)$ and all $t \geq 1$ satisfying $|x-y|^\alpha < t \leq C_6 R^\alpha$ it holds that*

$$p^B(t,x,y) \geq C_5 t^{-d/\alpha}\,.$$

(iii) *For every $R \geq 1$, for all $x, y \in B(0,2R/3)$ and all $t \geq 1$ satisfying $|x-y|^\alpha \geq t$ and $t \leq C_6 R^\alpha$ it holds that*

$$p^B(t,x,y) \geq C_5 t R^{-d-\alpha}\,.$$

Proof. We first find an upper bound for $r^B(t,x,y)$. Suppose that $0 < R \leq 1$. Note that by combining Lemma 2.6 with Remark 2.7, if $|y-z| > R/3$ and $t < |y-z|^2$, then $p(t,z,y) \leq c(R/3)^{\alpha-2} p^{(\alpha)}(t,z,y) \leq c C_2 (R/3)^{\alpha-2}\, t\, |z-y|^{-d-\alpha}$. Let $x, y \in B(0,2R/3)$ and choose $t \leq R^2/10$. Then $|X_{\tau_B} - y| > R/3$ and $t \leq R^2/10 < |X_{\tau_B} - y|^2$, so on $\{t > \tau_B\}$

$$\begin{aligned} p(t-\tau_B, X_{\tau_B}, y) &\leq c C_2 (R/3)^{\alpha-2} (t-\tau_B) |X_{\tau_B} - y|^{-d-\alpha} \\ &\leq c C_2 (R/3)^{\alpha-2} t (R/3)^{-d-\alpha} \leq c_1 R^{-d-2} t\,. \end{aligned}$$

Note further that for $x \in B(0,2R/3)$ it holds that $\mathbb{P}^x(\tau_B < t) \leq \mathbb{P}^x(\tau_{B(x,R/3)} < t) \leq C_4 t/(R/3)^2$ by Lemma 3.1. Therefore, for all $x, y \in B(0,2R/3)$ and all $t \leq R^2/10$,

$$\begin{aligned} r^B(t,x,y) &= \mathbb{E}^x[p(t-\tau_B, X_{\tau_B}, y)\, 1_{(t>\tau_B)}] \\ &\leq c_1 R^{-d-2} t \mathbb{P}^x(t > \tau_B) \\ &\leq 9 c_1 R^{-d-2} t\, C_4 t R^{-2} = c_2 t^2 R^{-d-4}\,. \end{aligned}$$

Suppose now that $R \geq 1$. If $1 > |y-z| \geq R/3 \geq 1/3$, and $t < |y-z|^\alpha$, it holds by Remark 2.7 that $p(t,y,z) \leq c(1/3)^{\alpha-2} p^{(\alpha)}(t,y,z)$. If $1 > |y-z| \geq R/3 \geq 1/3$, and $t \geq |y-z|^\alpha$, then by Lemma 2.3, $p(t,y,z) \leq c p^{(\alpha)}(t,y,z)$. If $|y-z| \geq 1$, then the estimate $p(t,y,z) \leq c p^{(\alpha)}(t,y,z)$ follows from Theorem 2.13. Therefore, whenever $|y-z| \geq R/3$

and for all $t > 0$, $p(t,y,z) \le cp^{(\alpha)}(t,y,z) \le cC_2 t|y-z|^{-\alpha-d} \le cC_2 t(R/3)^{-\alpha-d}$. Let $x,y \in B(0,2R/3)$. Then on $\{t > \tau_B\}$

$$p(t-\tau_B, X_{\tau_B}, y) \le cC_2 t(R/3)^{-d-\alpha} \le c_3 R^{-d-\alpha} t\,.$$

Again by Lemma 3.1, $\mathbb{P}^x(\tau_B < t) \le \mathbb{P}^x(\tau_{B(x,R/3)} < t) \le C_4 t/(R/3)^\alpha$. It follows that

$$\begin{aligned} r^B(t,x,y) &= \mathbb{E}^x[p(t-\tau_B, X_{\tau_B}, y)\, 1_{(t>\tau_B)}] \\ &\le c_3 R^{-d-\alpha} t \mathbb{P}^x(t > \tau_B) \\ &\le 9c_3 R^{-d-\alpha} t\, C_4 t R^{-\alpha} = c_4 t^2 R^{-d-2\alpha}\,. \end{aligned}$$

(i) Suppose first that $0 < t < 1$ and $R \in (0,1]$. For all $x,y \in \mathbb{R}^d$ such that $|x-y| \le 1$, we have by Theorem 2.13 that $p(t,x,y) \ge c_5 \hat{p}^{(2)}(t, y-x)$. Therefore, for $x,y \in B(0,2R/3)$ and $|x-y|^2 \le t \le R^2/10$ it follows that

$$p(t,x,y) \ge c_5 (4\pi t)^{-d/2} e^{-|x-y|^2/t} \ge c_6 e^{-1}\, t^{-d/2} = c_7 t^{-d/2}\,.$$

It follows that for $x,y \in B(0,2R/3)$ and $|x-y|^2 \le t \le R^2/10$

$$\begin{aligned} p^B(t,x,y) &= p(t,x,y) - r^B(t,x,y) \\ &\ge c_7 t^{-d/2} - c_2 t^2 R^{-d-4} \\ &= c_7 t^{-d/2}\Big(1 - \frac{c_2}{c_7} t^{(d+4)/2} R^{-d-4}\Big) \ge \frac{c_7}{2} t^{-d/2} \end{aligned}$$

provided that $1 - (c_2/c_7) t^{(d+4)/2} R^{-d-4} > 1/2$. This last condition is satisfied if

$$t < \left(\frac{c_7}{2c_2}\right)^{2/(d+4)} R^2 = c_8 R^2\,.$$

Suppose now that $0 < t < 1$ and $R \ge 1$. The same argument as above shows that for $x,y \in B(0,2R/3)$ and $|x-y|^2 \le t \le R^\alpha/10$ it holds that $p(t,x,y) \ge c_7 t^{-d/2}$, and consequently

$$\begin{aligned} p^B(t,x,y) &= p(t,x,y) - r^B(t,x,y) \\ &\ge c_7 t^{-d/2} - c_2 t^2 R^{-d-2\alpha} \\ &= c_7 t^{-d/2}\Big(1 - \frac{c_2}{c_7} t^{(d+4)/2} R^{-d-2\alpha}\Big) \ge \frac{c_7}{2} t^{-d/2} \end{aligned}$$

provided that $1 - (c_2/c_7) t^{(d+4)/2} R^{-d-2\alpha} > 1/2$. This last condition is satisfied if

$$t < \left(\frac{c_7}{2c_2}\right)^{2/(d+4)} R^{2(d+2\alpha)/(d+4)} = c_9 R^{2(d+2\alpha)/(d+4)}\,.$$

Note that $\alpha < 2(d+2\alpha)/(d+4)$. Therefore, if $t < c_9 R^\alpha$, then $t < c_9 R^{2(d+2\alpha)/(d+4)}$, and consequently $p^B(t,x,y) \ge (c_7/2) t^{-d/2}$.

Choose $c_{11} = \min(1/10, c_8, c_9)$. Then we have proved that for every $R > 0$, for all $x,y \in B(0,2R/3)$ and all $0 < t < 1$ satisfying $|x-y|^2 < t \le c_{11}(R^2 \wedge R^\alpha)$ it holds that

$$p^B(t,x,y) \ge \frac{c_7}{2} t^{-d/2}\,.$$

(ii) Let $R \ge 1$ and $x, y \in B(0, 2R/3)$. Suppose that $|x-y|^\alpha < t$. By Theorem 2.13, $p(t,x,y) \ge C_3^{-1} p^{(\alpha)}(t,x,y) \ge C_3^{-1} C_2 \min(t^{-d/\alpha}, t|x-y|^{-d-\alpha}) \ge c_{12} t^{-d/\alpha}$. By combining with the upper bound for $r^B(t,x,y)$, it follows that

$$\begin{aligned} p^B(t,x,y) &= p(t,x,y) - r^B(t,x,y) \\ &\ge c_{12} t^{-d/\alpha} - c_4 t^2 R^{-d-2\alpha} \\ &= c_{12} t^{-d/\alpha} (1 - \frac{c_4}{c_{12}} t^{d/\alpha+2} R^{-d-2\alpha}) \ge \frac{c_{12}}{2} t^{-d/\alpha} \end{aligned}$$

provided that $1 - (c_4/c_{12}) t^{d/\alpha+2} R^{-d-2\alpha} > 1/2$. This last condition is satisfied if

$$t < \left(\frac{c_{12}}{2c_4} \right)^{\alpha/(d+2\alpha)} R^\alpha = c_{13} R^\alpha .$$

(iii) Let $R \ge 1$ and $x, y \in B(0, 2R/3)$. Suppose that $|x-y|^\alpha \ge t$. Again by Theorem 2.13, $p(t,x,y) \ge C_3^{-1} p^{(\alpha)}(t,x,y) \ge C_3^{-1} C_2 \min(t^{-d/\alpha}, t|x-y|^{-d-\alpha}) = C_3^{-1} C_2 t |x-y|^{-d-\alpha} \ge c_{14} t R^{-d-\alpha}$. By combining with the upper bound for $r^B(t,x,y)$, it follows that

$$\begin{aligned} p^B(t,x,y) &= p(t,x,y) - r^B(t,x,y) \\ &\ge c_{14} t R^{-d-\alpha} - c_4 t^2 R^{-2\alpha-d} \\ &\ge c_{14} t R^{-d-\alpha} (1 - \frac{c_4}{c_{14}} t R^{-\alpha}) \\ &\ge \frac{c_{14}}{2} t R^{-d-\alpha} \end{aligned}$$

provided that $1 - (c_4/c_{14}) t R^{-\alpha} \ge 1/2$. This is satisfied if

$$t < \frac{c_{14}}{2c_4} R^\alpha = c_{15} R^\alpha .$$

We finish the proof of the lemma by choosing $C_5 = \min(c_5/2, c_{12}/2, c_{14}/2)$ and $C_6 = \min(c_{11}, c_{13}, c_{15})$. □

Let $N = \lfloor 2/C_6 \rfloor$ where C_6 is the constant from Lemma 3.2, and $\lfloor \cdot \rfloor$ denotes the smallest integer function. The proof of the next result follows the proof of Theorem 2.7 in [5].

PROPOSITION 3.3. *Let $\delta \in (0,1)$. There exists a constant $C_7 = C_7(\delta) > 0$ such that for all $0 < R \le (2N/\delta)^{1/\alpha}$, all $x, y \in B(0, R/2)$ and all $0 < t \le R^2 \wedge R^\alpha$ it holds that*

$$p^B(t,x,y) \ge C_7 t^{-d/2} e^{-|x-y|^2/(C_7 t)} .$$

PROOF. Let $R \le (2N/\delta)^{1/\alpha}$, $x, y \in B(0, R/2)$ and $t \le R^2 \wedge R^\alpha$. Suppose first that $t < |x-y|^2$ and define $k = \lfloor 4N|x-y|^2/(\delta t) \rfloor$. Then $k \ge 2N/\delta \ge 2/(C_6\delta)$, and therefore $t/k \le (C_6 \delta t)/2 \le (1/2) C_6 \delta (R^2 \wedge R^\alpha) \le C_6 (R^2 \wedge R^\alpha)$. Moreover,

$$\frac{C_6\delta}{2} (R^2 \wedge R^\alpha) \le \frac{C_6\delta}{2} \left[\left(\frac{N}{\delta} \right)^{1/\alpha} \right]^\alpha \le \frac{C_6\delta}{2} \frac{N}{\delta} \le 1 ,$$

implying that $t/k \le 1$. For $l = 1, 2, \dots, k-1$ let

$$z_l = x + \frac{l}{k}(y-x)\,.$$

From $k \ge 2N|x-y|^2/(\delta t)$ it follows that $|x-y|^2 \le \delta k t/(2N)$. Therefore

$$|z_l - z_{l-1}| = \frac{|x-y|}{k} \le \sqrt{\frac{\delta}{2}}\frac{\sqrt{kt/N}}{k} = \sqrt{\frac{\delta}{2N}}\sqrt{\frac{t}{k}}\,.$$

Define

$$S = \prod_{l=1}^{k-1} B\left(z_l, \sqrt{\frac{\delta}{2N}}\sqrt{\frac{t}{k}}\right)\,.$$

Note that

$$\sqrt{\frac{\delta}{2N}}\sqrt{\frac{t}{k}} \le \sqrt{\frac{\delta}{2N}}\sqrt{C_6 R^2} \le C_6 R \le \frac{1}{10}R\,,$$

implying that for every $l = 1, 2, \dots, k-1$,

$$B\left(z_l, \sqrt{\frac{\delta}{2N}}\sqrt{\frac{t}{k}}\right) \subset B(0, 2R/3)\,.$$

For $\zeta_l \in B(z_l, \sqrt{\delta/(2N)}\sqrt{t/k})$ and $\zeta_{l-1} \in B(z_{l-1}, \sqrt{\delta/(2N)}\sqrt{t/k})$, $l = 2, 3, \dots, k-1$, we have that $|\zeta_l - \zeta_{l-1}| \le |\zeta_l - z_l| + |z_l - z_{l-1}| + |z_{l-1} - \zeta_{l-1}| \le 3\sqrt{\delta/(2N)}\sqrt{t/k} \le \sqrt{t/k}$. Therefore,

$$|\zeta_l - \zeta_{l-1}|^2 \le \frac{t}{k} \le C_6(R^2 \wedge R^\alpha)\,,$$

implying by Lemma 3.2(i) that $p^B(t/k, \zeta_{l-1}, \zeta_l) \ge C_5 (t/k)^{-d/2}$. Hence

$$\begin{aligned}
p^B(t,x,y) &= \int_B \int_B \cdots \int_B p^B\left(\frac{t}{k}, x, \zeta_1\right) p^B\left(\frac{t}{k}, \zeta_1, \zeta_2\right) \cdots p^B\left(\frac{t}{k}, \zeta_{k-1}, y\right) d\zeta_1\, d\zeta_2 \dots d\zeta_{k-1} \\
&\ge \int \cdots \int_S p^B\left(\frac{t}{k}, x, \zeta_1\right) p^B\left(\frac{t}{k}, \zeta_1, \zeta_2\right) \cdots p^B\left(\frac{t}{k}, \zeta_{k-1}, y\right) d\zeta_1\, d\zeta_2 \dots d\zeta_{k-1} \\
&\ge |S| \left(C_5 \left(\frac{t}{k}\right)^{-d/2}\right)^k = \left(|B(0,1)| \left(\frac{\delta t}{2Nk}\right)^{d/2}\right)^{k-1} \left(C_5 \left(\frac{t}{k}\right)^{-d/2}\right)^k \\
&= \left(\frac{C_5 \delta^{d/2} |B(0,1)|}{2^{d/2} N^{d/2}}\right)^k \frac{2^{d/2} N^{d/2}}{\delta^{d/2} |B(0,1)|} k^{d/2} t^{-d/2} \\
&\ge \exp\left(-k \log\left(\frac{2^{d/2} N^{d/2}}{C_5 \delta^{d/2} |B(0,1)|}\right)\right) \frac{2^{d/2} N^{d/2}}{\delta^{d/2} |B(0,1)|} N^{d/2} t^{-d/2} \\
&\ge c_{16} \exp\left(-\log\left(\frac{2^{d/2} N^{d/2}}{C_5 \delta^{d/2} |B(0,1)|}\right) \frac{4N|x-y|^2}{\delta t}\right) t^{-d/2} \\
&\ge c_{17} t^{-d/2} \exp\left(-\frac{|x-y|^2}{c_{17} t}\right)\,.
\end{aligned}$$

Assume now that $t \geq |x-y|^2$ and define $k = \lfloor 4N/\delta \rfloor$. Then again $k \geq 2N/\delta \geq 2/(C_6\delta)$ implying $t/k \leq C_6 t \leq C_6(R^2 \wedge R^\alpha)$ and $t/k \leq 1$. The same argument as above gives the following estimate

$$\begin{aligned}
p^B(t,x,y) &\geq \left(\frac{C_5\delta^{d/2}|B(0,1)|}{2^{d/2}N^{d/2}}\right)^k \frac{2^{d/2}N^{d/2}}{\delta^{d/2}|B(0,1)|} k^{d/2} t^{-d/2} \\
&\geq \left(\frac{C_5\delta^{d/2}|B(0,1)|}{2^{d/2}N^{d/2}}\right)^{2N} \frac{2^{d/2}N^{d/2}}{\delta^{d/2}|B(0,1)|} (2N)^{d/2} t^{-d/2} \\
&= c_{18} t^{-d/2} \geq c_{18} t^{-d/2} \exp\left(-\frac{|x-y|^2}{c_{18}t}\right).
\end{aligned}$$

The claim follows by taking $C_7 = \min(c_{17}, c_{18})$. □

Proposition 3.4. *Let $\delta \in (0,1)$. There exists a constant $C_8 = C_8(\delta) > 0$ such that for all $R \geq (2N/\delta)^{1/\alpha}$, all $x, y \in B(0, R/2)$ and all t satisfying $\delta R^\alpha < t < R^\alpha$ it holds that*

$$p^B(t,x,y) \geq C_8 t^{-d/\alpha}.$$

Proof. Let $R \geq (2N/\delta)^{1/\alpha}$ and $\delta R^\alpha < t < R^\alpha$. First note that $\delta R^\alpha \geq 2N$, implying $t \geq 2N$. Define $k = N$. Then clearly $t/k = t/N \geq 1$. Also, since $N > 1/C_6$, we have that $t/k \leq C_6 t \leq C_6 R^\alpha$.

Suppose that $u, v \in B(0, R/2)$. If $|u-v|^\alpha \leq t/k$, then by Lemma 3.2(ii) it follows that

$$p^B(t/k, u, v) \geq C_5 \left(\frac{t}{k}\right)^{-d/\alpha} \geq C_5 t^{-d/\alpha}.$$

If $|u-v|^\alpha \geq t/k$, then by Lemma 3.2(iii) it follows that $p^B(t/k,u,v) \geq C_5(t/k)R^{-d-\alpha}$. But since $\delta R^\alpha < t$, we have that $R < (t/\delta)^{1/\alpha}$, implying

$$p^B(t/k,u,v) \geq C_5 \frac{t}{k}\left(\frac{t}{\delta}\right)^{-(d+\alpha)/\alpha} = C_5 \frac{\delta^{1+d/\alpha}}{N} t^{-d/\alpha} = c_{19} t^{-d/\alpha}.$$

Hence, for any $u, v \in B(0,R/2)$ it holds that $p^B(t/k,u,v) \geq c_{19} t^{-d/\alpha}$. Define

$$S = \prod_{l=1}^{k-1} B\left(0, \frac{t^{1/\alpha}}{k}\right).$$

Note that $B(0, t^{1/\alpha}/k) \subset B(0, R/2)$ since $t < R^\alpha$. Hence

$$\begin{aligned}
p^B(t,x,y) &= \int_B \int_B \cdots \int_B p^B\left(\frac{t}{k}, x, \zeta_1\right) p^B\left(\frac{t}{k}, \zeta_1, \zeta_2\right) \cdots p^B\left(\frac{t}{k}, \zeta_{k-1}, y\right) d\zeta_1\, d\zeta_2 \ldots d\zeta_{k-1} \\
&\geq \int \cdots \int_S p^B\left(\frac{t}{k}, x, \zeta_1\right) p^B\left(\frac{t}{k}, \zeta_1, \zeta_2\right) \cdots p^B\left(\frac{t}{k}, \zeta_{k-1}, y\right) d\zeta_1\, d\zeta_2 \ldots d\zeta_{k-1} \\
&\geq |S| \left(c_{19}\left(\frac{t}{k}\right)^{-d/\alpha}\right)^k = \left(|B(0,1)| \left(\frac{t^{1/\alpha}}{k}\right)^d\right)^{k-1} \left(c_{19}\left(\frac{t}{k}\right)^{-d/\alpha}\right)^k \\
&= |B(0,1)|^{k-1} c_{19}^k k^{dk(1/\alpha-1)+d} t^{-d/\alpha} = c_{20} t^{-d/\alpha},
\end{aligned}$$

where c_{20} depends on δ. Choose $C_8 = c_{20}$. $\square$

COROLLARY 3.5. *Let $\delta \in (0,1)$. There exists a constant $C_9 = C_9(\delta) > 0$ such that for all $R > 0$, all $x, y \in B(0, R/2)$ and all $t \in (\delta(R^2 \wedge R^\alpha), R^2 \wedge R^\alpha)$ it holds that*

$$p^B(t,x,y) \geq \frac{C_9}{R^d}\,.$$

PROOF. Suppose first that $R \leq 1$. By Proposition 3.3,

$$p^B(t,x,y) \geq C_7 t^{-d/2} e^{-|x-y|^2/(C_7 t)}\,.$$

We use that $|x-y|^2/t \leq 1/\delta$ and $t^{-d/2} \geq R^{-d}$ to obtain the estimate with $c_{21} = C_7 \exp(-1/(C_7\delta))$. If $1 \leq R \leq (2N/\delta)^{1/\alpha}$, then again by Proposition 3.3,

$$p^B(t,x,y) \geq C_7 t^{-d/2} e^{-|x-y|^2/(C_7 t)}\,.$$

Now we use the estimate

$$\frac{|x-y|^2}{t} \leq \frac{R^2}{\delta R^\alpha} \leq \frac{1}{\delta}\left(\frac{2N}{\delta}\right)^{(2-\alpha)/\alpha} = c_{22}\,,$$

and $t \leq R^\alpha \leq R^2$, to obtain that

$$p^B(t,x,y) \geq C_7 e^{-c_{22}/C_7} R^{-d} = c_{23} R^{-d}\,.$$

Finally, let $R \geq (2N/\delta)^{1/\alpha}$. Then by Proposition 3.4 it holds that

$$p^B(t,x,y) \geq C_8 t^{-d/\alpha} \geq C_8 R^{-d}\,.$$

The proof is finished by choosing $C_9 = \min(c_{21}, c_{22}, C_8)$.

$\square$

4. PARABOLIC HARNACK INEQUALITY

In this section we are going to prove the parabolic Harnack inequality following closely the approach from [4]. Let us first introduce the space-time process $Z_s = (T_0 + s, X_s)$. The law of the space-time process starting from $(t,x) \in [0,\infty) \times \mathbb{R}^d$ will be denoted by $\mathbb{P}^{(t,x)}$.

DEFINITION 4.1. Let $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and let $r_1, r_2 > 0$. We say that a nonnegative function $q : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ is *parabolic* in $[t, t+r_1] \times B(x, r_2)$ if for any $[s_1, s_2] \subset [t, t+r_1)$ and $B(y,r) \subset \overline{B(y,r)} \subset B(x, r_2)$ we have

$$q(s,z) = \mathbb{E}^{(s,z)}(q(Z_{\tau_{[s_1,s_2)\times B(y,r)}})), \quad \forall (s,z) \in [s_1, s_2) \times B(y,r),$$

where $\tau_{[s_1,s_2)\times B(y,r)} = \inf\{s : Z_s \notin [s_1, s_2) \times B(y,r)\}$.

For $t \geq 0$, $x \in \mathbb{R}^d$ and $R > 0$, define

$$Q(t,x,R) = [t, t + (R^2 \wedge R^\alpha)] \times B(x,R)\,.$$

For $A \subset [0,\infty) \times \mathbb{R}^d$, let $\sigma_A = \inf\{t > 0;\, Z_t \in A\}$ and $A_s = \{y \in \mathbb{R}^d; (s,y) \in A\}$.

The idea for the proof of the next result comes from [6].

LEMMA 4.2. *Let $\delta \in (0,1]$. There exists a constant $C_{10} = C_{10}(\delta) > 0$ such that for all $R > 0$, any $z \in \mathbb{R}^d$, any $v \in B(z, R/3)$ and any $A \subset Q(0,z,R/2) \cap ([\delta(R^2 \wedge R^\alpha), \infty) \times \mathbb{R}^d)$,*

$$\mathbb{P}^{(0,v)}(\sigma_A < \tau_R) \geq C_{10} \frac{|A|}{R^d (R^2 \wedge R^\alpha)}\,,$$

where $\tau_R = \tau_{Q(0,z,R)}$.

PROOF. We are going to estimate the expected time that the space-time process Z spends in A before exiting $Q(0,z,R)$. Let $X^{B(z,R)}$ denote the process X killed upon exiting the ball $B(z,R)$ and let $p^{B(z,R)}$ be its transition density. Then

$$\begin{aligned}
\mathbb{E}^{(0,v)} \int_0^{\tau_R} 1_A(s, X_s)\, ds &= \mathbb{E}^{(0,v)} \int_0^\infty 1_A(s, X_s^{B(z,R)})\, ds = \int_0^{R^2 \wedge R^\alpha} \mathbb{P}^{(0,v)}((s, X_s^{B(z,R)}) \in A)\, ds \\
&= \int_0^{R^2 \wedge R^\alpha} \mathbb{P}^v(X_s^{B(z,R)} \in A_s)\, ds = \int_{\delta(R^2 \wedge R^\alpha)}^{R^2 \wedge R^\alpha} \int_{A_s} p^{B(z,R)}(s,v,y)\, dy\, ds \\
&\geq \int_{\delta(R^2 \wedge R^\alpha)}^{R^2 \wedge R^\alpha} \int_{A_s} \frac{C_9}{R^d}\, dy\, ds = C_9 \frac{|A|}{R^d}\,,
\end{aligned}$$

where the inequality follows from Corollary 3.5 by using that $s \in (\delta(R^2 \wedge R^\alpha), R^2 \wedge R^\alpha)$ and $v, y \in B(z, R/2)$. On the other hand,

$$\begin{aligned}
\mathbb{E}^{(0,v)} \int_0^{\tau_R} 1_A(s, X_s)\, ds &= \int_0^\infty \mathbb{P}^{(0,v)}\Big(\int_0^{\tau_R} 1_A(s, X_s)\, ds > u\Big)\, du \\
&= \int_0^{R^2 \wedge R^\alpha} \mathbb{P}^{(0,v)}\Big(\int_0^{\tau_R} 1_A(s, X_s)\, ds > u\Big)\, du \\
&\leq \int_0^{R^2 \wedge R^\alpha} \mathbb{P}^{(0,v)}\Big(\int_0^{\tau_R} 1_A(s, X_s)\, ds > 0\Big)\, du \\
&\leq (R^2 \wedge R^\alpha)\, \mathbb{P}^{(0,v)}(\sigma_A < \tau_R)\,.
\end{aligned}$$

The last two displays prove the lemma. □

Define $U(t,x,r) = \{t\} \times B(x,r)$.

LEMMA 4.3. *Let $\delta \in (0,1)$. There exists $C_{11} = C_{11}(\delta) > 0$ such that for all $R > 0$, any $z \in \mathbb{R}^d$, $(t,x) \in Q(0,z,R/3)$, $v \in B(z,R/3)$, $r \leq R/4$ and $t \geq \delta(R^2 \wedge R^\alpha)$,*

$$\mathbb{P}^{(0,v)}(\sigma_{U(t,x,r/3)} < \tau_{Q(0,z,R)}) \geq C_{11} \frac{(r/3)^{d+2}}{R^{d+2}}\,.$$

PROOF. Note that

$$\begin{aligned}\mathbb{P}^{(0,v)}(\sigma_{U(t,x,r/3)} < \tau_{Q(0,z,R)}) &= \mathbb{P}^v(X_t^{B(z,R)} \in B(x,r/3)) \\ &= \int_{B(x,r/3)} p^{B(z,R)}(t,v,y)\,dy \geq C_9 \frac{|B(x,r/3)|}{R^d} \\ &= C_{11}\frac{(r/3)^d}{R^d} \geq C_{11}\frac{(r/3)^{d+2}}{R^{d+2}}\,,\end{aligned}$$

which proves the corollary. Note that the first inequality above follows from Corollary 3.5, because $v, y \in B(z, R/2)$ and $t \geq \delta(R^2 \wedge R^\alpha)$. □

LEMMA 4.4. *There exists a constant C_{12} such that for any $x \in \mathbb{R}^d$, $r > 0$, $y \in B(x, r/3)$ and any bounded non-negative function h on $[0,\infty) \times \mathbb{R}^d$ that is supported in $[0,\infty) \times B(x, 2r)^c$,*

$$\mathbb{E}^{(0,x)}[h(\tau_r, X_{\tau_r})] \leq C_{12}\mathbb{E}^{(0,y)}[h(\tau_r, X_{\tau_r})]\,, \tag{4}$$

where $\tau_r = \tau_{Q(0,x,r)}$.

PROOF. A Lévy system (N, H) of the process X is given by $N(x, dy) = \frac{c\,dy}{|x-y|^{d+\alpha}}$ and $H_t = t$ for some positive constant c. Thus the proof of the lemma is the same as that of Lemma 4.9 in [4]. The fact that our process X has a continuous component does not play any role since the function h is supported in $[0,\infty) \times B(x, 2r)^c$. □

With these lemmas observed, the next theorem can be proved in a manner similar to that in Proposition 4.3 in [4].

THEOREM 4.5. *For every $\delta \in (0, 1/18)$ there exists a constant $C_{13} = C_{13}(\delta) > 0$ such that for all $R > 0$, for every $z \in \mathbb{R}^d$ and every non-negative function q on $[0,\infty) \times \mathbb{R}^d$ that is parabolic and bounded on $[0, 4(R^2 \wedge R^\alpha)] \times B(z, 2R)$,*

$$\sup_{(t,y)\in Q(\delta(R^2\wedge R^\alpha),z,R/3)} q(t,y) \leq C_{13} \inf_{y\in B(z,R/3)} q(0,y)\,.$$

PROOF. Without loss of generality we may assume that

$$\inf_{y\in B(z,R/3)} q(0,y) = 1/2\,.$$

Let $v \in B(z, R/3)$ be such that $q(0, v) \leq 1$. For any $x \in \mathbb{R}^d$ and $t \geq 0$, consider $Q(t, x, r)$ for $r \leq R/4$ and let $\tau_r = \tau_{Q(t,x,r)}$. Suppose that $C \subset Q^\delta(t, x, r/3) := Q(t, x, r/3) \cap ([t + \delta(r^2 \wedge r^\alpha), \infty) \times \mathbb{R}^d) = [t + \delta(r^2 \wedge r^\alpha), t + (r/3)^2 \wedge (r/3)^\alpha] \times B(x, r/3)$. Then by Lemma 4.2,

$$\mathbb{P}^{(t,x)}(\sigma_C < \tau_r) \geq C_{10}\frac{|C|}{r^d(r^2 \wedge r^\alpha)}\,.$$

Note that there exists a constant c_0 such that $c_0 r^d(r^2\wedge r^\alpha)\le |Q^\delta(t,x,r/3)|\le c_0^{-1}r^d(r^2\wedge r^\alpha)$. Hence, there exists a constant $c_1=c_1(\delta)>0$ such that for all $C\subset Q^\delta(t,x,r/3)$ satisfying $|C|/|Q^\delta(t,x,r/3)|\ge 2/3$ we have

$$\mathbb{P}^{(t,x)}(\sigma_C<\tau_r)\ge c_1\,. \tag{5}$$

Define

$$\eta=\frac{c_1}{3}\quad\text{and}\quad \xi=\frac{1}{3}\wedge(C_{12}\eta)\,, \tag{6}$$

where C_{12} is the constant from Lemma 4.4.

Suppose there is some point $(t,x)\in Q(\delta(R^2\wedge R^\alpha),z,R/3)$ such that $q(t,x)>K$, where K is a constant to be determined later. Define

$$c_2=\max\left(\left(\frac{3}{c_0C_{10}\xi}\right)^{1/(d+\alpha)},\left(\frac{3}{c_0C_{10}\xi}\right)^{1/(d+2)},\frac{3\cdot 2^{1/(d+2)}}{(C_{11}\xi)^{1/(d+2)}}\right)\,,$$

where C_{10} and C_{11} are constants from Lemma 4.2 and Lemma 4.3 respectively. Choose

$$r=c_2RK^{-1/(d+2)}\,. \tag{7}$$

Then an easy computation shows that

$$\frac{|Q^\delta(0,x,r/3)|}{R^d(R^2\wedge R^\alpha)}\ge\frac{3}{C_{10}\xi K}\,,\qquad \frac{r^{d+2}}{R^{d+2}}\ge\frac{2\cdot 3^{d+2}}{C_{11}\xi K}\,, \tag{8}$$

Let $U=\{t\}\times B(x,r/3)$. Suppose that $q\ge\xi K$ on U. Let $Q=Q(0,z,R)$. Then by Lemma 4.3

$$1\ge q(0,v)=\mathbb{E}^{(0,v)}[q(Z_{\sigma_U\wedge\tau_R})]\ge\xi K\mathbb{P}^{(0,v)}(\sigma_U<\tau_R)\ge\xi K\frac{C_{11}(r/3)^{d+2}}{R^{d+2}}\,,$$

which contradicts the choice of r in the second inequality in (8). Thus, there exists at least one point in U at which q takes a value less than ξK.

We next claim that

$$\mathbb{E}^{(t,x)}[q(\tau_r,X_{\tau_r}):\,X_{\tau_r}\notin B(x,2r)]\le\eta K\,, \tag{9}$$

where $\tau_r=\tau_{Q(t,x,r)}$. If not, by Lemma 4.4, for all $y\in B(x,r/3)$,

$$\begin{aligned} q(t,y) &\ge \mathbb{E}^{(t,y)}[q(\tau_r,X_{\tau_r}):\,X_{\tau_r}\notin B(x,2r)]\\ &\ge C_{12}^{-1}\mathbb{E}^{(t,x)}[q(\tau_r,X_{\tau_r}):\,X_{\tau_r}\notin B(x,2r)]\\ &\ge C_{12}^{-1}\eta K\ge\xi K\,. \end{aligned}$$

But this contradicts the already proven fact that there exists at least one point in U at which q takes a value less than ξK. Therefore, (9) holds true.

Let A be any compact subset of

$$\tilde A:=\{(s,y)\in Q^\delta(t,x,r/3);\,q(s,y)\ge\xi K\}\,.$$

Note that $\tilde{A} \subset Q(0, z, R/2)$. By Lemma 4.2

$$1 \geq q(0, v) \geq \mathbb{E}^{(0,v)}[q(Z_{\sigma_A}) : \sigma_A < \tau_Q] \geq \xi K \mathbb{P}^{(0,v)}(\sigma_A < \tau_Q) \geq \xi K \frac{C_{10}|A|}{R^d(R^2 \wedge R^\alpha)} .$$

By the first inequality in (8)

$$\frac{|A|}{|Q^\delta(t, x, r/3)|} \leq \frac{R^d(R^2 \wedge R^\alpha)}{C_{10}\,|Q^\delta(t, x, r/3)|\,\xi K} \leq \frac{1}{3} . \tag{10}$$

Since (10) holds for every compact subset A of $\tilde{A}$, it holds for $\tilde{A}$ in place of A.

Let $C := Q^\delta(t, x, r/3) \setminus \tilde{A}$. Then by (10), $|C|/|Q^\delta(t, x, r/3)| \geq 2/3$. Let

$$M = \sup_{(s,y) \in Q(t,x,2r)} q(s, y) .$$

Then

$$\begin{aligned} q(t, x) &= \mathbb{E}^{(t,x)}[q(\sigma_C, X_{\sigma_C}) : \sigma_C < \tau_r] \\ &\quad + \mathbb{E}^{(t,x)}[q(\sigma_C, X_{\sigma_C}) : \tau_r \leq \sigma_C, X_{\tau_r} \notin B(x, 2r)] \\ &\quad + \mathbb{E}^{(t,x)}[q(\sigma_C, X_{\sigma_C}) : \tau_r \leq \sigma_C, X_{\tau_r} \in B(x, 2r)] . \end{aligned}$$

The first term on the right is bounded by $\xi K \mathbb{P}^{(t,x)}(\sigma_C < \tau_r)$, the second term is according to (9) bounded by ηK, and the third term is bounded by $M \mathbb{P}^{(t,x)}(\sigma_C \geq \tau_r)$. Therefore,

$$K \leq q(t, x) \leq \xi K \mathbb{P}^{(t,x)}(\sigma_C < \tau_r) + \eta K + M \mathbb{P}^{(t,x)}(\sigma_C \geq \tau_r) .$$

Note that by (5), $\mathbb{P}^{(t,x)}(\sigma_C < \tau_r) \geq c_1$. Hence by use of (6),

$$\frac{M}{K} \geq \frac{1 - \eta - \xi \mathbb{P}^{(t,x)}(\sigma_C < \tau_r)}{\mathbb{P}^{(t,x)}(\sigma_C \geq \tau_r)} \geq \frac{1 - \eta - \xi c_1}{1 - c_1} \geq \frac{1 - 2c_1/3}{1 - c_1} = 1 + 2\beta ,$$

where $\beta = c_1/6(1 - c_1)$. Hence, there exists a point $(t_1, x_1) \in Q(t, x, 2r) \subset \hat{Q}(0, z, R) := [0, 3(R^2 \wedge R^\alpha)] \times B(z, R)$ such that $q(t_1, x_1) \geq (1 + \beta)K =: K_1$. Note that $0 \leq t_1 - t \leq (2r)^2 \wedge (2r)^\alpha$ and $|x_1 - x| \leq 2r$.

Iterate the above procedure to obtain a sequence of points $\{(t_k, x_k)\}$ in the following way. Using above argument (with (t_1, x_1) and K_1 instead of (t, x) and K), there exists $(t_2, x_2) \in Q(t_1, x_1, 2r_1)$ such that $q(t_2, x_2) \geq (1 + \beta)K_1 =: K_2$. Continue this procedure to obtain a sequence of points $\{(t_k, x_k)\}$ such that $(t_{k+1}, x_{k+1}) \in Q(t_k, x_k, 2r_k)$ and $q(t_{k+1}, x_{k+1}) \geq (1 + \beta)^{k+1} K_1 =: K_{k+1}$. We have that $0 \leq t_{k+1} - t_k \leq (2r_k)^2 \wedge (2r_k)^\alpha$, $|x_{k+1} - x_k| \leq 2r_k$. Moreover, by (7),

$$r_k \leq c_2 R K_k^{-1/(d+2)} \leq c_2 (1 + \beta)^{-k/(d+2)} K^{-1/(d+2)} R .$$

Note that

$$\begin{aligned}
\sum_k r_k &\leq c_2 K^{-1/(d+2)} R \sum_k ((1+\beta)^{-1/(d+2)})^k = \frac{c_2 K^{-1/(d+2)} R}{1-(1+\beta)^{-1/(d+2)}}\,, \\
\sum_k (2r_k)^2 &\leq (2c_2 K^{-1/(d+2)} R)^2 \sum_k ((1+\beta)^{-2/(d+2)})^k = \frac{(2c_2 K^{-1/(d+2)} R)^2}{1-(1+\beta)^{-2/(d+2)}}\,, \\
\sum_k (2r_k)^\alpha &\leq (2c_2 K^{-1/(d+2)} R)^\alpha \sum_k ((1+\beta)^{-2/(d+2)})^k = \frac{(2c_2 K^{-1/(d+2)} R)^\alpha}{1-(1+\beta)^{-2/(d+2)}}\,.
\end{aligned}$$

Therefore, we may choose K large enough so that $(t_k, x_k) \in \hat{Q}(0, z, R)$ for all k. This is a contradiction because $q(t_k, x_k) \geq (1+\beta)^k K$ goes to infinity as $k \to \infty$. $\square$

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Department of Mathematics
University of Illinois
Urbana, IL 61801, USA
E-mail address: `rsong@math.uiuc.edu`

Department of Mathematics
University of Zagreb
Zagreb, Croatia
E-mail address: `vondra@math.hr`