

Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings

Soobin Cho* Panki Kim[†] Renming Song[‡] and Zoran Vondraček[§]

Abstract

In this paper we discuss non-local operators with killing potentials, which may not be in the standard Kato class. We first discuss factorization of their Dirichlet heat kernels in metric measure spaces. Then we establish explicit estimates of the Dirichlet heat kernels under critical killings in $C^{1,1}$ open subsets of \mathbb{R}^d or in $\mathbb{R}^d \setminus \{0\}$. The decay rates of our explicit estimates come from the values of the multiplicative constants in the killing potentials. Our method also provides an alternative and unified proof of the main results of [18, 19, 20].

AMS 2010 Mathematics Subject Classification: Primary 60J35, 60J50, 60J75;

Keywords and phrases: Dirichlet heat kernel, transition density, critical killings, potential, additive functional, Kato class

1 Introduction

Stability of Dirichlet heat kernel estimates under certain Feynman-Kac transforms was studied in the recent paper [22]. To be more precise, let X be a Hunt process on a Borel set $D \subset \mathbb{R}^d$ that admits a jointly continuous transition density $p_D(t, x, y)$ with respect to the Lebesgue measure. Let $\alpha \in (0, 2)$ and $\gamma \in [0, \alpha \wedge d)$, and define

$$q_\gamma(t, x, y) := \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^\gamma \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^\gamma \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

where $\delta_D(x)$ denotes the distance between $x \in D$ and D^c . Assume that $p_D(t, x, y)$ is comparable to $q_\gamma(t, x, y)$ for $(t, x, y) \in (0, 1] \times D \times D$. Examples of processes satisfying this assumption include killed symmetric stable processes in $C^{1,1}$ open sets D (with $\gamma = \alpha/2$,

*This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. NRF-2015R1A4A1041675)

[†]This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. NRF-2015R1A4A1041675)

[‡]Research supported in part by a grant from the Simons Foundation (#429343, Renming Song)

[§]Research supported in part by the Croatian Science Foundation under the project 4197

cf. [18]), and, when $\alpha \in (1, 2)$, censored α -stable processes in any $C^{1,1}$ open sets D (with $\gamma = \alpha - 1$, cf. [19]). Consider the following Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}_x [\exp(-A_t) f(X_t)],$$

where A is a continuous additive functional of X with Revuz measure μ . If \mathcal{L} denotes the L^2 -infinitesimal generator of X , then, informally, the semigroup (T_t) has the L^2 -infinitesimal generator $\mathcal{A}f(x) := (\mathcal{L} - \mu)f(x)$. Under the assumption that μ belongs to some appropriate Kato class, one of the main results of [22] implies that the semigroup (T_t) admits a continuous density $q^D(t, x, y)$ which is comparable to $q_\gamma(t, x, y)$ for all $(t, x, y) \in (0, 1] \times D \times D$. Hence a Kato class perturbation preserves the Dirichlet heat kernel estimates and is in this sense subcritical. Related results on the stability of the heat kernel estimates (without boundary condition) under Kato class perturbations were obtained earlier in [14, 47, 50].

Kato class perturbations of the Laplacian have been studied earlier and more thoroughly, e.g. [1, 5, 39, 46], with the same conclusion that Kato class perturbations preserve the (Dirichlet) heat kernel estimates. It is well known since [2] that, in the case of the Laplacian in the whole space, the inverse square potential $\kappa(x) = c|x|^{-2}$ is critical, and, in the case of the Dirichlet Laplacian in a domain D , the potential $\kappa(x) = c\delta_D(x)^{-2}$ is critical. Criticality of the potentials above can be explained as follows. In the whole space case, consider the symmetric form

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 + \frac{cu(x)^2}{|x|^2} \right) dx, \quad u \in C_0^\infty(\mathbb{R}^d).$$

Due to Hardy's inequality, this form is non-negative if and only if $c \geq -((d-2)/2)^2$ (here $d \geq 3$), and the operator $\Delta - c|x|^{-2}$ is defined as the L^2 -generator of the form above. In the case of a bounded smooth domain D , consider the symmetric form

$$\mathcal{E}(u, u) = \int_D \left(|\nabla u(x)|^2 + \frac{cu(x)^2}{\delta_D(x)^2} \right) dx, \quad u \in C_0^\infty(D).$$

Again due to Hardy's inequality, this form is non-negative if and only if $c \geq -1/4$, and the operator $\Delta - c\delta_D(x)^{-2}$ is defined as the L^2 -generator of the form above. In both cases above, when $c < 0$, the potential κ above can be interpreted as creation, and, when $c > 0$, the potential κ can be interpreted as killing. Note that in both cases, even when $c > 0$, the potential κ above does not belong to the Kato class. There exists a large body of literature on the heat kernel estimates of critical perturbations of the (Dirichlet) Laplacian, e.g., [3, 32, 35, 42, 43, 44]. In this paper we use probabilistic methods to study sharp two-sided heat kernel estimates for critical perturbations of the fractional Laplacian in a smooth domain D , as well as the fractional Laplacian in \mathbb{R}^d . When the potential involves both killing and creation, there is no Markov process associated with the corresponding Schrödinger type operator. Since our argument depends crucially on properties of Markov processes, we will only deal with killing type potentials. To deal with critical perturbations involving creation, different methods are needed, cf. [9, 12].

Let us describe some of our results in more detail. Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a $C^{1,1}$ open set and let $X = (X_t, \mathbb{P}_x)$ be the reflected α -stable process in \overline{D} , $\alpha \in (0, 2)$, cf. [7]. When

$\alpha \in (0, 1]$, we denote by X^D the process X restricted to D (X does not hit ∂D in this case), while for $\alpha \in (1, 2)$, X^D denotes the censored α -stable process in D . In this paragraph, we will only describe some of our results in the case $\alpha \in (1, 2)$. So in the rest of this paragraph, we will assume that $\alpha \in (1, 2)$. Similar results are also valid in the case $\alpha \in (0, 1]$. Consider the potential $\kappa(x) = c_1 \delta_D(x)^{-\alpha}$ where $c_1 \in [0, \infty)$ (the family of potentials we study is in fact larger, see (3.6) for the full picture). The criticality of this type of potentials can also be interpreted using the fractional Hardy inequality in, for instance, [28, 8]. Let

$$T_t f(x) := \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_s^D) ds} f(X_t^D) \right], \quad x \in D, t > 0,$$

be the Feynman-Kac semigroup of X^D via the multiplicative functional $e^{-\int_0^t \kappa(X_s^D) ds}$. Alternatively, we can think of (T_t) as the semigroup corresponding to the Schrödinger operator $\mathcal{L} - \kappa$, where \mathcal{L} is the generator of X^D . We show that the semigroup (T_t) admits a continuous density $q^D(t, x, y)$ and that there exists $p \in [\alpha - 1, \alpha)$ depending on the constant c_1 such that $q^D(t, x, y)$ is comparable to

$$\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}} \right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}} \right)^p \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right), \quad (1.1)$$

for $(t, x, y) \in (0, 1] \times D \times D$, see Theorem 3.2. Moreover, the mapping $[0, \infty) \ni c_1 \mapsto p \in [\alpha - 1, \alpha)$ is one-to-one and onto. If $c_1 = 0$ (no killing), then $p = \alpha - 1$ and we recover the heat kernel estimates for the censored α -stable process obtained in [19]. When $p = \alpha/2$, the semigroup (T_t) corresponds to the α -stable process killed upon exiting D and we recover the heat kernel estimates from [18]. The novelty of our approach is that by changing the constant c_1 in the potential κ we can obtain a whole spectrum of boundary behaviors of the heat kernel. In particular, we construct semigroups whose heat kernels satisfy the assumptions from [22]. On the other hand, the interior estimates $t^{-d/\alpha} \wedge t|x - y|^{-d-\alpha}$, which correspond to the reflected α -stable process, remain unchanged.

Besides reflected α -stable processes on \overline{D} , we can also consider processes which are lower order perturbations of reflected α -stable processes. A typical example is the process X^β on \overline{D} with Dirichlet form of the type

$$\mathcal{E}^\beta(u, u) = \frac{1}{2} \int_D \int_D (u(x) - u(y))^2 \left(\frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{b}{|x - y|^{d+\beta}} \right) dx dy,$$

where D is a bounded $C^{1,1}$ open set, $\beta < \alpha < 2$, $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma((d + \alpha)/2) \Gamma(1 - \alpha/2)^{-1}$ and b is a non-negative constant (see Subsection 3.2 for the more general setup, in particular, b need not be non-negative). Let $X^{\beta, D}$ denote the process X^β killed upon exiting D . We prove that the Feynman-Kac semigroup of $X^{\beta, D}$ via the multiplicative functional $e^{-\int_0^t \kappa(X_s^D) ds}$ has a continuous transition density comparable to (1.1), where the critical potential $\kappa(x) = c_1 \delta_D(x)^{-\alpha}$ is the same as above.

Our final result concerns the isotropic α -stable process Z in \mathbb{R}^d , $d \geq 2$, and the singular potential $\kappa(x) = c_1 |x|^{-\alpha}$, $c_1 > 0$ (again, we in fact consider more general potentials – see (3.25)). We show that the Feynman-Kac semigroup of Z via the multiplicative functional $e^{-\int_0^t \kappa(Z_s) ds}$ admits a continuous density comparable to

$$\left(1 \wedge \frac{|x|}{t^{1/\alpha}} \right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}} \right)^p \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right),$$

with $p \in (0, \alpha)$ depending on c_1 , see Theorem 3.9. For related results, cf. [12, Theorem 1.1] and [36, Theorem 1.1].

Organization of the paper: the paper is divided into two major parts and an appendix. The first part is Section 2 and the setup is quite general there. We consider a Hunt process X on a locally compact separable metric space (\mathfrak{X}, ρ) . The process X is not necessarily symmetric and may not be conservative. We assume that the process X is in strong duality (with respect to a Radon measure m with full support) with another Hunt process \widehat{X} . We further assume that both X and \widehat{X} are Feller and strongly Feller, and that their semigroups admit a strictly positive and jointly continuous transition density $p(t, x, y)$. The main assumption is that $p(t, x, y)$ is, for small times, comparable to the function $\tilde{q}(t, x, y)$ defined in (2.6) in terms of the volume of balls in (\mathfrak{X}, ρ) and a strictly increasing function Φ satisfying a weak scaling condition (2.3). In Subsection 2.2 we argue that X and \widehat{X} satisfy the scale invariant parabolic Harnack inequality with explicit scaling in terms of Φ and use this to obtain interior lower bound on the transition density $p_D(t, x, y)$ of the process X^D - the process X killed upon exiting an open subset D of \mathfrak{X} . Subsection 2.3 contains the definition of the class $\mathbf{K}_T(D)$, $T > 0$, of possibly critical smooth measures μ on D , cf. Definition 2.12. Using the positive additive functional (A_t^μ) of X^D with Revuz measure μ , we define the Feynman-Kac semigroup of X^D associated with μ :

$$T_t^{\mu, D} f(x) = \mathbb{E}_x [\exp(-A_t^\mu) f(X_t^D)], \quad t \geq 0, x \in D.$$

The Hunt process corresponding to $(T_t^{\mu, D})$ is denoted by Y . We analogously define the dual semigroup $(\widehat{T}_t^{\mu, D})$ and denote the corresponding Hunt process by \widehat{Y} . We argue that $(T_t^{\mu, D})$ has a transition density (with respect to the measure m) $q^D(t, x, y)$ and that there exists $C_0 > 0$ such that $q^D(t, x, y) \leq C_0 \tilde{q}(t, x, y)$ for small t . The argument relies on the 3P inequality for $\tilde{q}(t, x, y)$ proved in Lemma 2.11. In Subsection 2.4 we prove some interior estimates for the transition density $q^U(t, x, y)$, where U is an open subset of D . Two examples of critical potentials are given in Subsection 2.5. Finally, in Subsection 2.6, we show that factorization of the transition density $q^D(t, x, y)$ in κ -fat open set D holds true. The result is proved in Theorem 2.22 and states that for small time t , $q^D(t, x, y)$ is comparable to $\mathbb{P}_x(\zeta > t) \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) \tilde{q}(t, x, y)$, $x, y \in D$. Here ζ and $\widehat{\zeta}$ are the lifetimes of Y and \widehat{Y} respectively. We note that this is a quite general result and, besides critical perturbations, includes also subcritical perturbations (or no perturbation at all). Criticality and subcriticality of the perturbation are hidden in the tail behavior of the lifetimes, $\mathbb{P}_x(\zeta > t)$ and $\widehat{\mathbb{P}}_y(\widehat{\zeta} > t)$. To prove Theorem 2.22, we follow the ideas in the proof of [21, Theorem 1.3], but we use Assumption **U**, cf. Subsection 2.6, instead of the boundary Harnack principle. Approximate factorization of heat kernels involving tails of lifetimes can be traced back to [49, 10]. If one can get explicit two-sided estimates on the survival probabilities, then one can combine them with the approximate factorization to get explicit two-sided estimates on the heat kernel. This is the strategy employed in [11, 21]. We will also use this strategy in Section 3, and as a by-product, give an alternative and unified proof of the main results of [18, 19, 20].

The second part is Section 3. In this section we assume that \mathfrak{X} is either the closure of a $C^{1,1}$ open subset D of \mathbb{R}^d or \mathbb{R}^d itself, $d \geq 2$, and we assume that the underlying process X is either a reflected α -stable(-like) process on \overline{D} (or a non-local perturbation of it), or

an α -stable process in \mathbb{R}^d (or a drift perturbation of it). The critical potentials have been already described above and are essentially of the form either $c_1\delta_D(x)^{-\alpha}$ or $c_1|x|^{-\alpha}$. The goal of this section is to estimate the tail of the lifetime $\mathbb{P}_x(\zeta > t)$ in terms of $\delta_D(x)$ and $|x|$ respectively. Then, as was done in [11, 21], together with the factorization obtained in Theorem 2.22, this gives sharp two-sided estimates of the transition density of the Feynman-Kac semigroup. The main step in obtaining the estimates of the lifetime consists of finding appropriate superharmonic and subharmonic function for the process Y . This relies on quite detailed computations of the generator of Y acting on some appropriate functions. Although similar methods have been already employed in some previous works, our calculations are quite involved and delicate. Section 3 also provides an alternative and unified proof of the main results of [18, 19, 20].

The paper ends with an appendix devoted to a result about a continuous additive functional for killed non-symmetric process.

Notation: We will use the symbol “:=” to denote a definition, which is read as “is defined to be.” In this paper, for $a, b \in \mathbb{R}$, we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We also use the convention $0^{-1} = +\infty$. For two non-negative functions f and g , the notation $f \asymp g$ means that there are strictly positive constants c_1 and c_2 such that $c_1g(x) \leq f(x) \leq c_2g(x)$ in the common domain of the definition of f and g .

Letters with subscripts $r_i, R_i, A_i, C_i, i = 0, 1, 2, \dots$, denote constants that will be fixed throughout the paper. Lower case letters c 's without subscripts denote strictly positive constants whose values are unimportant and which may change even within a line, while values of lower case letters with subscripts $c_i, i = 0, 1, 2, \dots$, are fixed in each proof, and the labeling of these constants starts anew in each proof. $c_i = c_i(a, b, c, \dots)$, $i = 0, 1, 2, \dots$, denote constants depending on a, b, c, \dots . The dependence on the dimension $d \geq 1$ may not be mentioned explicitly.

For a function space $\mathbb{H}(U)$ on an open set U in \mathfrak{X} , we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f \text{ has compact support}\}$, $\mathbb{H}_0(U) := \{f \in \mathbb{H}(U) : f \text{ vanishes at infinity}\}$ and $\mathbb{H}_b(U) := \{f \in \mathbb{H}(U) : f \text{ is bounded}\}$.

2 Factorization of Dirichlet heat kernels in metric measure spaces

2.1 Setup

We first spell out our assumptions on the state space: (\mathfrak{X}, ρ) is a locally compact separable metric space such that all bounded closed sets are compact and m is a Radon measure on \mathfrak{X} with full support. By $B(x, r) = \{y \in \mathfrak{X} : \rho(x, y) < r\}$ we denote the open ball of radius r centered at $x \in \mathfrak{X}$. For any open set V of \mathfrak{X} and $x \in \mathfrak{X}$, we denote by $\delta_V(x)$ the distance between x and $\mathfrak{X} \setminus V$.

Let $R_0 \in (0, \infty]$ be the largest number such that $\mathfrak{X} \setminus B(x, 2r) \neq \emptyset$ for all $x \in \mathfrak{X}$ and all $r < R_0$. We call R_0 the localization radius of (\mathfrak{X}, ρ) .

Let $V(x, r) := m(B(x, r))$. We assume that there exist constants $d \geq d_0 > 0$ such that

for every $M \geq 1$ there exists $\tilde{C}_M \geq 1$ with the property that

$$\tilde{C}_M^{-1} \left(\frac{R}{r}\right)^{d_0} \leq \frac{V(x, R)}{V(x, r)} \leq \tilde{C}_M \left(\frac{R}{r}\right)^d \quad \text{for all } x \in \mathfrak{X} \text{ and } 0 < r \leq R < MR_0. \quad (2.1)$$

Observe that

$$V(x, n_0 r) \geq 2V(x, r) \quad \text{for all } x \in \mathfrak{X} \text{ and } r \in (0, R_0/n_0), \quad (2.2)$$

where $n_0 := (2\tilde{C}_1)^{1/d_0}$.

Now we spell out the assumptions on the processes we are going to work with. We assume that $X = (X_t, \mathbb{P}_x)$ is a Hunt process admitting a (strong) dual Hunt process $\hat{X} = (\hat{X}_t, \hat{\mathbb{P}}_x)$ with respect to the measure m . For the definition of (strong) duality, see [6, Section VI.1]. We further assume that the transition semigroups (P_t) and (\hat{P}_t) of X and \hat{X} are both Feller and strongly Feller, and that all semipolar sets are polar. The condition that semipolar sets are polar is known as Hunt's hypothesis (H). This guarantees the duality between the killed processes when the original processes are duals (since X never hits irregular points). See [16, p.481] and the end of [29, Section 13.6].

In the sequel, all objects related to the dual process \hat{X} will be denoted by a hat. We also assume that X and \hat{X} admit a strictly positive and jointly continuous transition density $p(t, x, y)$ with respect to m so that

$$P_t f(x) = \int_{\mathfrak{X}} p(t, x, y) f(y) m(dy) \quad \text{and} \quad \hat{P}_t f(x) = \int_{\mathfrak{X}} p(t, y, x) f(y) m(dy).$$

We will make some assumptions on the transition density $p(t, x, y)$. To do this, we first introduce some notation.

Let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be a strictly increasing function satisfying the following weak scaling condition: there exist constants $\delta_l, \delta_u \in (0, \infty)$, $a_l \in (0, 1]$, $a_u \in [1, \infty)$ such that

$$a_l \left(\frac{R}{r}\right)^{\delta_l} \leq \frac{\Phi(R)}{\Phi(r)} \leq a_u \left(\frac{R}{r}\right)^{\delta_u}, \quad r \leq R < R_0. \quad (2.3)$$

Remark 2.1. *Since the function Φ is strictly increasing, for every $\tilde{R} \in (0, \infty)$, there exist $\tilde{a}_l \in (0, 1]$ and $\tilde{a}_u \in [1, \infty)$ such that*

$$\tilde{a}_l \left(\frac{R}{r}\right)^{\delta_l} \leq \frac{\Phi(R)}{\Phi(r)} \leq \tilde{a}_u \left(\frac{R}{r}\right)^{\delta_u}, \quad 0 < r \leq R \leq \tilde{R}. \quad (2.4)$$

Indeed, if $R_0 \leq \tilde{R} < \infty$, then for $r \vee R_0 \leq R \leq \tilde{R}$, we have

$$\frac{\Phi(R)}{\Phi(r)} \leq \frac{\Phi(\tilde{R})}{\Phi(R_0/2)} \frac{\Phi(R_0/2)}{\Phi(r \wedge (R_0/2))} \leq a_u \frac{\Phi(\tilde{R})}{\Phi(R_0/2)} \left(\frac{R_0/2}{r \wedge (R_0/2)}\right)^{\delta_u} \leq a_u \frac{\Phi(\tilde{R})}{\Phi(R_0/2)} \left(\frac{R}{r}\right)^{\delta_u}$$

and

$$\frac{\Phi(R)}{\Phi(r)} \geq \frac{\Phi(R_0/2)}{\Phi(r \vee (R_0/2))} \geq a_l \frac{\Phi(R_0/2)}{\Phi(\tilde{R})} \left(\frac{R_0/2}{r}\right)^{\delta_l} \geq a_l \frac{\Phi(R_0/2)}{\Phi(\tilde{R})} \left(\frac{R_0}{2\tilde{R}}\right)^{\delta_l} \left(\frac{R}{r}\right)^{\delta_l}.$$

We will use (2.4) instead of (2.3) whenever necessary. From (2.3) we can also get the scaling condition for the inverse of Φ :

$$a_u^{-1/\delta_u} \left(\frac{R}{r}\right)^{1/\delta_u} \leq \frac{\Phi^{-1}(R)}{\Phi^{-1}(r)} \leq a_l^{-1/\delta_l} \left(\frac{R}{r}\right)^{1/\delta_l}, \quad 0 < r \leq R < \Phi(R_0). \quad (2.5)$$

Define

$$\tilde{q}(t, x, y) := \frac{1}{V(x, \Phi^{-1}(t))} \wedge \frac{t}{V(x, \rho(x, y))\Phi(\rho(x, y))}, \quad t > 0, \quad x, y \in \mathfrak{X}. \quad (2.6)$$

Remark 2.2. *It is easy to see that*

$$\tilde{q}(t, x, y) \asymp \tilde{q}(t, y, x) \asymp \frac{1}{V(x, \Phi^{-1}(t))} \wedge \frac{t}{V(y, \rho(x, y))\Phi(\rho(x, y))}.$$

See [25, Remark 1.12]. Moreover, by integrating $\tilde{q}(t, x, y)$ over the set $\{y : \rho(x, y) \leq \Phi^{-1}(t)\}$, one easily gets that for all $t > 0$ and $x \in \mathfrak{X}$,

$$\int_{\mathfrak{X}} \tilde{q}(t, x, y) m(dy) \geq 1. \quad (2.7)$$

We will assume that there exists a constant $C_0 \geq 1$ such that

$$C_0^{-1} \tilde{q}(t, x, y) \leq p(t, x, y) \leq C_0 \tilde{q}(t, x, y), \quad (t, x, y) \in (0, \tilde{T}) \times \mathfrak{X} \times \mathfrak{X} \quad (2.8)$$

for some $\tilde{T} \in (0, \infty]$. Then (2.7) and the lower bound in (2.8) yield that

$$1 \leq \int_{\mathfrak{X}} \tilde{q}(t, x, y) m(dy) \leq C_0 \quad (2.9)$$

for all $(t, x) \in (0, \tilde{T}) \times \mathfrak{X}$. The processes X (and \hat{X}) may not be conservative so the lifetimes may be finite. We add an extra point ∂ (which is called the cemetery point) to \mathfrak{X} and assume our processes stay at the cemetery point after their lifetimes. If $\tilde{T} = \infty$, we assume $R_0 = m(\mathfrak{X}) = \infty$. Note that, if $\tilde{T} = \infty$ and both X and \hat{X} admit no killing inside \mathfrak{X} , then it follows that $R_0 = m(\mathfrak{X}) = \infty$, and X and \hat{X} are conservative (see the proof of [37, Proposition 2.5], which still works under the non-symmetric setting). All functions h on \mathfrak{X} will be automatically extended to $\mathfrak{X} \cup \{\partial\}$ by setting $h(\partial) = 0$.

Remark 2.3. *When $\tilde{T} \in (0, \infty)$, the value of \tilde{T} is not important. That is, when $\tilde{T} \in (0, \infty)$, for every $T > 0$ there exists a constant $\bar{C}_0 = \bar{C}_0(T) \geq 1$ such that*

$$\bar{C}_0^{-1} \tilde{q}(t, x, y) \leq p(t, x, y) \leq \bar{C}_0 \tilde{q}(t, x, y), \quad (t, x, y) \in (0, T) \times \mathfrak{X} \times \mathfrak{X}. \quad (2.10)$$

This is a consequence of the semigroup property of $p(t, x, y)$, (2.1), (2.5) and (2.8). Indeed, assume $T \geq \tilde{T}$, let $n := \lfloor 2T/\tilde{T} \rfloor \geq 2$ and fix it. It follows from (2.1), (2.5) and (2.8) that

$$\tilde{q}(t/n, z, w) \asymp \tilde{q}(t/n^2, z, w) \asymp p(t/n^2, z, w), \quad (t, z, w) \in (0, T) \times \mathfrak{X} \times \mathfrak{X}$$

and

$$p(t/n, z, w) \asymp \tilde{q}(t/n, z, w) \asymp \tilde{q}(t, z, w), \quad (t, z, w) \in (0, T) \times \mathfrak{X} \times \mathfrak{X}.$$

Thus by the semigroup property of $p(t, x, y)$, we have for each $(t, x, y) \in (0, T) \times \mathfrak{X} \times \mathfrak{X}$,

$$\begin{aligned} p(t, x, y) &= \int_{\mathfrak{X}} \cdots \int_{\mathfrak{X}} p(t/n, x, z_1) \cdots p(t/n, z_{n-1}, y) m(dz_1) \cdots m(dz_{n-1}) \\ &\asymp \int_{\mathfrak{X}} \cdots \int_{\mathfrak{X}} \tilde{q}(t/n, x, z_1) \cdots \tilde{q}(t/n, z_{n-1}, y) m(dz_1) \cdots m(dz_{n-1}) \\ &\asymp \int_{\mathfrak{X}} \cdots \int_{\mathfrak{X}} p(t/n^2, x, z_1) \cdots p(t/n^2, z_{n-1}, y) m(dz_1) \cdots m(dz_{n-1}) \\ &= p(t/n, x, y) \asymp \tilde{q}(t, x, y). \end{aligned}$$

Let $C_0(\mathfrak{X})$ stand for the Banach space of bounded continuous functions on \mathfrak{X} vanishing at infinity. Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\widehat{\mathcal{L}}, \mathcal{D}(\widehat{\mathcal{L}}))$ be the generators of (P_t) and (\widehat{P}_t) in $C_0(\mathfrak{X})$ respectively. We assume the following Urysohn-type condition.

Assumption A: There is a linear subspace \mathcal{D} of $\mathcal{D}(\mathcal{L}) \cap \mathcal{D}(\widehat{\mathcal{L}})$ satisfying the following condition: For any compact K and open U with $K \subset U \subset \mathfrak{X}$, there is a nonempty collection $\mathcal{D}(K, U)$ of functions $f \in \mathcal{D}$ satisfying the conditions (i) $f(x) = 1$ for $x \in K$; (ii) $f(x) = 0$ for $x \in \mathfrak{X} \setminus U$; (iii) $0 \leq f(x) \leq 1$ for $x \in \mathfrak{X}$, and (iv) the boundary of the set $\{x : f(x) > 0\}$ has zero m measure.

Assumption A implies that there exists a kernel $J(x, dy) = J(x, y)m(dy)$ (satisfying $J(x, \{x\}) = 0$ for all $x \in \mathfrak{X}$) such that X satisfies the following Lévy system formula (see [16, p.482]): for every stopping time T ,

$$\mathbb{E}_x \sum_{s \in (0, T]} f(X_{s-}, X_s) = \mathbb{E}_x \int_0^T \int_{\mathfrak{X}} f(X_s, z) J(X_s, dz) ds. \quad (2.11)$$

Here $f : \mathfrak{X} \times \mathfrak{X} \rightarrow [0, \infty]$, $f(x, x) = 0$ for all $x \in \mathfrak{X}$. The kernel $J(x, dy) = J(x, y)m(dy)$ is called the jumping kernel of X .

Since J satisfies

$$\int_{\mathfrak{X}} f(y) J(x, dy) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x f(X_t)}{t} \quad (2.12)$$

for all bounded continuous function f on \mathfrak{X} and $x \in \mathfrak{X} \setminus \text{supp}(f)$, we have from (2.8) that

$$C_0^{-1} \frac{1}{V(x, \rho(x, y)) \Phi(\rho(x, y))} \leq J(x, y) \leq C_0 \frac{1}{V(x, \rho(x, y)) \Phi(\rho(x, y))}. \quad (2.13)$$

Similarly, \widehat{X} has a jumping kernel $\widehat{J}(x, dy) = \widehat{J}(x, y)m(dy)$ with $\widehat{J}(x, y) = J(y, x)$.

There are plenty of examples of processes satisfying the assumptions of this subsection. Reflected stable-like processes in a closed d -set $D \subset \mathbb{R}^d$ satisfy the assumptions of this subsection, see [16, 22]. Unimodal Lévy processes in \mathbb{R}^d with Lévy exponents satisfying weak upper and lower scaling conditions at infinity, in particular, isotropic stable processes, satisfy the assumptions of this subsection, see, for example, [13, 21]. Another typical example is given at the end of this section.

2.2 Preliminaries

In this section, we will argue that X and \widehat{X} satisfy the scale-invariant parabolic Harnack inequality with explicit scaling in terms of Φ . Note that X may not be symmetric and may not be conservative.

Let $\tau_U^X := \inf\{t > 0 : X_t \notin U\}$ be the first exit time from U for X . If D is an open subset of \mathfrak{X} , the killed process X^D is defined by $X_t^D = X_t$ if $t < \tau_D^X$ and $X_t^D = \partial$ if $t \geq \tau_D^X$, where ∂ is the cemetery point added to \mathfrak{X} .

Similarly, we define the killed process \widehat{X}^D . It is well known that X^D and \widehat{X}^D are strong duals of each other with respect to m_D , the restriction of m to D (see [16, p.481] and the end of [29, Section 13.6]). For $t > 0$, $x, y \in D$, define

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}_x[p(t - \tau_D^X, X_{\tau_D^X}, y) : \tau_D^X < t < \zeta^X], \quad (2.14)$$

where ζ^X is the lifetime of X . By the strong Markov property, $p_D(t, x, y)$ is the transition density of X^D and, by the continuity of $p(t, x, y)$, (2.10), the Feller and the strong Feller properties of X and \widehat{X} , it is easy to see that $p_D(t, x, y)$ is jointly continuous (see [30, pp.34–35] and [40, Lemma 2.2 and Proposition 2.3]).

The following lemma is basically [4, Lemma 3.8], except that we require neither symmetry nor conservativeness.

Lemma 2.4. *Suppose that there exist positive constants r, t and p such that*

$$\mathbb{P}_x(X_s \notin B(x, r), s < \zeta^X) \leq p, \quad x \in \mathfrak{X}, s \in [0, t]. \quad (2.15)$$

Then

$$\mathbb{P}_x\left(\sup_{0 \leq s \leq t} \rho(X_s, X_0) > 2r, t < \zeta^X\right) \leq 2p, \quad x \in \mathfrak{X}.$$

Proof. Let $S := \inf\{s > 0 : \rho(X_s, X_0) > 2r\}$ Then using the strong Markov property of X and (2.15),

$$\begin{aligned} \mathbb{P}_x\left(\sup_{0 \leq s \leq t} \rho(X_s, X_0) > 2r, t < \zeta^X\right) &= \mathbb{P}_x(S \leq t < \zeta^X) \\ &\leq \mathbb{P}_x(\rho(X_t, X_0) > r, t < \zeta^X) + \mathbb{P}_x(S \leq t < \zeta^X, \rho(X_t, X_0) \leq r) \\ &\leq p + \mathbb{P}_x(S \leq t < \zeta^X, \rho(X_t, X_0) \leq r) \\ &\leq p + \mathbb{E}_x[\mathbf{1}_{S \leq t < \zeta^X} \mathbb{P}_{X_S}(\rho(X_{t-S}, X_0) > r, t - S < \zeta^X)] \leq 2p. \end{aligned}$$

□

Combining this lemma with (2.10) and (2.14), we can repeat the argument of the proof of [17, Proposition 2.3] word for word to get the following result. Note that conservativeness is not needed.

Proposition 2.5. *For every $a > 0$, there exist constants $c > 0$ and $\varepsilon \in (0, 1/2)$ such that for all $x_0 \in \mathfrak{X}$ and $r \in (0, aR_0)$,*

$$p_{B(x_0, r)}(t, x, y) \geq \frac{c}{V(x, \Phi^{-1}(t))} \quad \text{for } x, y \in B(x_0, \varepsilon\Phi^{-1}(t)) \text{ and } t \in (0, \Phi(\varepsilon r)). \quad (2.16)$$

Proof. Note that by Remarks 2.2 and 2.3, there exists $c_1 \geq 1$ such that

$$c_1^{-1}\tilde{q}(t, y, x) \leq p(t, x, y) \leq c_1\tilde{q}(t, x, y), \quad (t, y, x) \in (0, \Phi(aR_0)) \times \mathfrak{X} \times \mathfrak{X}. \quad (2.17)$$

Let $\varepsilon \in (0, 1/2)$ be a small constant which will be chosen later. Observe that for every $x, y \in B(x_0, \varepsilon\Phi^{-1}(t))$ and $t \in (0, \Phi(\varepsilon r))$,

$$\rho(X_{\tau_{B(x_0, r)}^X}, y) \geq \rho(X_{\tau_{B(x_0, r)}^X}, x_0) - \rho(x_0, y) \geq r - \varepsilon\Phi^{-1}(t) \geq (\varepsilon^{-1} - \varepsilon)\Phi^{-1}(t).$$

Thus, by (2.14), (2.17), (2.6), (2.1) and (2.4), we have that for every $x, y \in B(x_0, \varepsilon\Phi^{-1}(t))$ and $t \in (0, \Phi(\varepsilon r))$,

$$\begin{aligned} p_{B(x_0, r)}(t, x, y) &= p(t, x, y) - \mathbb{E}_x [p(t - \tau_{B(x_0, r)}^X, X_{\tau_{B(x_0, r)}^X}, y) : \tau_{B(x_0, r)}^X < t < \zeta^X] \\ &\geq c_1^{-1}\tilde{q}(t, y, x) - c_1\mathbb{E}_x \left[\frac{t - \tau_{B(x_0, r)}^X}{V(y, \rho(X_{\tau_{B(x_0, r)}^X}, y))\Phi(\rho(X_{\tau_{B(x_0, r)}^X}, y))} : \tau_{B(x_0, r)}^X < t < \zeta^X \right] \\ &\geq c_1^{-1} \frac{1}{V(y, \Phi^{-1}(t))} - c_1 \frac{t}{V(y, (\varepsilon^{-1} - \varepsilon)\Phi^{-1}(t))\Phi((\varepsilon^{-1} - \varepsilon)\Phi^{-1}(t))} \\ &\geq (c_1^{-1} - c_1\tilde{C}_a\tilde{a}_l^{-1}(\varepsilon^{-1} - \varepsilon)^{-(d_0 + \delta_l)}) \frac{1}{V(y, \Phi^{-1}(t))}. \end{aligned}$$

Therefore, by choosing ε sufficiently small so that $c_1^{-1} - c_1\tilde{C}_a\tilde{a}_l^{-1}(\varepsilon^{-1} - \varepsilon)^{-(d_0 + \delta_l)} \geq 2^{-1}c_1^{-1}$ and using (2.1), we conclude the result as

$$p_{B(x_0, r)}(t, x, y) \geq 2^{-1}c_1^{-1} \frac{1}{V(y, \Phi^{-1}(t))} \geq c_2 \frac{1}{V(x, \Phi^{-1}(t))}.$$

□

Let $\Xi_s := (V_s, X_s)$ be the time-space process of X , where $V_s = V_0 - s$. The law of the time-space process $s \mapsto \Xi_s$ starting from (t, x) will be denoted as $\mathbb{P}^{(t, x)}$.

Definition 2.6. A non-negative Borel function $h(t, x)$ on $\mathbb{R} \times \mathfrak{X}$ is said to be *parabolic* (or *caloric*) on $(a, b] \times B(x_0, r)$ with respect to X if for every relatively compact open subset U of $(a, b] \times B(x_0, r)$, $h(t, x) = \mathbb{E}^{(t, x)}[h(\Xi_{\tau_{\bar{U}}})] = \mathbb{E}^{(t, x)}[h(\Xi_{\tau_{\bar{U}}}^{\Xi}) : \tau_{\bar{U}}^{\Xi} < \zeta^X]$ for every $(t, x) \in U \cap ([0, \infty) \times \mathfrak{X})$, where $\tau_{\bar{U}}^{\Xi} := \inf\{s > 0 : \Xi_s \notin U\}$.

Theorem 2.7. For every $a > 0$, there exist $c > 0$ and $c_1, c_2 \in (0, 1)$ depending on d, \tilde{T} and a such that for all $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $R \in (0, aR_0)$ and every non-negative function u on $[0, \infty) \times \mathbb{R}^d$ that is parabolic on $(t_0, t_0 + 4c_1\Phi(R)) \times B(x_0, R)$ with respect to X or \hat{X} ,

$$\sup_{(t_1, y_1) \in Q_-} u(t_1, y_1) \leq c \inf_{(t_2, y_2) \in Q_+} u(t_2, y_2),$$

where $Q_- = (t_0 + c_1\Phi(R), t_0 + 2c_1\Phi(R)) \times B(x_0, c_2R)$ and $Q_+ = [t_0 + 3c_1\Phi(R), t_0 + 4c_1\Phi(R)] \times B(x_0, c_2R)$.

Proof. Note that by (2.13), there exists $c > 0$ such that for all $x \neq y \in \mathfrak{X}$ with $r \leq \rho(x, y)/2 < R_0$,

$$J(x, y) \leq \frac{c}{V(x, r)} \int_{B(x, r)} J(z, y) m(dz) \text{ and } J(x, y) \leq \frac{c}{V(y, r)} \int_{B(x, r)} J(y, z) m(dz).$$

Thus, using this, Proposition 2.5 and (2.8), we see that the proof of Theorem 2.7 is almost identical to the proof for the symmetric case in [26, Theorem 4.3]. We emphasize that the conservativeness is not used in the proofs of [26, Lemmas 3.7, 4.1, 4.2 and Theorem 4.3]. We omit the details. \square .

Theorem 2.7 clearly implies the elliptic Harnack inequality. Using Theorem 2.7, we have the following result. In the remainder of this section, D will always stand for an open subset of \mathfrak{X} .

Proposition 2.8. *For all $a, b > 0$, there exists $c = c(a, b) > 0$ such that for every open set $D \subset \mathfrak{X}$,*

$$p_D(t, x, y) \geq \frac{c}{V(x, \Phi^{-1}(t))}$$

for all $(t, x, y) \in (0, aR_0) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq b\Phi^{-1}(t) \geq 4\rho(x, y)$.

Proof. See the proof of [38, Proposition 3.4]. \square

The proof of the next result is also standard.

Proposition 2.9. *For all $a, b > 0$, there exists $c = c(a, b) > 0$ such that for every open set $D \subset \mathfrak{X}$, $p_D(t, x, y) \geq ctJ(x, y)$ for all $(t, x, y) \in (0, aR_0) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq b\Phi^{-1}(t)$ and $b\Phi^{-1}(t) \leq 4\rho(x, y)$.*

Proof. See the proofs of [21, Lemma 3.4 and Proposition 3.5]. \square

Combining the two propositions above we get

Proposition 2.10. *For all $a, b > 0$, there exists $c = c(a, b) > 0$ such that for every open set $D \subset \mathfrak{X}$, $p_D(t, x, y) \geq c\tilde{q}(t, x, y)$ for all $t \in (0, aR_0)$, $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq b\Phi^{-1}(t)$.*

2.3 3P inequality and Feynman-Kac perturbations

We first prove the following 3P inequality.

Lemma 2.11. *For every $a \in (0, \infty)$, there exists $c > 0$ such that for all $0 < s < t < aR_0$,*

$$\frac{\tilde{q}(s, x, z)\tilde{q}(t-s, z, y)}{\tilde{q}(t, x, y)} \leq c(\tilde{q}(s, x, z) + \tilde{q}(t-s, z, y)), \quad x, y, z \in \mathfrak{X}. \quad (2.18)$$

Proof. Note that, by the triangle inequality, either $\rho(x, z) \geq 2^{-1}\rho(x, y)$ or $\rho(z, y) \geq 2^{-1}\rho(x, y)$. Since the argument is the same, we only give the proof for the case $\rho(z, y) \geq 2^{-1}\rho(x, y)$. Thus we assume that $\rho(z, y) \geq 2^{-1}\rho(x, y)$ and set

$$\bar{q}(t, x, y, r) := tV(x, \Phi^{-1}(t)) + \Phi(r)V(y, r), \quad t, r > 0 \text{ and } x, y \in \mathfrak{X}. \quad (2.19)$$

Note that by Remark 2.2 and the fact $1 \wedge (1/r) \asymp 1/(1+r)$ for $r > 0$, we have that for all $t > 0$ and $x, y \in \mathfrak{X}$,

$$\tilde{q}(t, x, y) \asymp \frac{t}{\bar{q}(t, x, x, \rho(x, y))} \quad (2.20)$$

and

$$\bar{q}(t, x, x, \rho(x, y)) \asymp \bar{q}(t, y, y, \rho(x, y)) \asymp \bar{q}(t, x, y, \rho(x, y)) \asymp \bar{q}(t, y, x, \rho(x, y)). \quad (2.21)$$

We claim that

$$\bar{q}(t, x, x, \rho(x, y)) \leq c(\bar{q}(t-s, y, y, \rho(z, y)) + \bar{q}(s, x, x, \rho(x, z))). \quad (2.22)$$

The assertion of the lemma follows easily from (2.22). Indeed, (2.22) implies that

$$\begin{aligned} \frac{\tilde{q}(s, x, z)\tilde{q}(t-s, z, y)}{\tilde{q}(t, x, y)} &\asymp \frac{s(t-s)}{t} \frac{\bar{q}(t, x, x, \rho(x, y))}{\bar{q}(t-s, y, y, \rho(z, y))\bar{q}(s, x, x, \rho(x, z))} \\ &\leq c(s \wedge (t-s)) \left(\frac{1}{\bar{q}(s, x, x, \rho(x, z))} + \frac{1}{\bar{q}(t-s, y, y, \rho(z, y))} \right) \\ &\leq c \left(\frac{s}{\bar{q}(s, x, x, \rho(x, z))} + \frac{t-s}{\bar{q}(t-s, y, y, \rho(z, y))} \right) \asymp \tilde{q}(s, x, x, \rho(x, z)) + \tilde{q}(t-s, y, y, \rho(z, y)). \end{aligned}$$

We now prove the claim (2.22) by considering two cases separately.

(1) $\rho(x, y) \leq \Phi^{-1}(t)$: In this case, $\bar{q}(t, x, x, \rho(x, y)) \asymp tV(y, \Phi^{-1}(t)) \asymp tV(x, \Phi^{-1}(t))$. By (2.1) and (2.5), if $t/2 \geq s$,

$$tV(y, \Phi^{-1}(t)) \leq c(t-s)V(y, \Phi^{-1}(t-s)) \leq c[\bar{q}(t-s, y, y, \rho(z, y)) + \bar{q}(s, x, x, \rho(x, z))].$$

Similarly, if $t/2 < s$,

$$tV(x, \Phi^{-1}(t)) \leq csV(x, \Phi^{-1}(s)) \leq c(\bar{q}(t-s, y, y, \rho(z, y)) + \bar{q}(s, x, x, \rho(x, z))).$$

Thus, by (2.21), (2.22) holds true.

(2) $\rho(x, y) > \Phi^{-1}(t)$: The assumption $\rho(z, y) \geq 2^{-1}\rho(x, y)$ and (2.1) imply that

$$\frac{V(x, 2\rho(y, z))}{V(y, \rho(y, z))} \leq \frac{V(y, 2\rho(x, y) + 2\rho(y, z))}{V(y, \rho(y, z))} \leq \frac{V(y, 6\rho(y, z))}{V(y, \rho(y, z))} \leq c. \quad (2.23)$$

Combining (2.23) and (2.1) we get that for $x, y, z \in D$ and $0 < s < t < aR_0$,

$$\begin{aligned} \bar{q}(t, x, x, \rho(x, y)) &\asymp \Phi(\rho(x, y)) V(x, \rho(x, y)) \\ &\leq \Phi(\rho(x, z) + \rho(z, y)) V(x, \rho(x, z) + \rho(z, y)) \\ &\leq \Phi(2(\rho(x, z) \vee \rho(z, y))) V(x, 2(\rho(x, z) \vee \rho(z, y))) \\ &\leq \Phi(2\rho(z, y)) V(x, 2\rho(z, y)) + \Phi(2\rho(x, z)) V(x, 2\rho(x, z)) \\ &\leq c[\Phi(\rho(z, y)) V(y, \rho(z, y)) + \Phi(\rho(x, z)) V(x, \rho(x, z))] \\ &\leq c[\bar{q}(t-s, y, y, \rho(z, y)) + \bar{q}(s, x, x, \rho(x, z))]. \end{aligned}$$

We have proved (2.22). \square

Recall that, for an open set $D \subset \mathfrak{X}$, a measure μ on D is said to be a smooth measure of X^D with respect to the reference measure m_D if there is a positive continuous additive functional A of X^D such that for any bounded non-negative Borel function f on D ,

$$\int_D f(x)\mu(dx) = \lim_{t \downarrow 0} \mathbb{E}_m \left[\frac{1}{t} \int_0^t f(X_s^D) dA_s \right], \quad (2.24)$$

cf. [45]. The additive functional A is called the positive continuous additive functional of X^D with Revuz measure μ with respect to the reference measure m_D .

It is known (see [34]) that for any $x \in D$, $\alpha \geq 0$ and bounded non-negative Borel function f on D ,

$$\mathbb{E}_x \int_0^\infty e^{-\alpha t} f(X_t^D) dA_t = \int_0^\infty e^{-\alpha t} \int_D p_D(t, x, y) f(y) \mu(dy) dt,$$

and we have for any $x \in D$, $t > 0$ and non-negative Borel function f on D ,

$$\mathbb{E}_x \int_0^t f(X_s^D) dA_s = \int_0^t \int_D p_D(s, x, y) f(y) \mu(dy) ds. \quad (2.25)$$

We first introduce our class of possibly critical perturbations. For an open set $D \subset \mathfrak{X}$, a smooth Radon measure μ of X^D , $t > 0$ and $a \geq 0$, we define

$$N_a^{D, \mu}(t) := \sup_{x \in \mathfrak{X}} \int_0^t \int_{z \in D: \delta_D(z) > a\Phi^{-1}(t)} \tilde{q}(s, x, z) \mu(dz) ds. \quad (2.26)$$

Definition 2.12. Let μ be a smooth measure for both X^D and \widehat{X}^D with respect to the reference measure m_D and let $T \in (0, \infty]$. The measure μ is said to be in the class $\mathbf{K}_T(D)$ if

- (1) $\sup_{t < T} N_a^{D, \mu}(t) < \infty$ for all $a \in (0, 1]$;
- (2) $\lim_{t \rightarrow 0} N_0^{U, \mu}(t) = 0$ for every relatively compact open set U of D .

For $\mu \in \mathbf{K}_T(D)$, using condition (2) in the definition above, one can show that, for any relatively compact open subset U of D , $A_{t \wedge \tau_U^X}$ is a positive continuous additive functional of X^U with Revuz measure μ_U , where μ_U is the measure μ restricted to U . See Appendix for the proof.

Remark 2.13. Note that by the semigroup property, it is easy to check that

$$N_a^{D, \mu}(t) \leq N_a^{D, \mu}(s) + \overline{C}_0(T)^2 N_a^{D, \mu}(t - s), \quad 0 < s < t \leq T,$$

where $\overline{C}_0(T)$ is the constant in (2.10). Thus, if μ is in the class $\mathbf{K}_1(D)$, then $\sup_{t < T} N_a^{D, \mu}(t) < \infty$ for all $a > 0$ and $T \in (0, \infty)$.

For $\mu \in \mathbf{K}_1(D)$, we denote by A_t^μ the positive continuous additive functional of X^D with Revuz measure μ and denote by \widehat{A}_t^μ the positive continuous additive functional of \widehat{X}^D with Revuz measure μ . For any non-negative Borel function f on D , we define

$$T_t^{\mu, D} f(x) = \mathbb{E}_x \left[\exp(-A_t^\mu) f(X_t^D) \right], \quad \widehat{T}_t^{\mu, D} f(x) = \widehat{\mathbb{E}}_x \left[\exp(-\widehat{A}_t^\mu) f(\widehat{X}_t^D) \right], \quad t \geq 0, x \in D.$$

The semigroup $(T_t^{\mu,D} : t \geq 0)$ (respectively $(\widehat{T}_t^{\mu,D} : t \geq 0)$) is called the Feynman-Kac semigroup of X^D (respectively \widehat{X}^D) associated with μ . By [51, Theorem 6.10(2)], $T_t^{\mu,D}$ and $\widehat{T}_t^{\mu,D}$ are duals of each other with respect to the measure m_D so that

$$\int_D T_t^{\mu,D} f(x) g(x) m(dx) = \int_D f(x) \widehat{T}_t^{\mu,D} g(x) m(dx). \quad (2.27)$$

Let Y (\widehat{Y} , respectively) be a Hunt process on D corresponding to the transition semigroup $(T_t^{\mu,D})$ ($(\widehat{T}_t^{\mu,D})$, respectively). For an open subset $U \subset D$, we denote by Y^U (\widehat{Y}^U , respectively) the process Y (\widehat{Y} , respectively) killed upon exiting U .

Suppose that $U \subset D$ is a relatively compact open subset of D . Since for any relatively compact open set U , $A_t^{\mu,U} := A_{t \wedge \tau_U^X}^\mu$ is a positive continuous additive functional of X^U with Revuz measure μ_U , the transition semigroup of Y^U is $(T_t^{\mu,U})$. For simplicity, in the sequel we denote this semigroup as $(T_t^{\mu,U})$. Moreover, for any $t \geq 0, x \in U$,

$$T_t^{\mu,U} f(x) = \mathbb{E}_x [f(Y_t^U)] = \mathbb{E}_x \left[\exp \left(- A_{t \wedge \tau_U^X}^\mu \right) f(X_t^U) \right]$$

and

$$\mathbb{E}_x \int_0^t f(X_s^U) dA_s^{\mu,U} = \int_0^t \int_U p_U(s, x, y) f(y) \mu(dy) ds.$$

It follows from Definition 2.12(2) that, for all relatively compact open subset U of D ,

$$\limsup_{t \rightarrow 0} \int_0^t \int_U p_U(s, x, y) \mu(dy) ds = 0,$$

i.e., μ_U is in the standard Kato class of X^U . Thus, according to the discussion in [22, Section 1.2], we have for any non-negative bounded Borel function f on U ,

$$T_t^{\mu,U} f(x) = \mathbb{E}_x [f(X_t^U)] + \mathbb{E}_x \left[f(X_t^U) \sum_{n=1}^{\infty} (-1)^n \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} dA_{s_1}^{\mu,U} \cdots dA_{s_{n-1}}^{\mu,U} dA_{s_n}^{\mu,U} \right].$$

Define $p_U^0(t, x, y) := p_U(t, x, y)$ and, for $k \geq 1$,

$$p_U^k(t, x, y) = - \int_0^t \int_U p_U(s, x, z) p_U^{k-1}(t-s, z, y) \mu(dz) ds. \quad (2.28)$$

Repeating the discussion in [22, Section 1.2], one can conclude that

$$T_t^{\mu,U} f(x) = \int_U q^U(t, x, y) f(y) m(dy), \quad (t, x) \in (0, \infty) \times U,$$

where $q^U(t, x, y) := \sum_{k=0}^{\infty} p_U^k(t, x, y)$.

By Lemma 2.11, we have that for any μ in $\mathbf{K}_1(D)$, any relatively compact open set U of D and any $(t, x, y) \in (0, 1] \times U \times U$,

$$\int_0^t \int_U \tilde{q}(t-s, x, z) \tilde{q}(s, z, y) \mu(dz) ds$$

$$\leq c\tilde{q}(t, x, y) \sup_{u \in \mathfrak{X}} \int_0^t \int_U \tilde{q}(s, u, z) \mu(dz) ds = c\tilde{q}(t, x, y) N_0^{U, \mu}(t). \quad (2.29)$$

Using (2.29) and the semigroup property, it is standard to show that $p_U^k(t, x, y)$ is continuous in (t, y) for each fixed x , continuous in (t, x) for each fixed y , and $\sum_{k=0}^{\infty} p_U^k(t, x, y)$ converges absolutely and uniformly so that $q^U(t, x, y)$ is continuous in (t, y) for each fixed x , and also continuous in (t, x) for each fixed y (for example, see [22]).

Define $q^D(t, x, y) := \lim_{n \rightarrow \infty} q^{D_n}(t, x, y)$, where $D_n \subset D$ are bounded increasing open sets such that $\overline{D_n} \subset D_{n+1}$ and $\cup_{n=1}^{\infty} D_n = D$. Then, using the monotone convergence theorem and

$$q^{D_n}(t, x, y) \leq p_{D_n}(t, x, y) \leq p(t, x, y) \leq \overline{C}_0(T) \tilde{q}(t, x, y), \quad t < T,$$

we see that $q^D(t, x, y)$ is the transition density of Y and $q^D(t, x, y) \leq \overline{C}_0(T) \tilde{q}(t, x, y)$ for $t < T$. Therefore, we obtain the following

Proposition 2.14. *Suppose that D is an open set in \mathfrak{X} and $\mu \in \mathbf{K}_1(D)$. Then the Hunt process Y on D corresponding to the transition semigroup $(T_t^{\mu, D})$ has a transition density $q^D(t, x, y)$ with respect to m such that for each $T \in (0, \infty)$, $q^D(t, x, y) \leq \overline{C}_0(T) \tilde{q}(t, x, y)$ for $t < T$. Furthermore, if D is relatively compact, then $q^D(t, x, y)$ is continuous in (t, y) for each fixed x , and continuous in (t, x) for each fixed y . If $\mu \in \mathbf{K}_{\infty}(D)$ and $\tilde{T} = \infty$, then the estimate $q^D(t, x, y) \leq c\tilde{q}(t, x, y)$ holds for every $t > 0$.*

2.4 Interior estimates

In this subsection, we prove some interior estimates for the transition density $q^U(t, x, y)$, where U is an open subset of D . Recall that we assume $R_0 = m(\mathfrak{X}) = \infty$ when $\tilde{T} = \infty$.

Theorem 2.15. *Suppose that $\mu \in \mathbf{K}_1(D)$. Then for every $T \in (0, \infty)$ and $a \in (0, 1]$, there exists a constant $c := c(a, \Phi, C_0, M, \sup_{t \leq T} N_{2^{-1}a}^{D, \mu}(t)) > 0$ such that for every open $U \subset D$,*

$$q^U(t, x, y) \geq c\tilde{q}(t, x, y) \quad (2.30)$$

for all $t \in (0, T)$, $x, y \in U$ with $\delta_U(x) \wedge \delta_U(y) \geq a\Phi^{-1}(t)$. Moreover, if $\mu \in \mathbf{K}_{\infty}(D)$ and $\tilde{T} = \infty$, then (2.30) holds for all $t > 0$.

Proof. Fix $t \in (0, T)$, $x, y \in U$ with $\delta_U(x) \wedge \delta_U(y) \geq a\Phi^{-1}(t)$ and choose a bounded open set $V \subset \{z \in U : \delta_U(z) > 2^{-1}a\Phi^{-1}(t)\}$ containing x, y such that $\delta_V(x) \wedge \delta_V(y) \geq 2^{-2}a\Phi^{-1}(t)$ (for example, one can take $V = \{z \in U : \delta_U(z) > 2^{-1}a\Phi^{-1}(t)\} \cap B(x, 2(\rho(x, y) + \Phi^{-1}(t)))$). Note that $q^U(t, x, w) \geq q^V(t, x, w)$ for all $w \in V$ and $w \mapsto q^V(t, x, w)$ is continuous.

For $w \in V$, let

$$\tilde{p}_V^1(t, x, w) := \int_0^t \left(\int_V p_V(t-s, x, z) p_V(s, z, w) \mu(dz) \right) ds.$$

Then for any bounded Borel function f on V , by the Markov property of X^V , we have

$$\mathbb{E}_x [A_t^{\mu V} f(X_t^V)] = \mathbb{E}_x \left[\int_0^t \mathbb{E}_{X_s^V} [f(X_{t-s}^V)] dA_s^{\mu V} \right] = \int_V \tilde{p}_V^1(t, x, w) f(w) m(dw). \quad (2.31)$$

Observe that, since $\delta_D(z) \geq \delta_U(z) > 2^{-1}a\Phi^{-1}(t)$ for $z \in V$, by (2.14), (2.8), Lemma 2.11 and Proposition 2.10, we have that for $w \in B(y, 2^{-3}a\Phi^{-1}(t))$,

$$\begin{aligned} \tilde{p}_V^1(t, x, w) &\leq \bar{C}_0^2 \int_0^t \int_V \tilde{q}(t-s, x, z) \tilde{q}(s, z, w) \mu(dz) ds \\ &\leq \bar{C}_0^2 \int_0^t \int_{z \in D: \delta_D(z) > 2^{-1}a\Phi^{-1}(t)} \tilde{q}(t-s, x, z) \tilde{q}(s, z, w) \mu(dz) ds \\ &\leq \bar{C}_0^2 c \left(\sup_{s \leq T} N_{2^{-1}a}^{D, \mu}(s) \right) \tilde{q}(t, x, w) \leq \bar{C}_0^2 c \left(\sup_{s \leq T} N_{2^{-1}a}^{D, \mu}(s) \right) C_*^{-1} p_V(t, x, w) \\ &=: (k/2) p_V(t, x, w). \end{aligned}$$

Hence, for $w \in B(y, 2^{-3}a\Phi^{-1}(t))$, we have $p_V(t, x, w) - k^{-1} \tilde{p}_V^1(t, x, w) \geq 2^{-1} p_V(t, x, w)$, which implies that for any $r < 2^{-3}a\Phi^{-1}(t)$,

$$\frac{1}{2} \mathbb{E}_x [\mathbf{1}_{B(y, r)}(X_t^V)] \leq \mathbb{E}_x [(1 - A_t^{\mu_V}/k) \mathbf{1}_{B(y, r)}(X_t^V)]. \quad (2.32)$$

Using the elementary fact that $1 - A_t^{\mu_V}/k \leq \exp(-A_t^{\mu_V}/k)$, we get that for any $r < 2^{-3}a\Phi^{-1}(t)$,

$$\frac{1}{V(y, r)} \mathbb{E}_x [(1 - A_t^{\mu_V}/k) \mathbf{1}_{B(y, r)}(X_t^V)] \leq \frac{1}{V(y, r)} \mathbb{E}_x [\exp(-A_t^{\mu_V}/k) \mathbf{1}_{B(y, r)}(X_t^V)].$$

Thus, by (2.32), (2.31) and Hölder's inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{1}{V(y, r)} \mathbb{E}_x [\mathbf{1}_{B(y, r)}(X_t^V)] &\leq \frac{1}{V(y, r)} \mathbb{E}_x [\exp(-A_t^{\mu_V}/k) \mathbf{1}_{B(y, r)}(X_t^V)] \\ &\leq \left(\frac{1}{V(y, r)} \mathbb{E}_x [\exp(-A_t^{\mu_V}) \mathbf{1}_{B(y, r)}(X_t^V)] \right)^{1/k} \left(\frac{1}{V(y, r)} \mathbb{E}_x [\mathbf{1}_{B(y, r)}(X_t^V)] \right)^{1-1/k}. \end{aligned}$$

Therefore,

$$\frac{1}{2^k} \frac{1}{V(y, r)} \mathbb{E}_x [\mathbf{1}_{B(y, r)}(X_t^V)] \leq \frac{1}{V(y, r)} \mathbb{E}_x [\exp(-A_t^{\mu_V}) \mathbf{1}_{B(y, r)}(X_t^V)].$$

Since $w \rightarrow q^V(t, x, w)$ is continuous by Proposition 2.14, we conclude by sending $r \downarrow 0$ and applying Proposition 2.10 again that for every $t \in (0, T]$, $x, y \in U$ with $\delta_U(x) \wedge \delta_U(y) \geq a\Phi^{-1}(t)$,

$$q^U(t, x, y) \geq q^V(t, x, y) \geq 2^{-k} p_V(t, x, y) \geq c 2^{-k} \tilde{q}(t, x, y).$$

□

Let $\tau_U := \inf\{s > 0 : Y_s \notin U\}$ and $\hat{\tau}_U := \inf\{s > 0 : \hat{Y}_s \notin U\}$. Since the proofs for the dual processes are same, throughout the paper we give the proofs for Y only.

Corollary 2.16. (1) Suppose that $\mu \in \mathbf{K}_1(D)$. For any positive constants R, T and a , there exists $c_1 = c_1(a, T) > 0$ such that for all $t \in (0, T)$ and $B(x, \Phi^{-1}(t)) \subset D$,

$$\inf_{z \in B(x, a\Phi^{-1}(t)/2)} \mathbb{P}_z(\tau_{B(x, a\Phi^{-1}(t))} > t) \wedge \inf_{z \in B(x, a\Phi^{-1}(t)/2)} \hat{\mathbb{P}}_z(\hat{\tau}_{B(x, a\Phi^{-1}(t))} > t) \geq c_1 \quad (2.33)$$

and

$$\mathbb{E}_x[\tau_{B(x,r)}] \wedge \widehat{\mathbb{E}}_x[\widehat{\tau}_{B(x,r)}] \geq c_1\Phi(r). \quad (2.34)$$

Moreover, there exist $r_1, c_2 > 0$ such that for all $r \in (0, r_1]$ and $B(x, r) \subset D$,

$$\mathbb{E}_x[\tau_{B(x,r)}] \vee \widehat{\mathbb{E}}_x[\widehat{\tau}_{B(x,r)}] \leq c_2\Phi(r). \quad (2.35)$$

(2) If $\mu \in \mathbf{K}_\infty(D)$ and $\widetilde{T} = \infty$ (and $R_0 = \infty$), then (2.33)–(2.35) hold for all $r, t > 0$.

Proof. (1) For any $z \in B(x, a\Phi^{-1}(t)/2)$, we have by Theorem 2.15 that

$$\begin{aligned} \mathbb{P}_z(\tau_{B(x, a\Phi^{-1}(t))} > t) &\geq \mathbb{P}_z(\tau_{B(z, a\Phi^{-1}(t)/6)} > t) \\ &= \int_{B(z, a\Phi^{-1}(t)/6)} q^{B(z, a\Phi^{-1}(t)/6)}(t, z, y)m(dy) \\ &\geq c \int_{B(z, a\Phi^{-1}(t)/12)} \frac{1}{V(z, \Phi^{-1}(t))} \wedge \frac{t}{V(z, \rho(z, y))\Phi(\rho(z, y))} m(dy) \geq c_0. \end{aligned}$$

(2.34) is clear from (2.33). In fact,

$$\mathbb{E}_x[\tau_{B(x,r)}] \geq \Phi(r)\mathbb{P}_x(\tau_{B(x,r)} > \Phi(r)) \geq c_1\Phi(r).$$

We now prove (2.35). By the semigroup property and Proposition 2.14, we have that for $t > 2^k s$ and $s \leq 1$,

$$\begin{aligned} q^{B(x,r)}(t, x, y) &= \int_{B(x,r)} q^{B(x,r)}(t - 2^k s, x, z)q^{B(x,r)}(2^k s, z, y)m(dz) \\ &\leq \sup_{z, y \in B(x,r)} q^{B(x,r)}(2^k s, z, y) = \sup_{z, y \in B(x,r)} \int_{B(x,r)} q^{B(x,r)}(2^{k-1} s, z, w)q^{B(x,r)}(2^{k-1} s, w, y)dw \\ &\leq \left(\sup_{z, y \in B(x,r)} q^{B(x,r)}(2^{k-1} s, z, y) \right)^2 V(x, r) \\ &\leq \dots \leq \left(\sup_{z, y \in B(x,r)} q^{B(x,r)}(s, z, y) \right)^{2^k} V(x, r)^{2^{k-1} + \dots + 2 + 1} \\ &\leq \left(\sup_{z \in B(x,r)} \frac{c}{V(z, \Phi^{-1}(s))} \right)^{2^k} V(x, r)^{2^k - 1}. \end{aligned}$$

Using this, we have that for all $A > 0$ and $r \leq \Phi^{-1}(1/A)$,

$$\begin{aligned} \mathbb{E}_x[\tau_{B(x,r)}] &\leq A\Phi(r) + \sum_{k=0}^{\infty} \int_{A2^k\Phi(r)}^{A2^{k+1}\Phi(r)} \int_{B(x,r)} q^{B(x,r)}(t, x, y)m(dy)dt \\ &\leq A\Phi(r) \left(1 + \sum_{k=0}^{\infty} 2^k \left(\sup_{z, y \in B(x,r), t \geq A2^k\Phi(r)} q^{B(x,r)}(t, z, y) \right) V(x, r) \right) \\ &\leq A\Phi(r) \left(1 + \sum_{k=0}^{\infty} 2^k \left(\sup_{z \in B(x,r)} \frac{cV(x, r)}{V(z, \Phi^{-1}(A\Phi(r)))} \right)^{2^k} \right). \end{aligned}$$

Using (2.1) and (2.5), for all $z \in B(x, r)$, $A > 1$ and $r \leq \Phi^{-1}(1/A)$,

$$\begin{aligned} \frac{cV(x, r)}{V(z, \Phi^{-1}(A\Phi(r)))} &= c \frac{V(x, \Phi^{-1}(A\Phi(r)))}{V(z, \Phi^{-1}(A\Phi(r)))} \frac{V(x, r)}{V(x, \Phi^{-1}(A\Phi(r)))} \\ &\leq c \frac{V(z, r + \Phi^{-1}(A\Phi(r)))}{V(z, \Phi^{-1}(A\Phi(r)))} c_2 \left(\frac{r}{\Phi^{-1}(A\Phi(r))} \right)^{d_0} \\ &\leq c \frac{V(z, 2\Phi^{-1}(A\Phi(r)))}{V(z, \Phi^{-1}(A\Phi(r)))} c_2 \left(\frac{\Phi^{-1}(\Phi(r))}{\Phi^{-1}(A\Phi(r))} \right)^{d_0} \leq c_3 2^d c_2 c_4 A^{-d_0/\delta_u}. \end{aligned}$$

Choose $A > 1$ large so that $\varepsilon := c_3 2^d c_2 c_4 A^{-d_0/\delta_u} < 1$. Then for $r \leq r_1 := \Phi^{-1}(1/A)$

$$\mathbb{E}_x[\tau_{B(x, r)}] \leq A\Phi(r) \left(1 + \sum_{k=0}^{\infty} 2^k \varepsilon^{2^k} \right) \leq c\Phi(r).$$

(2) The proof is similar to the proof of (1). We omit the details. \square

2.5 Examples of critical potentials

In this subsection, we give two examples of critical potentials.

Example 2.17. Suppose $\mu(dz) = q(z)m(dz)$ with $0 \leq q(z) \asymp 1/\Phi(\delta_D(z) \wedge 1)$. Since q is bounded on every relatively compact open set $U \subset D$, $N_0^{U, \mu}(t) \leq C_0 t \|q\|_{L^\infty(U)} \rightarrow 0$ as $t \rightarrow 0$. Moreover, for $x \in D$, $a \in (0, 1]$ and $t < 1$,

$$\begin{aligned} &\int_0^t \int_{z \in D: \delta_D(z) > a\Phi^{-1}(t)} \tilde{q}(s, x, z) q(z) m(dz) ds \\ &\leq ct + c \int_0^t \int_{z \in D: 1 > \delta_D(z) > a\Phi^{-1}(t)} \Phi(\delta_D(z))^{-1} \tilde{q}(s, x, z) m(dz) ds \\ &\leq ct + c \frac{1}{\Phi(a\Phi^{-1}(t))} \int_0^t \int_D \tilde{q}(s, x, z) m(dz) ds \leq ct + c \frac{t}{\Phi(a\Phi^{-1}(t))} < c < \infty. \end{aligned} \quad (2.36)$$

Thus $\sup_{t < 1} N_a^{D, \mu}(t) < \infty$ for all $a \in (0, 1]$ and so μ is in the class $\mathbf{K}_1(D)$.

Example 2.18. Suppose $\tilde{T} = \infty$ and $\mu(dz) = q(z)m(dz)$ with $0 \leq q(z) \asymp 1/\Phi(\delta_D(z))$. Then $R_0 = \infty$ and for all $a \in (0, 1]$ and $t < \infty$,

$$N_a^{D, \mu}(t) \leq c \frac{1}{\Phi(a\Phi^{-1}(t))} \sup_{x \in \mathfrak{X}} \int_0^t \int_D \tilde{q}(s, x, z) m(dz) ds \leq \frac{ct}{\Phi(a\Phi^{-1}(t))} < c < \infty. \quad (2.37)$$

Thus μ is in the class $\mathbf{K}_\infty(D)$.

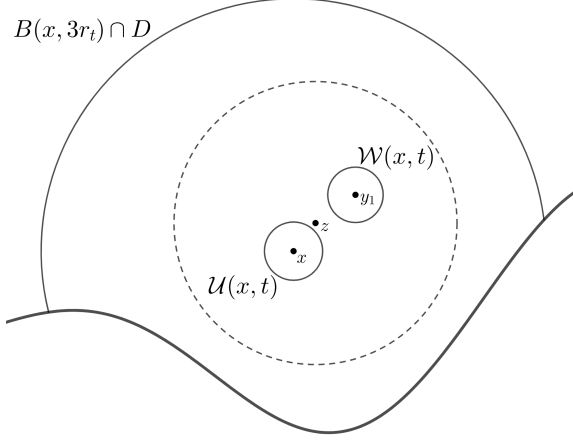


Figure 1. $\rho(x, z) \leq 3\kappa r_t/2$.

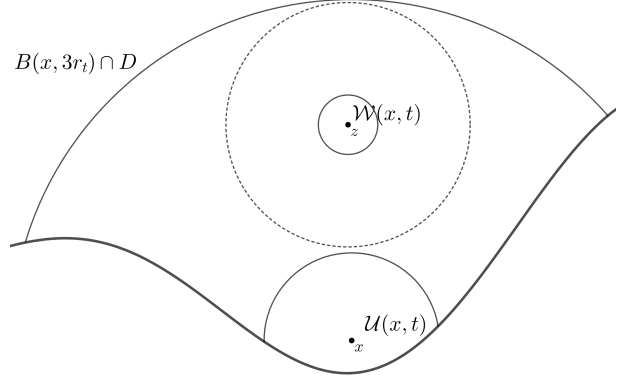


Figure 2. $\rho(x, z) > 3\kappa r_t/2$.

2.6 Factorization of Dirichlet heat kernel in κ -fat open set

Recall that $\mathcal{D}(K, U)$ is the subset of \mathcal{D} in Assumption **A**. Let

$$A(z_0, p, q) := \{x \in \mathfrak{X} : p < \rho(x, z_0) < q\} \quad \text{and} \quad \bar{A}(z_0, p, q) := \{x \in \mathfrak{X} : p \leq \rho(x, z_0) \leq q\}.$$

Note that, due to our assumption that all bounded closed sets are compact, $\bar{A}(z_0, p, q)$ is compact. Thus by Assumption **A**, for any $1/2 < b < a < 1$, the set $\mathcal{D}(\bar{A}(z_0, br, ar), A(z_0, r/2, r))$ is non-empty. We now add the final assumption saying that there exist proper bump functions in each non-empty set $\mathcal{D}(\bar{A}(z_0, br, ar), A(z_0, r/2, r))$ providing scale-invariant control on the action of the generator.

Assumption U: There exists $r_0 \in (0, \infty]$ such that for any $1/2 < b < a < 1$, there exists $c = c(a, b)$ such that for every $z_0 \in \mathfrak{X}$ and $r < r_0$,

$$\inf_{f \in \mathcal{D}(\bar{A}(z_0, br, ar), A(z_0, r/2, r))} \sup_{x \in \mathfrak{X}} \max(\mathcal{L}f(x), \widehat{\mathcal{L}}f(x)) \leq \frac{c}{\Phi(r)}. \quad (2.38)$$

This assumption is used in connection with Dynkin's formula in Lemma 2.20 to get a scale-invariant estimate of the exit probability.

Definition 2.19. Let $0 < \kappa \leq 1/2$. We say that an open set D is κ -fat if there is $R_1 \in (0, \infty]$ such that for all $x \in \bar{D}$ and all $r \in (0, R_1)$, there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$. The pair (R_1, κ) is called the characteristics of the κ -fat open set D .

In the remainder of this subsection, $T > 0$ is a fixed constant and D is a fixed κ -fat open set with characteristics (R_1, κ) . Without loss of generality, we can assume that $R_1 \leq R_0 \wedge r_0 \wedge r_1$, where r_1 is the constant in Corollary 2.16(1). For $(t, x) \in (0, T] \times D$, set $r_t = \Phi^{-1}(t)R_1/(3\Phi^{-1}(T)) \leq R_1/3$. An open neighborhood $\mathcal{U}(x, t)$ of $x \in D$ and an open ball $\mathcal{W}(x, t) \subset D \setminus \mathcal{U}(x, t)$ are defined as follows:

By the definition of κ -fat open set, we can find $z = z_{x,t} \in D$ such that $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$.

(i) If $\rho(x, z) \leq 3\kappa r_t/2$, we choose $y_1 \in \mathfrak{X}$ such that $\kappa r_t/n_0 \leq \rho(x, y_1) \leq \kappa r_t$, where $n_0 > 1$ is the constant in (2.2). Then we define $\mathcal{U}(x, t) = B(x, \kappa r_t/(4n_0))$ and $\mathcal{W}(x, t) =$

$B(y_1, \kappa r_t/(4n_0))$. We can easily check that $\mathcal{U}(x, t) \cup \mathcal{W}(x, t) \subset B(x, 3\kappa r_t/2) \subset B(z, 3\kappa r_t) \subset D$ and $\mathcal{U}(x, t) \cap \mathcal{W}(x, t) = \emptyset$.

(ii) If $\rho(x, z) > 3\kappa r_t/2$, we define $\mathcal{U}(x, t) = B(x, \kappa r_t) \cap D$ and $\mathcal{W}(x, t) = B(z, \kappa r_t/(4n_0))$.

Note that in either case, we have,

$$\kappa r_t/(2n_0) \leq \rho(u, v) \leq 4r_t \quad \text{for all } u \in \mathcal{U}(x, t) \text{ and } v \in \mathcal{W}(x, t). \quad (2.39)$$

See Figures 1 and 2 for some illustration of the sets $\mathcal{U}(x, t)$ and $\mathcal{W}(x, t)$.

It follows from [51, Theorem I.3.4] that the Lévy system of Y is the same as that of X , hence the following Lévy system formula is valid: for any $f : D \times D \rightarrow [0, \infty]$ vanishing on the diagonal and every stopping time S ,

$$\mathbb{E}_x \sum_{t \in (0, S]} f(Y_{t-}, Y_t) = \mathbb{E}_x \int_0^S \int_D f(Y_t, z) J(Y_t, z) m(dz) dt. \quad (2.40)$$

Recall that $\tau_U = \inf\{s > 0 : Y_s \notin U\}$ and $\widehat{\tau}_U = \inf\{s > 0 : \widehat{Y}_s \notin U\}$. Note that $\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) = \mathbb{P}_x(\tau_{\mathcal{U}(x,t)} < \zeta)$, where ζ is the lifetime of Y .

Lemma 2.20. *Suppose that $\mu \in \mathbf{K}_1(D)$. There exists $c = c(T) > 0$ such that for all $(t, x) \in (0, T] \times D$ and $z = z_{x,t} \in D$ with $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$ and $\rho(x, z) > 3\kappa r_t/2$, we have*

$$\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \leq c \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \frac{\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))}{\mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))} \quad (2.41)$$

and

$$\widehat{\mathbb{P}}_x(\widehat{Y}_{\widehat{\tau}_{\mathcal{U}(x,t)}} \in D) \leq c \widehat{\mathbb{P}}_y(\widehat{Y}_{\widehat{\tau}_{\mathcal{U}(x,t)}} \in D) \frac{\widehat{\mathbb{P}}_x(\widehat{Y}_{\widehat{\tau}_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))}{\widehat{\mathbb{P}}_y(\widehat{Y}_{\widehat{\tau}_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))}, \quad (2.42)$$

for every $y \in B(x, \kappa r/2) \cap D$.

Proof. Fix $(t, x) \in (0, T] \times D$ and assume that $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$ and $\rho(x, z) > 3\kappa r_t/2$. Recall that $\mathcal{U}(x, t) = B(x, \kappa r_t) \cap D$ and $\mathcal{W}(x, t) = B(z, \kappa r_t/(4n_0))$. Define $D_1 := B(x, 9\kappa r_t/8) \cap D$ and $D_2 := B(x, 9\kappa r_t/8)^c \cap D$. Take any

$$f \in \mathcal{D}(\overline{A}(z, \kappa r_t, 9\kappa r_t/8), A(z, 5\kappa r_t/8, 5\kappa r_t/4)).$$

Then, by Dynkin's formula for X (see [16, (2.11)] and the proof of [16, (4.6)]), we have for all $y \in B(x, \kappa r_t/2) \cap D$,

$$\begin{aligned} \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D_1) &= \mathbb{E}_y \left[\exp(-A_{\tau_{\mathcal{U}(x,t)}}^\mu) : X_{\tau_{\mathcal{U}(x,t)}} \in D_1 \right] \\ &\leq \mathbb{E}_y \left[f(X_{\tau_{\mathcal{U}(x,t)}}) \exp(-A_{\tau_{\mathcal{U}(x,t)}}^\mu) \right] - f(y) \\ &= \mathbb{E}_y \left[\int_0^{\tau_{\mathcal{U}(x,t)}} \mathcal{L}f(X_s) \exp(-A_s^\mu) ds \right] + \mathbb{E}_y \left[\int_0^{\tau_{\mathcal{U}(x,t)}} f(X_s) d \exp(-A_s^\mu) \right] \end{aligned}$$

$$\leq \left(\sup_{z \in \mathfrak{X}} \mathcal{L}f(z) \right) \mathbb{E}_y \left[\int_0^{\tau_{\mathcal{U}(x,t)}^X} \exp(-A_s^\mu) ds \right] = \left(\sup_{z \in \mathfrak{X}} \mathcal{L}f(z) \right) \mathbb{E}_y[\tau_{\mathcal{U}(x,t)}].$$

By Assumption **U**, taking infimum over f on both sides gives

$$\mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D_1) \leq \frac{c_1}{\Phi(r_t)} \mathbb{E}_y[\tau_{\mathcal{U}(x,t)}], \quad y \in B(x, \kappa r_t/2) \cap D,$$

for some constant $c_1 > 0$.

On the other hand, by (2.39) and (2.40), we have that for all $y \in B(x, \kappa r_t/2) \cap D$,

$$\begin{aligned} \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D_2) &\geq \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) \\ &= \mathbb{E}_y \left[\int_0^{\tau_{\mathcal{U}(x,t)}} \int_{\mathcal{W}(x,t)} J(Y_s, w) m(dw) ds \right] \\ &\asymp \mathbb{E}_y[\tau_{\mathcal{U}(x,t)}] \int_{\mathcal{W}(x,t)} \frac{1}{V(y, \rho(y, w)) \Phi(\rho(y, w))} m(dw) \asymp \frac{1}{\Phi(r_t)} \mathbb{E}_y[\tau_{\mathcal{U}(x,t)}]. \end{aligned}$$

Thus, using that $J(y, w) \asymp J(v, w)$ for $(w, y, v) \in D_2 \times (B(x, \kappa r_t/2) \cap D) \times \mathcal{U}(x, t)$, we conclude that for all $y \in B(x, \kappa r_t/2) \cap D$,

$$\mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \asymp \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D_2) \asymp \frac{1}{\Phi(r_t)} \mathbb{E}_y[\tau_{\mathcal{U}(x,t)}] \asymp \mathbb{E}_y[\tau_{\mathcal{U}(x,t)}] \int_{D_2} J(y, w) m(dw).$$

Finally, we have

$$\begin{aligned} \frac{\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D)}{\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))} &\leq c_2 \Phi(r_t) \int_{D_2} J(x, w) m(dw) \\ &\leq c_2^2 \frac{\mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D)}{\mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))}, \quad y \in B(x, \kappa r_t/2) \cap D. \end{aligned}$$

□

Recall that ζ is the lifetime of Y . We also denote by $\widehat{\zeta}$ the lifetime of \widehat{Y} .

Lemma 2.21. *Suppose that $\mu \in \mathbf{K}_1(D)$. For all $M, T \geq 1$, we have that, for all $t \in (0, T)$ and $x \in D$,*

$$\begin{aligned} \mathbb{P}_x(\zeta > t) &\asymp \mathbb{P}_x(\zeta > t/M) \asymp \mathbb{P}_x(\tau_{\mathcal{U}(x,t)} > t) \asymp \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \\ &\asymp \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) \asymp t^{-1} \mathbb{E}_x[\tau_{\mathcal{U}(x,t)}] \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathbb{P}}_x(\widehat{\zeta} > t) &\asymp \widehat{\mathbb{P}}_x(\widehat{\zeta} > t/M) \asymp \widehat{\mathbb{P}}_x(\widehat{\tau}_{\mathcal{U}(x,t)} > t) \asymp \mathbb{P}_x(\widehat{Y}_{\tau_{\mathcal{U}(x,t)}} \in D) \\ &\asymp \mathbb{P}_x(\widehat{Y}_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) \asymp t^{-1} \widehat{\mathbb{E}}_x[\widehat{\tau}_{\mathcal{U}(x,t)}], \end{aligned}$$

where $\mathcal{U}(x, t)$ and $\mathcal{W}(x, t)$ are the open sets defined in the beginning of this subsection and the comparison constants depend only on $d_0, d, \delta_l, \delta_u, T, M, R_1$ and κ .

Proof. Fix $t \in (0, T]$, $x \in D$ and set $r := r_t = \Phi^{-1}(t)R_1/3\Phi^{-1}(T)$.

Case (i): $\rho(x, z) \leq 3\kappa r/2$. By (2.33), we have

$$1 \geq \mathbb{P}_x(\zeta > t/M) \geq \mathbb{P}_x(\zeta > t) \geq \mathbb{P}_x(\tau_{\mathcal{U}(x,t)} > t) = \mathbb{P}_x(\tau_{B(x, \kappa r/(4n_0))} > t) \geq c > 0.$$

On the other hand, by (2.39), (2.40) and (2.34),

$$\begin{aligned} 1 &\geq \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \geq \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) = \mathbb{E}_x \left[\int_0^{\tau_{\mathcal{U}(x,t)}} \int_{\mathcal{W}(x,t)} J(Y_s, v)m(dv)ds \right] \\ &\geq c_1 \frac{m(\mathcal{W}(x, t))\mathbb{E}_x[\tau_{\mathcal{U}(x,t)}]}{V(x, 3r)\Phi(3r)} \geq c_2 \frac{\mathbb{E}_x[\tau_{\mathcal{U}(x,t)}]}{\Phi(3r)} \geq c_3 t^{-1} \mathbb{E}_x[\tau_{B(x, \kappa r/(4n_0))}] \geq c_4. \end{aligned}$$

Therefore, we arrive at the assertion of the lemma in this case.

Case (ii): $\rho(x, z) > 3\kappa r/2$. Note that

$$\mathbb{P}_x(\zeta > t/M) \leq \mathbb{P}_x(\tau_{\mathcal{U}(x,t)} > t/M) + \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \leq Mt^{-1}\mathbb{E}_x[\tau_{\mathcal{U}(x,t)}] + \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D).$$

Fix a $y \in D$ such that $B(y, \kappa^2 r/2) \subset B(x, \kappa r/2) \cap D$. Then by Lemma 2.20 we have

$$\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \leq c_5 \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \frac{\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))}{\mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t))}.$$

By (2.39), (2.40) and (2.34),

$$\begin{aligned} \mathbb{P}_y(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) &\geq \mathbb{P}_y(Y_{\tau_{B(y, \kappa^2 r/2)}} \in \mathcal{W}(x, t)) \\ &= \mathbb{E}_y \left[\int_0^{\tau_{B(y, \kappa^2 r/2)}} \int_{\mathcal{W}(x,t)} J(Y_s, v)m(dv)ds \right] \geq c_6 \frac{V(x, r)\mathbb{E}_y[\tau_{B(y, \kappa^2 r/2)}]}{V(x, 3r)\Phi(3r)} \geq c_7 \end{aligned}$$

and

$$\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) \asymp \frac{V(x, r)\mathbb{E}_x[\tau_{\mathcal{U}(x,t)}]}{V(x, r)\Phi(r)} = c_8 t^{-1} \mathbb{E}_x[\tau_{\mathcal{U}(x,t)}].$$

It follows that

$$\mathbb{P}_x(\zeta > t/M) \leq c_9 t^{-1} \mathbb{E}_x[\tau_{\mathcal{U}(x,t)}] \asymp \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)) \asymp \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D).$$

Note that $B(x, (3 - 2\kappa)r) \cap D \supset \mathcal{U}(x, t) \cup \mathcal{W}(x, t)$ for every $(t, x) \in (0, T] \times D$. Thus by (2.33),

$$\begin{aligned} \mathbb{P}_x(\zeta > t) &\geq \mathbb{P}_x(\tau_{B(x, 3r) \cap D} > t) \geq \mathbb{E}_x \left[\inf_{w \in \mathcal{W}(x,t)} \mathbb{P}_w(\tau_{B(x, 3r) \cap D} > t) : Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t) \right] \\ &\geq \mathbb{E}_x \left[\inf_{w \in \mathcal{W}(x,t)} \mathbb{P}_w(\tau_{B(w, \kappa r)} > t) : Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t) \right] \geq c_{10} \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)). \end{aligned}$$

The proof is now complete. \square

Theorem 2.22. *Let D be a κ -fat set with characteristics (R_1, κ) . Suppose that $\mu \in \mathbf{K}_1(D)$. Then for all $T > 0$, there exists $c \geq 1$ such that for all $(t, x, y) \in (0, T) \times D \times D$,*

$$c^{-1}\mathbb{P}_x(\zeta > t)\widehat{\mathbb{P}}_y(\widehat{\zeta} > t)\widetilde{q}(t, x, y) \leq q^D(t, x, y) \leq c\mathbb{P}_x(\zeta > t)\widehat{\mathbb{P}}_y(\widehat{\zeta} > t)\widetilde{q}(t, x, y). \quad (2.43)$$

Proof. Fix $t \in (0, T]$ and set $r := \Phi^{-1}(t)R_1/(3\Phi^{-1}(T))$.

(1) We first prove the upper bound. By Lemma 2.21, for any $x, y \in D$ with $\rho(x, y) \leq 4r$, we have

$$\begin{aligned} q^D(t/2, x, y) &= \int_D q^D(t/4, x, w)q^D(t/4, w, y)m(dw) \\ &\leq C_0 \int_D q^D(t/4, x, w)\widetilde{q}(t/4, w, y)m(dw) \\ &\leq c_1\mathbb{P}_x(\zeta > t/4)V(y, \Phi^{-1}(t))^{-1} \leq c_2\mathbb{P}_x(\zeta > t)p(t/2, x, y). \end{aligned}$$

Now, we assume $\rho(x, y) > 4r$. Let $U_1 := \mathcal{U}(x, t)$ be the set defined before, $U_3 := \{w \in D : \rho(x, w) > \rho(x, y)/2\}$, and $U_2 := D \setminus (U_1 \cup U_3)$. Since $x \in U_1$, $y \in U_3$ and $U_1 \cap U_3 = \emptyset$, by the strong Markov property, we have

$$\begin{aligned} q^D(t/2, x, y) &= \mathbb{E}_x[q^D(t/2 - \tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t/2, Y_{\tau_{U_1}} \in U_2] \\ &\quad + \mathbb{E}_x[q^D(t/2 - \tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t/2, Y_{\tau_{U_1}} \in U_3] =: I + II. \end{aligned}$$

First, note that for every $u \in U_2$, $\rho(u, y) \geq \rho(x, y) - \rho(x, u) \geq \rho(x, y)/2$, which implies that

$$V(y, \rho(x, y)) \leq V(u, \rho(x, y) + \rho(u, y)) \leq V(u, 3\rho(u, y)).$$

Therefore, using (2.1), for all $(s, u) \in (0, t/2] \times U_2$,

$$q^D(s, u, y) \leq c_3\widetilde{q}(s, u, y) \leq c_4 \frac{t}{V(y, \rho(x, y))\Phi(\rho(x, y))} \leq c_5p(t/2, x, y).$$

Now it follows from Lemma 2.21 that

$$I \leq c_5p(t/2, x, y)\mathbb{P}_x(Y_{\tau_{U_1}} \in D) \asymp \mathbb{P}_x(\zeta > t)p(t/2, x, y).$$

On the other hand, for all $u \in U_1$ and $w \in U_3$, we have $\rho(u, x) \leq \kappa r < 4^{-1}\kappa\rho(x, y)$ and $\rho(u, w) \geq \rho(x, w) - \rho(x, u) \geq \rho(x, y)/2 - \kappa r \geq \rho(x, y)/4$, which implies that

$$V(x, \rho(x, y)) \leq V(u, \rho(x, y) + \rho(u, x)) \leq V(u, (1 + \kappa/4)\rho(x, y)) \leq V(u, 8\rho(u, w)).$$

Thus, by (2.1), (2.4), Lemma 2.21 and the Lévy system formula, and using the assumption $\rho(x, y) > 4r$,

$$\begin{aligned} II &= \int_0^{t/2} \int_{U_1} \int_{U_3} q^{U_1}(s, x, u)J(u, w)q^D(t/2 - s, w, y)m(dw)m(du)ds \\ &\leq c_6 \frac{1}{V(x, \rho(x, y))\Phi(\rho(x, y))} \int_0^{t/2} \mathbb{P}_x(\tau_{U_1} > s)\widehat{\mathbb{P}}_y(\widehat{\zeta} > t/2 - s)ds \end{aligned}$$

$$\begin{aligned}
&\leq c_6 \frac{1}{V(x, \rho(x, y))\Phi(\rho(x, y))} \int_0^\infty \mathbb{P}_x(\tau_{U_1} > s) ds \\
&= 2c_6 \frac{t/2}{V(x, \rho(x, y))\Phi(\rho(x, y))} t^{-1} \mathbb{E}_x[\tau_{U_1}] \asymp \mathbb{P}_x(\zeta > t) p(t/2, x, y).
\end{aligned}$$

Thus for all $x, y \in D$, $q^D(t/2, x, y) \leq c_7 \mathbb{P}_x(\zeta > t) p(t/2, x, y)$, and, similarly, for all $x, y \in D$, $q^D(t/2, x, y) \leq c_7 \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) p(t/2, x, y)$.

Finally, by the semigroup property, we conclude that

$$\begin{aligned}
q^D(t, x, y) &= \int_D q^D(t/2, x, w) q^D(t/2, w, y) m(dw) \\
&\leq c_7^2 \mathbb{P}_x(\zeta > t) \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) \int_x p(t/2, x, w) p(t/2, w, y) m(dw) \\
&\leq c_7^2 C_0 \mathbb{P}_x(\zeta > t) \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) \tilde{q}(t, x, y).
\end{aligned}$$

(2) For the lower bound, we use the notation \mathcal{W} as before. By the semigroup property,

$$\begin{aligned}
q^D(t, x, y) &= \int_D \int_D q^D(t/3, x, u) q^D(t/3, u, w) q^D(t/3, w, y) m(dw) m(du) \\
&\geq \int_{\mathcal{W}(x, t/3)} \int_{\mathcal{W}(y, t/3)} q^D(t/3, x, u) q^D(t/3, u, w) q^D(t/3, w, y) m(dw) m(du).
\end{aligned}$$

First, observe that for all $(u, w) \in \mathcal{W}(x, t/3) \times \mathcal{W}(y, t/3)$,

$$\delta_D(u), \delta_D(w) \geq \kappa(a_l/3)^{1/\delta_l} r/4, \quad (\rho(x, y) - 6(3a_u)^{1/\delta_u} r)_+ \leq \rho(u, w) \leq \rho(x, y) + 6(3a_u)^{1/\delta_u} r.$$

Here is an explanation of the last inequality above, the others being similar. By the triangle inequality and symmetry, it suffices to show that $\rho(u, x) \leq 3(3a_u)^{1/\delta_u} r$. Since $\mathcal{W}(x, t/3) \subset B(x, 3r_{t/3})$, this will be so provided that $r_{t/3} \leq (3a_u)^{1/\delta_u} r$. But this immediately follows from (2.5) by estimating $\Phi^{-1}(t/3)/\Phi^{-1}(t)$. By considering cases $\rho(x, y) > 12(3a_u)^{1/\delta_u} r$ and $\rho(x, y) \leq 12(3a_u)^{1/\delta_u} r$ separately, we get from Theorem 2.15 and (2.6) that for all $(u, w) \in \mathcal{W}(x, t/3) \times \mathcal{W}(y, t/3)$,

$$q^D(t/3, u, w) \asymp \tilde{q}(t/3, u, w) \asymp \left(\frac{1}{V(u, \Phi^{-1}(t))} \wedge \frac{t}{V(u, \rho(w, u))\Phi(\rho(w, u))} \right) \asymp \tilde{q}(t, x, y).$$

Next, let $c_9 := \kappa(a_l/3)^{1/\delta_l}/8$. By Theorem 2.15 and (2.6), for all $(s, u) \in (t/6, t/3) \times \mathcal{W}(x, t/3)$ and $w \in B(u, c_9 r)$, we have $q^D(s, u, w) \asymp \tilde{q}(s, u, w) \asymp V(u, r)^{-1}$. Moreover, by (2.39) and (2.13), for all $u \in \mathcal{W}(x, t/3)$ and $(v, w) \in \mathcal{U}(x, t/3) \times B(u, c_9 r)$,

$$J(v, w) \asymp \frac{1}{V(v, r)\Phi(r)} \asymp \frac{1}{V(x, r)\Phi(r)}.$$

Thus, by the Lévy system formula and Lemma 2.21, for every $u \in \mathcal{W}(x, t/3)$, we have

$$q^D(t/3, x, u) \geq \mathbb{E}_x[q^D(t/3 - \tau_{\mathcal{U}(x, t/3)}, Y_{\tau_{\mathcal{U}(x, t/3)}}, u) : \tau_{\mathcal{U}(x, t/3)} < t/3, Y_{\tau_{\mathcal{U}(x, t/3)}} \in B(u, c_9 r)]$$

$$\begin{aligned}
&\geq \int_0^{t/6} \int_{\mathcal{U}(x,t/3)} \int_{B(u,c_9r)} q^{\mathcal{U}(x,t/3)}(s,x,v) J(v,w) q^D(t/3-s,w,u) m(dw) m(dv) ds \\
&\geq c_{10} \frac{1}{V(x,r)\Phi(r)} \int_{t/6}^{t/3} \int_{B(u,c_9r)} \mathbb{P}_x(\tau_{\mathcal{U}(x,t/3)} > s) V(u,r)^{-1} m(dw) ds \\
&\geq c_{11} \frac{t}{V(x,r)\Phi(r)} \mathbb{P}_x(\tau_{\mathcal{U}(x,t/3)} > t/3) \asymp \frac{1}{V(x,r)} \mathbb{P}_x(\zeta > t).
\end{aligned}$$

Similarly for $w \in \mathcal{W}(y,t/3)$, $q^D(t/3,w,y) \geq c_{12} \frac{1}{V(y,r)} \widehat{\mathbb{P}}_y(\widehat{\zeta} > t)$. Therefore, we conclude that

$$\begin{aligned}
q^D(t,x,y) &\geq c_{13} \widetilde{q}(t,x,y) \int_{\mathcal{W}(x,t/3)} q^D(t/3,x,u) m(du) \int_{\mathcal{W}(y,t/3)} q^D(t/3,w,y) m(dw) \\
&\geq c_{14} \mathbb{P}_x(\zeta > t) \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) \widetilde{q}(t,x,y).
\end{aligned}$$

□

Using Theorem 2.15 and Corollary 2.16(2), the following global estimates can be proved by the same argument. We omit the proof.

Theorem 2.23. *Let D be a κ -fat set with characteristics (∞, κ) . Suppose that $\mu \in \mathbf{K}_1(D)$ and $R_0 = m(\mathfrak{X}) = \widetilde{T} = r_0 = \infty$, where r_0 is the constant in Assumption **U**. Then there exists $c_1(\kappa) > 0$ such that for all $(t,x,y) \in (0, \infty) \times D \times D$,*

$$c_1^{-1} \mathbb{P}_x(\zeta > t) \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) \widetilde{q}(t,x,y) \leq q^D(t,x,y) \leq c_1 \mathbb{P}_x(\zeta > t) \widehat{\mathbb{P}}_y(\widehat{\zeta} > t) \widetilde{q}(t,x,y). \quad (2.44)$$

Example 2.24. Suppose that (\mathfrak{X}, ρ, m) is an unbounded Ahlfors regular n -space for some $n \in (0, \infty)$, that is, for all $x \in \mathfrak{X}$ and $r \in (0, 1]$, $m(B(x,r)) \asymp r^n$. Assume that ρ is uniformly equivalent to the shortest-path metric in \mathfrak{X} . Suppose that there is a diffusion process ξ with a symmetric, continuous transition density $p^\xi(t,x,y)$ satisfying the following sub-Gaussian bounds

$$\begin{aligned}
&\frac{c_1}{t^{n/d_w}} \exp\left(-c_2 \left(\frac{\rho(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right) \leq p^\xi(t,x,y) \\
&\leq \frac{c_3}{t^{n/d_w}} \exp\left(-c_4 \left(\frac{\rho(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right),
\end{aligned} \quad (2.45)$$

for all $x, y \in \mathfrak{X}$ and $t \in (0, \infty)$. Here $d_w \geq 2$ is the walk dimension of the space \mathfrak{X} . Examples of ξ include Brownian motions on unbounded Riemannian manifolds, Brownian motions on Sierpinski gaskets, Sierpinski carpets or more general fractals. Let $\alpha \in (0, d_w)$ and let T be an (α/d_w) -stable subordinator independent of ξ . We define a process X by $X_t = \xi_{T_t}$. Then X is a symmetric Feller process. It is easy to check that X has a transition density $p(t,x,y)$ satisfying

$$p(t,x,y) \asymp \left(t^{-\frac{n}{\alpha}} \wedge \frac{t}{\rho(x,y)^{n+\alpha}}\right), \quad (2.46)$$

for all $x, y \in \mathfrak{X}$ and $t \in (0, \infty)$. It follows from [16, Appendix A] that Assumptions **A** and **U** above are also satisfied with $\Phi(r) = r^\alpha$. Therefore, by Theorem 2.22 and (2.46), if D is a κ -fat open set in \mathfrak{X} and $\mu \in \mathbf{K}_1(D)$, then for all $(t, x, y) \in (0, r_0) \times D \times D$,

$$q^D(t, x, y) \asymp \mathbb{P}_x(\zeta > t)\mathbb{P}_y(\zeta > t) \left(t^{-\frac{n}{\alpha}} \wedge \frac{t}{\rho(x, y)^{n+\alpha}} \right).$$

3 Dirichlet heat kernel estimates of regional fractional Laplacian with critical killing

In this section we assume that $d \geq 2$, \mathfrak{X} is either the closure of a $C^{1,1}$ open subset D of \mathbb{R}^d or \mathbb{R}^d itself, and the underlying process is either a reflected α -stable process in \bar{D} (or a non-local perturbation of it), or an α -stable process in \mathbb{R}^d (or a drift perturbation of it). We investigate Dirichlet heat kernel estimates under critical killing. We first recall the definition of reflected α -stable processes.

Let $0 < \alpha < 2$ and $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}$. Here Γ is the gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$, $\lambda > 0$. For a $C^{1,1}$ open subset D of \mathbb{R}^d , let $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ be the Dirichlet space on $L^2(D, dx)$ defined by

$$\begin{aligned} \bar{\mathcal{F}} &:= \left\{ u \in L^2(D); \int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy < \infty \right\}, \\ \bar{\mathcal{E}}(u, v) &:= \frac{1}{2} \mathcal{A}(d, -\alpha) \int_D \int_D \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy, \quad u, v \in \bar{\mathcal{F}}. \end{aligned}$$

It is well known that $W^{\alpha/2, 2}(D) = \bar{\mathcal{F}}$ and the Sobolev norm $\|\cdot\|_{\alpha/2, 2; D}$ is equivalent to $\sqrt{\bar{\mathcal{E}}_1}$ where $\bar{\mathcal{E}}_1 := \bar{\mathcal{E}} + (\cdot, \cdot)_{L^2(D)}$. As noted in [7], $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ is a regular Dirichlet form on \bar{D} and its associated Hunt process X lives on \bar{D} . We call the process X a reflected α -stable process in \bar{D} . When D is the whole \mathbb{R}^d , X is simply an α -stable process.

It follows from [24] that X admits a strictly positive and jointly continuous transition density $p(t, x, y)$ with respect to the Lebesgue measure dx and that

$$p(t, x, y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \quad (t, x, y) \in (0, 1) \times \bar{D} \times \bar{D}. \quad (3.1)$$

When $\alpha \in (1, 2)$, the killed process X^D is the censored stable process in D . When $\alpha \in (0, 1]$, it follows from [7, Section 2] that, starting from inside D , the process X neither hits nor approaches ∂D at any finite time. Thus, the killed process X^D is simply X restricted to D (without killing).

We will see that, for all $\alpha \in (0, 2)$, the killed isotropic α -stable process Z^D can be obtained from X^D through a Feynman-Kac perturbation of the form (3.8) with κ satisfying (3.6).

It follows from [19] that, when $\alpha \in (1, 2)$, the transition density $p_D^X(t, x, y)$ of X^D has the following estimates:

$$p_D^X(t, x, y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}} \right)^{\alpha-1} \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \quad (3.2)$$

for $(t, x, y) \in (0, 1) \times D \times D$.

It follows from [18] that the transition density $p_D^Z(t, x, y)$ of Z^D has the following estimates:

$$p_D^Z(t, x, y) \asymp \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \quad (3.3)$$

for $(t, x, y) \in (0, 1) \times D \times D$.

In Subsection 3.1, we will establish explicit Dirichlet heat kernel estimates under critical killing, which also provides an alternative and unified proof of (3.2) and (3.3).

In Subsection 3.2, we consider non-local perturbations of $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ when D is a bounded $C^{1,1}$ open set. Subsection 3.3 covers the case $D = \mathbb{R}^d \setminus \{0\}$ and drift perturbations.

3.1 $C^{1,1}$ open set

In this subsection, we assume that D is a $C^{1,1}$ open set in \mathbb{R}^d with characteristics (R_2, Λ) , and that X is a reflected α -stable process in \bar{D} . Without loss of generality, we will always assume that $\Lambda \geq 1$. It is easy to check that the process X satisfies the assumptions in Subsection 2.1 and Assumption **U**.

Let $\mathbb{R}_+^d := \{y = (y_1, \dots, y_d) \in \mathbb{R}^d : y_d > 0\}$. For $d \geq 2$ and $p \in (-1, \alpha)$, we define $w_p(y) = (y_d)^p$ for $y \in \mathbb{R}_+^d$ and $w_p(y) = 0$ otherwise. According to [7, (5.4)], we have

$$\mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}_+^d, |y-z| > \epsilon} \frac{w_p(y) - w_p(z)}{|y-z|^{d+\alpha}} dy = C(d, \alpha, p) z_d^{p-\alpha}, \quad z \in \mathbb{R}_+^d, \quad (3.4)$$

where $C(d, \alpha, p) := \mathcal{A}(d, -\alpha) \frac{\omega_{d-1}}{2} \beta(\frac{\alpha+1}{2}, \frac{d-1}{2}) \gamma(\alpha, p)$, $\beta(\cdot, \cdot)$ is the beta function, ω_{d-1} is the $(d-2)$ -dimensional Lebesgue measure of the unit sphere in \mathbb{R}^{d-1} and

$$\gamma(\alpha, p) = \int_0^1 \frac{(t^p - 1)(1 - t^{\alpha-p-1})}{(1-t)^{1+\alpha}} dt.$$

Observe that

$$\frac{d\gamma(\alpha, p)}{dp} = \int_0^1 \frac{(t^{\alpha-p-1} - t^p) |\log t|}{(1-t)^{1+\alpha}} dt$$

is positive for $p > (\alpha - 1)/2$ and thus $p \mapsto \gamma(\alpha, p)$ is strictly increasing on $((\alpha - 1)/2, \alpha)$. Moreover, we have

$$C(d, \alpha, \alpha - 1) = C(d, \alpha, 0) = 0 \quad \text{and} \quad \lim_{p \uparrow \alpha} C(d, \alpha, p) = \infty. \quad (3.5)$$

Let \mathcal{H}_α be the collection of non-negative functions κ on D with the property that there exist constants $C_1, C_2 \geq 0$ and $\eta \in [0, \alpha)$ such that $\kappa(x) \leq C_2$ for all $x \in D$ with $\delta_D(x) \geq 1$ and

$$|\kappa(x) - C_1 \delta_D(x)^{-\alpha}| \leq C_2 \delta_D(x)^{-\eta}, \quad (3.6)$$

for all $x \in D$ with $\delta_D(x) < 1$. If $\alpha \leq 1$, then we further assume that $C_1 > 0$. It follows from (3.5) that we can find a unique $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$ such that $C_1 = C(d, \alpha, p)$. For any $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$, define

$$\mathcal{H}_\alpha(p) := \{\kappa \in \mathcal{H}_\alpha : \text{the constant } C_1 \text{ in (3.6) is } C(d, \alpha, p)\}. \quad (3.7)$$

Note that $\mathcal{H}_\alpha = \cup_{p \in [\alpha-1, \alpha) \cap (0, \alpha)} \mathcal{H}_\alpha(p)$. We fix a $\kappa \in \mathcal{H}_\alpha(p)$ and let Y be a Hunt process on D corresponding to the Feynman-Kac semigroup of X^D through the multiplicative functional $e^{-\int_0^t \kappa(X_s^D) ds}$. That is,

$$\mathbb{E}_x[f(Y_t)] = \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_s^D) ds} f(X_t^D) \right], \quad t \geq 0, x \in D. \quad (3.8)$$

Since, by Example 2.17, $\kappa(x)dx \in \mathbf{K}_1(D)$, it follows from Theorem 2.22 that Y has a transition density $q^D(t, x, y)$ with the following estimate:

$$q^D(t, x, y) \asymp \mathbb{P}_x(\zeta > t) \mathbb{P}_y(\zeta > t) \left[t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right], \quad (3.9)$$

for $(t, x, y) \in (0, 1] \times D \times D$. To get explicit estimate of $\mathbb{P}_x(\zeta > t)$, we will estimate $\mathbb{P}_x(Y_{\tau_{\mathcal{U}}(x,t)} \in D)$ and use Lemma 2.21.

For $f \in C_c^2(D)$, define

$$L_\alpha f(x) := \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \int_{D, |y-x| > \epsilon} \frac{f(y) - f(x)}{|y-x|^{d+\alpha}} dy \quad \text{and} \quad Lf(x) := L_\alpha f(x) - \kappa(x)f(x). \quad (3.10)$$

The operator L coincides with the restriction to $C_c^2(D)$ of the generator of the transition semigroup of Y in $C_0(D)$.

Lemma 3.1. *Let $0 < p \leq q < \alpha$ and define*

$$h_q(x) := \delta_D(x)^q.$$

Then there exist $A_1 = A_1(q, d, \alpha, \Lambda, C_2, \eta, R_2) > 0$ and $A_2 = A_2(q, d, \alpha, \Lambda, C_2, \eta, R_2) \in (0, (R_2 \wedge 1)/4)$ such that the following inequalities hold:

(i) *If $q > p$, then*

$$A_1^{-1} \delta_D(x)^{q-\alpha} \leq Lh_q(x) \leq A_1 \delta_D(x)^{q-\alpha}$$

for every $x \in D$ with $0 < \delta_D(x) < A_2$.

(ii) *If $q = p$, then*

$$|Lh_p(x)| \leq A_1 (\delta_D(x)^{p-\eta} + |\log \delta_D(x)|)$$

for every $x \in D$ with $0 < \delta_D(x) < A_2$.

Proof. Without loss of generality, we assume $R_2 = 1$. Let $x \in D$ with $\delta_D(x) < 1/4$. Choose a point $z \in \partial D$ such that $\delta_D(x) = |x - z|$. Then there exist a $C^{1,1}$ function $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that $\psi(z) = \nabla \psi(z) = 0$ and an orthonormal coordinate system CS_z such that

$$D \cap B(z, 1) = \{y = (\tilde{y}, y_d) \text{ in } CS_z : y_d > \psi(\tilde{y})\} \cap B(z, 1),$$

and $z = 0$ and $x = (\tilde{x}, x_d) = (0, x_d)$ in CS_z . Note that

$$Lh_q(x) = L_\alpha h_q(x) - C(d, \alpha, p)x_d^{q-\alpha} - (\kappa(x) - C(d, \alpha, p)x_d^{-\alpha})h_q(x) =: I - II - III.$$

By our assumption, we have $|III| \leq C_2 x_d^{q-\eta}$.

For any open subset $U \subset \mathbb{R}^d$, define $\kappa_U(x) = \mathcal{A}(d, -\alpha) \int_{U^c} \frac{dy}{|y-x|^{d+\alpha}}$. Recall that $w_q(y) = (y_d)^q$ for $y \in \mathbb{R}_+^d$ and $w_q(y) = 0$ otherwise. Since $h_q(x) = w_q(x) = x_d^q$, by (3.4) we have

$$\begin{aligned} I &= \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \left[\int_{\mathbb{R}^d, |y-x| > \epsilon} \frac{h_q(y) - h_q(x)}{|y-x|^{d+\alpha}} dy \right] + \kappa_D(x) h_q(x) \\ &= \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \left[\int_{\mathbb{R}^d, |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy + \int_{\mathbb{R}^d, |y-x| > \epsilon} \frac{w_q(y) - w_q(x)}{|y-x|^{d+\alpha}} dy \right] + \kappa_D(x) w_q(x) \\ &= C(d, \alpha, q) x_d^{q-\alpha} + \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \left[\int_{\mathbb{R}^d, |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy \right] + (\kappa_D(x) - \kappa_{\mathbb{R}_+^d}(x)) w_q(x). \end{aligned}$$

According to [7, Lemma 5.6], if $1 < \alpha < 2$, then there is a constant $c = c(d, \alpha, \Lambda)$ such that $|\kappa_D(x) - \kappa_{\mathbb{R}_+^d}(x)| \leq c x_d^{1-\alpha}$. By a similar calculation as in [7, Lemma 5.6], one can show that for $\alpha \leq 1$, $|\kappa_D(x) - \kappa_{\mathbb{R}_+^d}(x)| \leq c(|\log x_d| \mathbf{1}_{\{\alpha=1\}} + 1)$. Thus, for any $0 < \alpha < 2$, we get

$$|(\kappa_D(x) - \kappa_{\mathbb{R}_+^d}(x)) w_q(x)| \leq c x_d^q (x_d^{1-\alpha} + |\log x_d|) \leq c. \quad (3.11)$$

Therefore, it remains to bound $I_\epsilon := \int_{\mathbb{R}^d, |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy$. Since D is a $C^{1,1}$ open set, it satisfies the inner and outer ball conditions. Thus we can assume that $B_1 = B(\mathbf{e}_d, 1) \subset D$ and $B_2 = B(-\mathbf{e}_d, 1) \subset D^c$ where $\mathbf{e}_d := (0, 1)$. We define $E := \{y = (\tilde{y}, y_d) : |\tilde{y}| < 1/4, |y_d| < 1/2\}$, $E_1 := \{y \in E : y_d > 2|\tilde{y}|^2\}$ and $E_2 := \{y \in E : -y_d > 2|\tilde{y}|^2\}$. Then it is easy to see that $E_1 \subset B_1 \cap E \subset D$ and $E_2 \subset B_2 \cap E \subset D^c$. Thus, since $h_q(y) = w_q(y) = 0$ for $y \in E_2$.

$$\begin{aligned} I_\epsilon &= \int_{E^c, |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy + \int_{E_1, |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy \\ &\quad + \int_{E \setminus (E_1 \cup E_2), |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\alpha}} dy =: J_{1,\epsilon} + J_{2,\epsilon} + J_{3,\epsilon}. \end{aligned}$$

First, since $x = (0, x_d)$ and $x_d = \delta_D(x) < 1/4$, we can see that $|y-x| \geq |y|/2$ for all $y \in E^c$. Indeed, if $y \in E^c$, then either $|y_d| \geq 1/2$ or $|\tilde{y}| \geq 1/4$. If $|y_d| \geq 1/2 > 2x_d$, then $|y-x|^2 = |\tilde{y}|^2 + |y_d - x_d|^2 \geq |\tilde{y}|^2 + |y_d|^2/4 \geq |y|^2/4$. If $|\tilde{y}| \geq 1/4$, then

$$|y-x|^2 - \frac{1}{4}|y|^2 = \frac{3}{4}|\tilde{y}|^2 + \frac{3}{4}|y_d - \frac{4}{3}x_d|^2 - \frac{1}{3}x_d^2 \geq \frac{3}{4}|\tilde{y}|^2 - \frac{1}{3}x_d^2 \geq \frac{3}{64} - \frac{1}{48} > 0.$$

Thus, since $|h_q(y) - w_q(y)| \leq 2|y|^q$, we get

$$|J_{1,\epsilon}| \leq 2^{1+d+\alpha} \int_{E^c} |y|^{q-d-\alpha} dy \leq c \int_{1/4}^\infty l^{q-\alpha-1} dl = c.$$

For $y \in E_1$, we have $\delta_D(y) \leq \delta_{B_2^c}(y) \leq y_d + 1 - \sqrt{1 - |\tilde{y}|^2} \leq y_d + |\tilde{y}|^2 < 2y_d$. Thus,

$$J_{2,\epsilon} \leq \int_{E_1, |y-x| > \epsilon} \frac{(y_d + |\tilde{y}|^2)^q - y_d^q}{|y-x|^{d+\alpha}} dy \leq \int_{E_1, |y-x| > \epsilon} \frac{q(1 \vee 2^{q-1})|\tilde{y}|^2 y_d^{q-1}}{|y-x|^{d+\alpha}} dy$$

$$\leq cx_d^{q+1-\alpha} \int_{B(0,1/x_d)} \frac{|\tilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du.$$

We have used the change of the variables $y = x_d u$ in the last inequality above. Note that

$$\begin{aligned} \int_{B(0,1/x_d)} \frac{|\tilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du &= \int_{B(0,2)} \frac{|\tilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du + \int_{B(0,1/x_d) \setminus B(0,2)} \frac{|\tilde{u}|^2 u_d^{q-1}}{|u - \mathbf{e}_d|^{d+\alpha}} du \\ &\leq c \int_{B(0,2)} |u - \mathbf{e}_d|^{2-d-\alpha} du + c \int_{B(0,1/x_d) \setminus B(0,2)} |u|^{q+1-d-\alpha} du \\ &\leq c \left(\int_0^2 l^{1-\alpha} dl + \int_2^{x_d^{-1}} l^{q-\alpha} dl \right). \end{aligned}$$

It follows that $J_{2,\epsilon} \leq c(1 + |\log x_d|)$. Besides, for $y \in E_1$, we have

$$\delta_D(y) \geq \delta_{B_1}(y) \geq 1 - \sqrt{(1 - y_d)^2 + |\tilde{y}|^2} \geq 1 - \sqrt{1 - (y_d - |\tilde{y}|^2)} \geq \frac{1}{2}(y_d - |\tilde{y}|^2) > \frac{1}{4}y_d$$

and

$$y_d - (1 - \sqrt{(1 - y_d)^2 + |\tilde{y}|^2}) = \frac{|\tilde{y}|^2}{\sqrt{(1 - y_d)^2 + |\tilde{y}|^2} + 1 - y_d} \leq \frac{|\tilde{y}|^2}{2(1 - y_d)} \leq |\tilde{y}|^2.$$

Therefore, by the mean value theorem,

$$\begin{aligned} J_{2,\epsilon} &\geq \int_{E_1, |y-x|>\epsilon} \frac{(1 - \sqrt{(1 - y_d)^2 + |\tilde{y}|^2})^q - y_d^q}{|y - x|^{d+\alpha}} dy \\ &\geq -q \int_{E_1, |y-x|>\epsilon} \frac{|\tilde{y}|^2 \left(\sup_{\lambda \in [0, |\tilde{y}|^2]} (1 - \sqrt{(1 - y_d)^2 + \lambda})^{q-1} \right)}{|y - x|^{d+\alpha}} dy \\ &\geq -q(4^{1-q} \vee 1) \int_{E_1, |y-x|>\epsilon} \frac{|\tilde{y}|^2 y_d^{q-1}}{|y - x|^{d+\alpha}} dy. \end{aligned}$$

Thus, we have $|J_{2,\epsilon}| \leq c(1 + |\log x_d|)$.

Lastly, let $m_{d-1}(dx)$ be the $(d-1)$ -dimensional Hausdorff measure. Then there exists a constant $c > 0$ such that for all $0 < l < 1$,

$$m_{d-1}(\{y : |\tilde{y}| = l, -2|\tilde{y}|^2 \leq y_d \leq 2|\tilde{y}|^2\}) \leq cl^d.$$

Since $|h_q(y)|, |w_q(y)| \leq 4^q |\tilde{y}|^{2q}$ for $y \in E \setminus (E_1 \cup E_2)$,

$$|J_{3,\epsilon}| \leq c \int_0^{1/4} \int_{|\tilde{y}|=l, y \in E \setminus (E_1 \cup E_2)} l^{2q-d-\alpha} m_{d-1}(dy) dl \leq c \int_0^{1/4} l^{2q-\alpha} dl \leq c.$$

Combining the above estimates, we conclude that $|I_\epsilon| \leq c(1 + \log |x_d|)$.

If $q > p$, we note that $C(d, \alpha, q) > C(d, \alpha, p)$ and $q - \alpha < 0 \wedge (q - \alpha + 1) \wedge (q - \eta)$. \square

Fix a $q \in (p, \alpha)$ such that $q < p - \eta + \alpha$. Then define $A_3 := A_1(p) \vee A_1(q)$, $A_4 := A_2(p) \wedge A_2(q)$, where A_1 and A_2 are the constants in Lemma 3.1, and

$$v_1(x) := h_p(x) + h_q(x).$$

By Lemma 3.1, for any $x \in D$ with $\delta_D(x) < A_4$, we have

$$Lv_1(x) \geq A_3^{-1} \delta_D(x)^{q-\alpha} - A_3(\delta_D(x)^{p-\eta} + |\log \delta_D(x)|).$$

Thus, there exist $A_5 \in (0, A_4)$ and $A_6 > 0$ such that

$$Lv_1(x) \geq 2A_6 \delta_D(x)^{q-\alpha} \quad \text{for all } x \in D \text{ with } \delta_D(x) < A_5. \quad (3.12)$$

Define $v_2(x) := h_p(x) - \frac{1}{2}h_q(x)$. By the same argument, we can find $A_7 \in (0, A_4)$ and $A_8 > 0$ such that $Lv_2(x) \leq -2A_8 \delta_D(x)^{q-\alpha}$ for all $x \in D$ with $\delta_D(x) < A_7$.

Now, we are ready to estimate $\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D)$. We continue to assume $R_2 = 1$. Note that D is a κ -fat open set with characteristics $(1, \kappa)$. Recall that r_t is defined as $r_t = \Phi^{-1}(t)R_1/(3\Phi^{-1}(T))$ in Subsection 2.6. Since in the current setting $\Phi(t) = t^{1/\alpha}$, we can take $r_t = t^{1/\alpha}/3$ in the definition of $\mathcal{U}(x, t)$. Let $A_9 \in (0, A_5/2]$ be a constant which will be chosen later. Without loss of generality we assume $\kappa < A_7 \wedge A_9$.

Fix $(t, x) \in (0, 1] \times D$. If $\delta_D(x) \geq \kappa t^{1/\alpha}/3$, then we have $\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D) \asymp 1$ in view of Lemma 2.21 and (2.34). Recall that $z_{x,t} \in D$ is a point such that $B(z, 3\kappa r_t) \subset B(x, 3r_t) \cap D$. Assume that $\delta_D(x) < \kappa t^{1/\alpha}/3$. In this case, we have $|x - z_{x,t}| \geq \delta_D(z_{x,t}) - \delta_D(x) > \kappa t^{1/\alpha} - \kappa t^{1/\alpha}/3 > \kappa t^{1/\alpha}/2$ and hence we should choose the second definition of $\mathcal{U}(x, t)$ so that $\mathcal{U}(x, t) = B(x, \kappa t^{1/\alpha}/3) \cap D$. Let $w \in \partial D$ be the point such that $|x - w| = \delta_D(x)$. Define $D^{\text{bdry}}(l) := \{y \in D : |y - w| < l\}$ and $D^{\text{int}}(l) := \{y \in D^{\text{bdry}}(2) : \delta_D(y) > l\}$. Note that $\mathcal{U}(x, t) \subset D^{\text{bdry}}(A_9) \subset D^{\text{bdry}}(2)$. Indeed, for every $y \in \mathcal{U}(x, t) \subset D$, we have $|y - w| \leq |y - x| + \delta_D(x) \leq \kappa/3 + \kappa/3 < A_9 < 1/8$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative radial function such that $\varphi(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} \varphi(y) dy = 1$. For $k \geq 1$, define $\varphi_k(y) := 6^{kd} \varphi(6^k y)$ and $f_k := \varphi_k * (v_1 \mathbf{1}_{D^{\text{int}}(5^{-k})})$. Since $6^{-k} < 5^{-k}$, we have $f_k \in C_c^\infty(D)$ and hence Lf_k is well defined everywhere. Pick any $z \in \mathcal{U}(x, t)$ (hence $\delta_D(z) < |z - w| < A_9$ by the calculation above) such that $\delta_D(z) > \max\{2^{-k/(q-p)}, 2^{-pk/(d+q)}\} =: a_k$ and observe that

$$\begin{aligned} Lf_k(z) &= L(\varphi_k * v_1)(z) - L(\varphi_k * v_1 - f_k)(z) \\ &= L(\varphi_k * v_1)(z) + \kappa(z)(\varphi_k * v_1 - f_k)(z) \\ &\quad - \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \int_{D, |y-z| > \epsilon} \frac{(\varphi_k * v_1)(y) - f_k(y) - (\varphi_k * v_1)(z) + f_k(z)}{|y-z|^{d+\alpha}} dy \\ &=: M_1(z) + M_2(z) + M_3(z) = M_1 + M_2 + M_3. \end{aligned}$$

To bound M_1 , we need some preparation. For $|u| < 6^{-k}$, let $w_u \in \partial D$ be a point such that $\delta_D(z - u) = |z - u - w_u|$. Note that by the triangle inequality and the assumption that $\delta_D(z) > a_k \geq 2^{-k}$, we have

$$(3^k - 1)|u| < (1 - 3^{-k})\delta_D(z) \leq \delta_D(z) - |u| \leq \delta_D(z - u) \leq \delta_D(z) + |u| \leq (1 + 3^{-k})\delta_D(z). \quad (3.13)$$

Let $\psi_u : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a $C^{1,1}$ function and CS_{w_u} an orthonormal coordinate system with origin at w_u such that $\psi_u(\tilde{0}) = 0$, $\nabla\psi_u(\tilde{0}) = \tilde{0}$, $\|\nabla\psi_u\|_\infty \leq \Lambda$, the coordinate of $z - u$ in CS_{w_u} is $(\tilde{0}, \delta_D(z - u))$ and $D \cap B(w_u, 1) = \{y^u = (\tilde{y}^u, y_d^u) \text{ in } CS_{w_u} : y_d^u > \psi_u(\tilde{y}^u)\} \cap B(w_u, 1)$. Define $D - u := \{y - u : y \in D\}$ for $u \in \mathbb{R}^d$. Using the coordinate system CS_{w_u} , we have that for all $q_0 \in [p, \alpha)$, $\epsilon \in (0, 1)$ and $|u| < 6^{-k}$,

$$\begin{aligned} & \left| \int_{D-u, |y-(z-u)| > \epsilon} \frac{h_{q_0}(y) - h_{q_0}(z-u)}{|y - (z-u)|^{d+\alpha}} dy \right| \\ & \leq \left| \int_{B(z-u, \epsilon)^c} \frac{h_{q_0}(y^u) - \delta_D(z-u)^{q_0}}{|y^u - (z-u)|^{d+\alpha}} dy^u \right| + \int_{(D-u)^c} \frac{|h_{q_0}(y^u) - \delta_D(z-u)^{q_0}|}{|y^u - (z-u)|^{d+\alpha}} dy^u \\ & \leq \left| \int_{B(z-u, \epsilon)^c} \frac{h_{q_0}(y^u) - (y_d^u \vee 0)^{q_0}}{|y^u - (z-u)|^{d+\alpha}} dy^u \right| + \left| \int_{B(z-u, \epsilon)^c} \frac{(y_d^u \vee 0)^{q_0} - \delta_D(z-u)^{q_0}}{|y^u - (z-u)|^{d+\alpha}} dy^u \right| \\ & \quad + \int_{B(z-u, \delta_D(z))^c} \frac{|y^u|^{q_0} + \delta_D(z-u)^{q_0}}{|y^u - (z-u)|^{d+\alpha}} dy^u \\ & =: N_1(z, u, \epsilon) + N_2(z, u, \epsilon) + N_3(z, u) = N_1 + N_2 + N_3. \end{aligned}$$

According to the proof of Lemma 3.1 and (3.13), we can see that for all $|u| < 6^{-k}$,

$$N_1 \leq c_1(1 + |\log \delta_D(z-u)|) \leq c_2(1 + \log(3/2) + |\log \delta_D(z)|) \quad \text{uniformly in } \epsilon \in (0, 1).$$

Moreover, by [7, p.120-121] and (3.13), we obtain $N_2 \leq c_3 \delta_D(z-u)^{q_0-\alpha} \leq c_4 \delta_D(z)^{q_0-\alpha}$ uniformly in $\epsilon \in (0, 1)$. Lastly, using the triangle inequality $|y^u| \leq |y^u - (z-u)^u| + |(z-u)^u| = |y^u - (z-u)^u| + \delta_D(z-u)$, we also have

$$\begin{aligned} N_3 & \leq c_5 \int_{\delta_D(z)}^\infty ((l + \delta_D(z-u))^{q_0} + \delta_D(z-u)^{q_0}) l^{-\alpha-1} dl \\ & \leq c_5 \int_{\delta_D(z)}^\infty ((l + 2\delta_D(z))^{q_0} + 2^{q_0} \delta_D(z)^{q_0}) l^{-\alpha-1} dl \leq c_6 \delta_D(z)^{q_0-\alpha}. \end{aligned}$$

Thus, we conclude that for all $q_0 \in [p, \alpha)$ there exists $c_7 = c_7(q_0) > 0$ such that for all $|u| < 6^{-k}$,

$$\left| \int_{D-u, |y-(z-u)| > \epsilon} \frac{h_{q_0}(y) - h_{q_0}(z-u)}{|y - (z-u)|^{d+\alpha}} dy \right| \leq c_7 \delta_D(z)^{q_0-\alpha} \quad \text{uniformly in } \epsilon \in (0, 1). \quad (3.14)$$

On the other hand, we also observe that by (3.13), for $|u| < 6^{-k}$,

$$\begin{aligned} & \int_{(D-u) \setminus D} \frac{dy}{|y - (z-u)|^{d+\alpha}} \leq \int_{((D-u) \setminus D) \cap B(w_u, 1)} \frac{dy^u}{|y^u - (z-u)|^{d+\alpha}} + \int_{B(w_u, 1)^c} \frac{dy^u}{|y^u - (z-u)|^{d+\alpha}} \\ & \leq \int_{|\tilde{y}^u| < \delta_D(z)/(2\Lambda)} \int_{\psi_u(\tilde{y}^u) - |u|}^{\psi_u(\tilde{y}^u)} (\delta_D(z-u) - y_d^u)^{-(d+\alpha)} dy_d^u d\tilde{y}^u \\ & \quad + \int_{\delta_D(z)/(2\Lambda) \leq |\tilde{y}^u| \leq 1} \int_{\psi_u(\tilde{y}^u) - |u|}^{\psi_u(\tilde{y}^u)} |\tilde{y}^u|^{-(d+\alpha)} dy_d^u d\tilde{y}^u + 2^{d+\alpha} \int_{B(0, 1)^c} |y|^{-d-\alpha} dy \\ & \leq |u| \left((\delta_D(z-u) - \delta_D(z)/2)^{-(d+\alpha)} \int_{|\tilde{y}^u| < \delta_D(z)/(2\Lambda)} d\tilde{y}^u + \int_{|\tilde{y}^u| \geq \delta_D(z)/(2\Lambda)} |\tilde{y}^u|^{-(d+\alpha)} d\tilde{y}^u \right) + c \end{aligned}$$

$$\leq A_{10}(6^{-k}\delta_D(z-u)^{-\alpha-1} + 1),$$

for some constant $A_{10} > 0$. In the second inequality above, we used the facts that the coordinates of $z-u$ in CS_{w_u} are $(\tilde{0}, \delta_D(z-u))$, and for all $y^u \in B(w_u, 1)^c$, we have $|y^u - (z-u)| \geq |y^u - w_u| - \delta_D(z-u) \geq 2^{-1}|y^u|$ since $w_u = 0$ in CS_{w_u} and $\delta_D(z-u) < 2\delta_D(z) \leq 2A_9 \leq A_5 < A_4 < 1/2$. Besides, in the third inequality above, we used the fact that for all $|\tilde{y}^u| < \delta_D(z)/(2\Lambda)$, we have $|\psi_u(\tilde{y}^u)| \leq \|\nabla\psi_u\|_\infty|\tilde{y}^u| \leq \delta_D(z)/2$. Since $2^{-k/(q-p)} \vee 2^{-k} \leq a_k < \delta_D(z) < A_9$ and $\lim_{k \rightarrow \infty} 6^{-k}a_k^{p-q-1} \leq \lim_{k \rightarrow \infty} 6^{-k}(2^{-k/(q-p)})^{p-q}(2^{-k})^{-1} \leq \lim_{k \rightarrow \infty} 6^{-k}4^k = 0$, by taking $A_9 < (A_6/(6\mathcal{A}(d, -\alpha)A_{10}))^{1/(p+\alpha-q)}$, for all k large enough, we have

$$\begin{aligned} & \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \int_{D-u, |y-(z-u)| > \epsilon} \frac{v_1(y) - v_1(z-u)}{|y - (z-u)|^{d+\alpha}} dy - \kappa(z-u)v_1(z-u) \\ &= Lv_1(z-u) + \mathcal{A}(d, -\alpha) \left(\int_{D \setminus (D-u)} \frac{v_1(z-u) - v_1(y)}{|y - (z-u)|^{d+\alpha}} dy - \int_{(D-u) \setminus D} \frac{v_1(z-u)}{|y - (z-u)|^{d+\alpha}} dy \right) \\ &\geq 2A_6\delta_D(z-u)^{q-\alpha} - \mathcal{A}(d, -\alpha)A_{10}(6^{-k}\delta_D(z-u)^{-\alpha-1} + 1)v_1(z-u) \\ &\geq 2A_6\delta_D(z-u)^{q-\alpha} - 2\mathcal{A}(d, -\alpha)A_{10}(6^{-k}\delta_D(z-u)^{p-\alpha-1} + \delta_D(z-u)^p) \\ &= 2 \left(A_6 - \mathcal{A}(d, -\alpha)A_{10}(6^{-k}\delta_D(z-u)^{p-q-1} + \delta_D(z-u)^{p-q+\alpha}) \right) \delta_D(z-u)^{q-\alpha} \\ &\geq 2 \left(A_6 - 2\mathcal{A}(d, -\alpha)A_{10}(6^{-k}a_k^{p-q-1} + A_9^{p-q+\alpha}) \right) \delta_D(z-u)^{q-\alpha} \geq A_6\delta_D(z-u)^{q-\alpha}. \end{aligned} \quad (3.15)$$

The first inequality above is valid since for all $|u| < 6^{-k}$ and $y \in D \setminus (D-u)$, by (3.13), it holds that $\delta_D(y) \leq |u| \leq \delta_D(z-u)$, implying $v_1(z-u) \geq v_1(y)$. Moreover, for k large enough, $\delta_D(z-u) \leq A_9 + 6^{-k} < A_5$ so we could use (3.12). We used the fact that $v_1(z-u) \leq 2\delta_D(z-u)^p$ in the second and (3.13) in the third inequality above.

Now, since the support of φ_k is contained in $B(0, 6^{-k})$, we have that for all sufficiently large k ,

$$\begin{aligned} M_1 &= \lim_{\epsilon \downarrow 0} \mathcal{A}(d, -\alpha) \int_{D, |y-z| > \epsilon} \int_{\mathbb{R}^d} \varphi_k(u) \frac{v_1(y-u) - v_1(z-u)}{|y-z|^{d+\alpha}} du dy \\ &\quad - \int_{\mathbb{R}^d} \kappa(z)\varphi_k(u)v_1(z-u) du \\ &= \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \varphi_k(u) \left(\mathcal{A}(d, -\alpha) \int_{D-u, |y-(z-u)| > \epsilon} \frac{v_1(y) - v_1(z-u)}{|y - (z-u)|^{d+\alpha}} dy - \kappa(z-u)v_1(z-u) \right) du \\ &\quad + \int_{\mathbb{R}^d} (\kappa(z-u) - \kappa(z))\varphi_k(u)v_1(z-u) du \\ &\geq A_6 \int_{\mathbb{R}^d} (\delta_D(z) + |u|)^{q-\alpha} \varphi_k(u) du + C_1 \int_{\mathbb{R}^d} (\delta_D(z-u)^{-\alpha} - \delta_D(z)^{-\alpha}) \varphi_k(u)v_1(z-u) du \\ &\quad - C_2 \int_{\mathbb{R}^d} (\delta_D(z-u)^{-\eta} + \delta_D(z)^{-\eta}) \varphi_k(u)v_1(z-u) du \\ &\geq A_6(1 + 3^{-k})^{q-\alpha} \delta_D(z)^{q-\alpha} - C_1(1 - (1 + 3^{-k})^{-\alpha}) \delta_D(z)^{-\alpha} (\varphi_k * v_1)(z) \\ &\quad - C_2(1 + (1 - 3^{-k})^{-\eta}) \delta_D(z)^{-\eta} (\varphi_k * v_1)(z). \end{aligned}$$

In the second equality above we used Fubini's theorem and the change of variables. In the first inequality we first used the dominated convergence theorem (which is applicable due to (3.14)) and then (3.15) and (3.6). In the second inequality above, we used (3.13).

Since $(\varphi_k * v_1)(z) \leq 2(1 + 3^{-k})^q \delta_D(z)^p$, $q < p - \eta + \alpha$ and $2^{-k/(q-p)} \leq a_k < \delta_D(z) < A_9$, by taking $A_9 < (A_6/(432C_2))^{1/(p-\eta+\alpha-q)}$, for all k large enough, we have

$$\begin{aligned} M_1 &\geq \frac{7}{8} A_6 \delta_D(z)^{q-\alpha} - 3(\alpha C_1 3^{-k} \delta_D(z)^{p-q} + 3C_2 \delta_D(z)^{p-\eta+\alpha-q}) \delta_D(z)^{q-\alpha} \\ &\geq \left(\frac{7}{8} A_6 - 3\alpha C_1 3^{-k} a_k^{p-q} - 9C_2 A_9^{p-\eta+\alpha-q} \right) \delta_D(z)^{q-\alpha} \geq \frac{5}{6} A_6 \delta_D(z)^{q-\alpha}. \end{aligned}$$

Note that for every $k \geq 2$, $u \in B(0, 6^{-k})$ and $y \in D$ such that $\delta_D(y) > 4^{-k}$ and $|y-w| \leq 1$, we have $\delta_D(y-u) \geq 4^{-k} - 6^{-k} > 5^{-k}$ and $|y-u-w| \leq |y-w| + |u| < 2$ and therefore

$$1 - \mathbf{1}_{D^{\text{int}}(5^{-k})}(y-u) = 0. \quad (3.16)$$

In particular, since φ_k is supported in $B(0, 6^{-k})$, $\delta_D(z) > 2^{-pk/(d+q)} > 4^{-k}$ and $|z-w| \leq |z-x| + |x-w| < 2t^{1/\alpha}/3 < 1$, for all $k \geq 2$, we have

$$M_2 = \kappa(z) \int_{\mathbb{R}^d} (1 - \mathbf{1}_{D^{\text{int}}(5^{-k})}(z-u)) v_1(z-u) \varphi_k(u) du = 0.$$

Finally, using (3.16), by taking A_9 sufficiently smaller than A_6 , for all k large enough, we have

$$\begin{aligned} |M_3| &\leq \mathcal{A}(d, -\alpha) \lim_{\epsilon \downarrow 0} \int_{D, |y-z| > \epsilon} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(1 - \mathbf{1}_{D^{\text{int}}(5^{-k})}(y-u)) v_1(y-u)}{|y-z|^{d+\alpha}} dudy \\ &\leq c_1 \left(\int_{D, \delta_D(y) \leq 4^{-k}} \int_{\mathbb{R}^d} \varphi_k(u) \frac{\delta_D(y-u)^p}{|y-z|^{d+\alpha}} dudy + \int_{D, |y-w| > 1} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(\delta_D(y-u) + 1)^q}{|y-z|^{d+\alpha}} dudy \right) \\ &\leq c_1 \left(\int_{D, \delta_D(y) \leq 4^{-k}} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(\delta_D(y) + |u|)^p}{|y-z|^{d+\alpha}} dudy + \int_{|y-w| > 1} \int_{\mathbb{R}^d} \varphi_k(u) \frac{(|y-w| + |u| + 1)^q}{(3^{-1}|y-w|)^{d+\alpha}} dudy \right) \\ &\leq c_2 \left(\int_{D, \delta_D(y) \leq 4^{-k}} \frac{4^{-pk}}{|y-z|^{d+\alpha}} dy \int_{\mathbb{R}^d} \varphi_k(u) du + \int_1^\infty l^{q-\alpha-1} dl \int_{\mathbb{R}^d} \varphi_k(u) du \right) \\ &\leq c_2 \left(\int_{D, \delta_D(y) \leq 4^{-k}, |y-z| \leq 1} \frac{4^{-pk}}{(\delta_D(z) - \delta_D(y))^{d+\alpha}} dy + \int_{|y-z| > 1} \frac{4^{-pk}}{|y-z|^{d+\alpha}} dy + \frac{1}{\alpha - q} \right) \\ &\leq c_3 (4^{-pk} \delta_D(z)^{-d-\alpha} + 1) \leq c_3 (4^{-pk} 2^{pk} + A_9^{\alpha-q}) \delta_D(z)^{q-\alpha} \leq \frac{A_6}{2} \delta_D(z)^{q-\alpha}. \end{aligned}$$

In the second inequality above, we have used the facts that $\delta_D(y)^q \leq \delta_D(y)^p$ for $\delta_D(y) < 1$ and $\delta_D(y)^q + 1 \geq \delta_D(y)^p$ for all y , since $q > p$. In the third inequality, we first estimate $|z-w| \leq |z-x| + |x-w| < 2/3 < (2/3)|y-w|$ by using that $|y-w| > 1$, which implies that $|y-z| \geq |y-w| - |z-w| \geq (1/3)|y-w|$. The estimate $\delta_D(y-u) \leq \delta_D(y) + |u| \leq |y-w| + |u|$, follows by the choice of $w \in \partial D$. In the fourth inequality, we have used the fact that the support of φ_k is contained in $B(0, 6^{-k})$. Besides, we have used the fact that $\int_{\mathbb{R}^d} \varphi_k(u) du = 1$ in the fifth inequality and $\delta_D(z) \leq |z-w| < A_9$ in the seventh inequality, and the assumption that $\delta_D(z) > a_k \geq 2^{-pk/(d+q)}$ in the sixth and seventh inequalities.

Thus, we conclude that, for all sufficiently large k , $Lf_k(z) \geq 3^{-1}A_6\delta_D(z)^{q-\alpha} \geq 0$ for all $z \in \mathcal{U}(x, t)$ such that $\delta_D(z) > a_k$. Recall that $f_k \in C_c^\infty(D)$ and hence contained in the domain of the generator of Y . Thus, by Dynkin's formula, we have that for all sufficiently large k ,

$$f_k(x) = \mathbb{E}_x[f_k(Y_{\tau_{\mathcal{U}(x,t) \cap D^{\text{int}}(a_k)}})] - \mathbb{E}_x \left[\int_0^{\tau_{\mathcal{U}(x,t) \cap D^{\text{int}}(a_k)}} Lf_k(Y_t) dt \right] \leq \mathbb{E}_x[f_k(Y_{\tau_{\mathcal{U}(x,t) \cap D^{\text{int}}(a_k)}})].$$

Since $f_k = \varphi_k * (v_1 \mathbf{1}_{D^{\text{int}}(5-k)}) \rightarrow v_1 \mathbf{1}_{D^{\text{bdry}}(2)} \leq v_1$ pointwise and $Y_{\tau_{\mathcal{U}(x,t) \cap D^{\text{int}}(a_k)}} \rightarrow Y_{\tau_{\mathcal{U}(x,t)}}$ (using $\mathcal{U}(x, t) \subset D^{\text{bdry}}(2)$), it follows from the bounded convergence theorem,

$$\begin{aligned} \delta_D(x)^p \leq v_1(x) &= \lim_{k \rightarrow \infty} f_k(x) \leq \lim_{k \rightarrow \infty} \mathbb{E}_x[f_k(Y_{\tau_{\mathcal{U}(x,t) \cap D^{\text{int}}(a_k)}})] \\ &= \mathbb{E}_x[v_1(Y_{\tau_{\mathcal{U}(x,t)}}) : Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2)] \leq \mathbb{E}_x[v_1(Y_{\tau_{\mathcal{U}(x,t)}})]. \end{aligned}$$

Recall that we have assumed $\kappa < A_7 \wedge A_9 < A_5 \wedge A_7$. Set $r = r(t) := (A_5 \wedge A_7)t^{1/\alpha} > \kappa t^{1/\alpha}$. Note that for every $n \geq 1$ and $u \in D^{\text{bdry}}(2^n r)$, we have $v_1(u) \leq (\delta_D(x) + 2^n r)^p + (\delta_D(x) + 2^n r)^q \leq 2^{(n+1)p} r^p + 2^{(n+1)q} r^q \leq 2^{(n+1)q+1} r^p$. Thus, we have

$$\begin{aligned} &\mathbb{E}_x[v_1(Y_{\tau_{\mathcal{U}(x,t)}})] \\ &\leq \mathbb{E}_x[v_1(Y_{\tau_{\mathcal{U}(x,t)}}) : Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)] + \sum_{n=0}^{\infty} \mathbb{E}_x[v_1(Y_{\tau_{\mathcal{U}(x,t)}}) : Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)] \\ &\leq c_0 r^p \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)) + c_0 \sum_{n=0}^{\infty} 2^{(n+1)q+1} r^p \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)) \end{aligned}$$

and that for every $n \geq 0$,

$$\begin{aligned} &\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)) \\ &\leq c_1 \mathbb{E}_x \int_0^{\tau_{\mathcal{U}(x,t)}} \int_{D^{\text{bdry}}(2^{n+1}r) \setminus D^{\text{bdry}}(2^n r)} |Y_s - z|^{-d-\alpha} dz ds \\ &\leq c_2 (2^{n+1}r)^d (2^n r)^{-d-\alpha} \mathbb{E}_x[\tau_{\mathcal{U}(x,t)}] = c_3 2^{-n\alpha} r^{-\alpha} \mathbb{E}_x[\tau_{\mathcal{U}(x,t)}]. \end{aligned}$$

Since

$$\mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)) \geq c_4 \mathbb{E}_x \left[\int_0^{\tau_{\mathcal{U}(x,t)}} \int_{D^{\text{bdry}}(r)} |Y_s - z|^{-d-\alpha} dz ds \right] \geq c_5 r^{-\alpha} \mathbb{E}_x[\tau_{\mathcal{U}(x,t)}],$$

we deduce that

$$\begin{aligned} \delta_D(x)^p &\leq c_0 r^p \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)) + c_6 \sum_{n=0}^{\infty} 2^{(n+1)q-n\alpha} r^p \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)) \\ &\leq c_7 r^p \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D^{\text{bdry}}(r)) \leq c_7 r^p \mathbb{P}_x(Y_{\tau_{\mathcal{U}(x,t)}} \in D), \end{aligned}$$

where in the second inequality we used the fact that $q < \alpha$.

By applying the similar argument to the function $g_k := \varphi_k * (v_2 \mathbf{1}_{D^{\text{int}}(5^{-k})})$, we also have that

$$\begin{aligned} \delta_D(x)^p \geq v_2(x) &= \lim_{k \rightarrow \infty} g_k(x) \geq \lim_{k \rightarrow \infty} \mathbb{E}_x [g_k(Y_{\tau_{\mathcal{U}(x,t)} \cap D^{\text{int}}(a_k)}))] \\ &= \mathbb{E}_x [(v_2 \mathbf{1}_{D^{\text{bdry}}(2)})(Y_{\tau_{\mathcal{U}(x,t)}})] \geq \frac{1}{2} r^p \mathbb{P}_x (Y_{\tau_{\mathcal{U}(x,t)}} \in \mathcal{W}(x, t)). \end{aligned}$$

The last inequality holds since $\mathcal{W}(x, t) \subset D^{\text{bdry}}(2)$.

Therefore, in view of Lemma 2.21, we get $\mathbb{P}_x(\zeta > t) \asymp \left(\frac{\delta_D(x)}{r}\right)^p$. Finally, from (3.9) we conclude that

Theorem 3.2. *Suppose that D is a $C^{1,1}$ open set in \mathbb{R}^d , $d \geq 2$, with characteristics (R_2, Λ) . For all $T > 0$, $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$ and $\eta \in [0, \alpha)$, there exists a constant $c = c(C_1, C_2, p, \alpha, d, \eta, T, R_2, \Lambda) \geq 1$ such that for all $\kappa \in \mathcal{H}_\alpha(p)$, the transition density $q^D(t, x, y)$ of the Hunt process Y on D corresponding to the Feynman-Kac semigroup of X^D via the multiplicative functional $e^{-\int_0^t \kappa(X_s^D) ds}$ satisfies that*

$$\begin{aligned} c^{-1} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right] \\ \leq q^D(t, x, y) \leq c \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right] \end{aligned}$$

for $(t, x, y) \in (0, T] \times D \times D$.

In the case $D = \mathbb{R}_+^d$ and $\kappa(x) = C(d, \alpha, p)x_d^{-\alpha}$, one can use the scaling property to get that the two-sided heat estimates in Theorem 3.2 is valid for all $t > 0$.

Remark 3.3. *Theorem 3.2 also holds in $d = 1$. In fact, let $D \subset \mathbb{R}$ be a union of open intervals with a localization radius r_0 and $C(1, \alpha, p) = \mathcal{A}(1, -\alpha)\gamma(\alpha, p)$. The first difference of the proof appears in the bound of $|III|$ in Lemma 3.1. We use the following calculation instead of [7, Lemma 5.6]:*

$$\begin{aligned} |\kappa_D(x) - \kappa_{\mathbb{R}_+}(x)| &\leq |\kappa_{(-r_0, 0)^c}(x) - \kappa_{(0, r_0)}(x)| = \mathcal{A}(1, -\alpha) \left[\int_{-\infty}^{-r_0} + \int_{r_0}^{\infty} \frac{dy}{|y-x|^{1+\alpha}} \right] \\ &= \frac{\mathcal{A}(1, -\alpha)}{\alpha} ((r_0 - x)^{-\alpha} + (r_0 + x)^{-\alpha}) \leq c, \end{aligned}$$

provided $\delta_D(x) < r_0/2$. Moreover, the bound for $|I_\epsilon|$ is easy in Lemma 3.1: Since $h_q(y) = w_q(y)$ for $y \in (-\infty, r_0)$, $I_\epsilon \leq c \int_{r_0}^{\infty} y^{q-1-\alpha} dy = c$.

Remark 3.4. *It follows from [7, pp.94–95] that Z^D can be obtained from X^D via a Feynman-Kac perturbation of the form $e^{-\int_0^t \kappa_D(X_s^D) ds}$. In view of (3.11), κ_D satisfies condition (3.6) with $C_1 = \frac{\mathcal{A}(d, -\alpha) \omega_{d-1}}{\alpha} \beta\left(\frac{\alpha+1}{2}, \frac{d-1}{2}\right)$. By direct calculation, we can see that $\gamma(\alpha, \alpha/2) = 1/\alpha$. This means that $C_1 = C(d, \alpha, \alpha/2)$. Thus Theorem 3.2 recovers (3.3). When $\alpha \in (1, 2)$, $C_1 = 0 = C(d, \alpha, \alpha - 1)$ is allowed. Thus, by taking $\kappa = 0$, Theorem 3.2 recovers (3.2) as well.*

We also remark here that Theorem 3.2 provides examples of processes studied in [22] (see (3.6) and [22, Proposition 4.1(ii)]).

3.2 Non-local perturbation in bounded $C^{1,1}$ open set

Recall that $\mathcal{A}(d, \alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$. We also recall that we write $y = (\tilde{y}, y_d)$ for $y \in \mathbb{R}^d$. For $u : \mathbb{R}_+^d \rightarrow [0, \infty)$, $\lambda \in (0, \infty)$ and $\beta \in (-\infty, 2)$, we define

$$L_{d,\lambda}^\beta u(x) := \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}_+^d : \varepsilon < |y-x| < \lambda\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\beta}}, \quad x \in \mathbb{R}_+^d.$$

In the remainder of this subsection, we will assume that $\beta \in (-\infty, 2)$. For any real number p and $y \in \mathbb{R}_+^d$, let $g(y) := y_d^p = \delta_{\mathbb{R}_+^d}(y)^p$.

Lemma 3.5. *For all positive p, λ and $\beta \in (-\infty, 2)$, there exist $c_1 = c_1(p, d, \beta, \lambda) > 0$ and $c_2 = c_2(p, d, \beta, \lambda) \in (0, \frac{1}{4})$ such that, for every $x \in \mathbb{R}_+^d$ with $0 < x_d < c_2$, the following inequalities hold:*

$$|L_{d,\lambda}^\beta g(x)| \leq c_1 \begin{cases} 1 & \text{if } p > \beta; \\ |\log x_d| & \text{if } p = \beta; \\ x_d^{p-\beta} & \text{if } p < \beta. \end{cases}$$

Proof. When $\beta \leq 0$, then clearly for $x \in \mathbb{R}_+^d$,

$$\int_{\mathbb{R}_+^d} \frac{|y_d^p - x_d^p|}{|y-x|^{d+\beta}} \mathbf{1}_{\{|y-x| < \lambda\}} dy \leq c \int_{B(0,\lambda)} |z|^{-d-\beta+p} dz \leq c \int_0^\lambda s^{-1-\beta+p} ds = c \lambda^{p-\beta}.$$

We now assume $\beta > 0$. For simplicity, take $x = (\tilde{0}, x_d)$ and denote $\mathbf{e}_d = (\tilde{0}, 1)$. Then by the change of variables $z = y/x_d$, we have

$$\begin{aligned} L_{d,\lambda}^\beta g(x) &= \text{p.v.} \int_{\mathbb{R}_+^d} \frac{y_d^p - x_d^p}{|y-x|^{d+\beta}} \mathbf{1}_{\{|y-x| < \lambda\}} dy \\ &= x_d^{p-\beta} \text{p.v.} \int_{\mathbb{R}^{d-1}} \int_0^\infty \frac{z_d^p - 1}{|z - \mathbf{e}_d|^{d+\beta}} \mathbf{1}_{\{|z - \mathbf{e}_d| < \lambda/x_d\}} dz_d d\tilde{z} =: x_d^{p-\beta} I_1. \end{aligned}$$

Using the change of variables $\tilde{z} = |z_d - 1| \tilde{u}$, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} \left(\text{p.v.} \int_0^\infty \frac{z_d^p - 1}{|z_d - 1|^{1+\beta}} \mathbf{1}_{\{|z_d - 1| < \lambda/(|\tilde{u}|^2 + 1)^{-1/2}/x_d\}} dz_d \right) d\tilde{u} \\ &=: \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} I_2 d\tilde{u}. \end{aligned}$$

Fix \tilde{u} and let $M := (|\tilde{u}|^2 + 1)^{1/2}$. Then

$$I_2 = \lim_{\varepsilon \rightarrow 0} \left(\int_{(1-\frac{\lambda}{Mx_d})_+}^{1-\varepsilon} \frac{z_d^p - 1}{|z_d - 1|^{1+\beta}} dz_d + \int_{1+\varepsilon}^{\frac{Mx_d+\lambda}{Mx_d}} \frac{z_d^p - 1}{|z_d - 1|^{1+\beta}} dz_d \right). \quad (3.17)$$

By using the change of variables $w = 1/z_d$, we get that, for $\varepsilon < \lambda/(Mx_d)$, the second integral in (3.17) is equal to

$$\int_{\frac{Mx_d}{Mx_d+\lambda}}^{\frac{1}{1+\varepsilon}} \frac{w^{\beta-1-p} - w^{\beta-1}}{(1-w)^{1+\beta}} dw = \int_{\frac{Mx_d}{Mx_d+\lambda}}^{1-\varepsilon} \frac{w^{\beta-1-p} - w^{\beta-1}}{(1-w)^{1+\beta}} dw + \int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{w^{\beta-1-p} - w^{\beta-1}}{(1-w)^{1+\beta}} dw.$$

Note that from [7, p.121], we see that

$$\left| \int_{1-\epsilon}^{\frac{1}{1+\epsilon}} \frac{w^{\beta-1-p} - w^{\beta-1}}{(1-w)^{1+\beta}} dw \right| \leq c\epsilon^{2-\beta}.$$

By writing the first integral in (3.17) as

$$\int_{\frac{Mx_d}{Mx_d+\lambda}}^{1-\epsilon} \frac{w^p - 1}{(1-w)^{1+\beta}} dw - \int_{(1-\frac{\lambda}{Mx_d})_+}^{\frac{Mx_d}{Mx_d+\lambda}} \frac{1-w^p}{(1-w)^{1+\beta}} dw,$$

and by using

$$(w^p - 1) + (w^{\beta-1-p} - w^{\beta-1}) = (1-w^p)(1-w^{p-(\beta-1)})w^{\beta-1-p}, \quad (3.18)$$

we have

$$I_2 = \lim_{\epsilon \rightarrow 0} \int_{\frac{Mx_d}{Mx_d+\lambda}}^{1-\epsilon} \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p} dw - \int_{(1-\frac{\lambda}{Mx_d})_+}^{\frac{Mx_d}{Mx_d+\lambda}} \frac{1-w^p}{(1-w)^{1+\beta}} dw =: I_{21} - I_{22}. \quad (3.19)$$

First, it is easy to see that

$$0 < I_{22} \leq \int_0^{\frac{Mx_d}{Mx_d+\lambda}} \frac{1-w^p}{(1-w)^{1+\beta}} dw \leq c \begin{cases} 1 & \text{if } \beta \in (0, 1); \\ \log(1 + Mx_d/\lambda) & \text{if } \beta = 1; \\ (1 + Mx_d/\lambda)^{\beta-1} & \text{if } \beta \in (1, 2). \end{cases}$$

Next, since $\beta < 2$, the fraction in I_{21} is integrable near 1. Thus,

$$I_{21} = \int_{\frac{Mx_d}{Mx_d+\lambda}}^1 \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p} dw.$$

Note that, if $\frac{Mx_d}{Mx_d+\lambda} \geq 1/4$, then clearly, $I_{21} \leq c < \infty$. If $\frac{Mx_d}{Mx_d+\lambda} < 1/4$, then

$$I_{21} \leq c + \int_{\frac{Mx_d}{Mx_d+\lambda}}^{1/2} \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p} dw \leq c + c \int_{\frac{Mx_d}{Mx_d+\lambda}}^1 w^{\beta-1-p} dw.$$

Thus

$$I_{21} \leq c \begin{cases} (1 + \lambda/(Mx_d))^{p-\beta} & \text{if } p > \beta; \\ \log(1 + \lambda/(Mx_d)) & \text{if } p = \beta. \end{cases}$$

Therefore, if $p > \beta$, then for small x_d ,

$$\begin{aligned} & |x_d^{p-\beta} I_1| \\ & \leq c x_d^{p-\beta} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} \end{aligned}$$

$$\begin{aligned}
& \times \begin{cases} (1 + \lambda/((|\tilde{u}|^2 + 1)^{1/2}x_d))^{p-\beta}d\tilde{u} & \text{if } \beta \in (0, 1); \\ ((1 + \lambda/((|\tilde{u}|^2 + 1)^{1/2}x_d))^{p-\beta} + \log(1 + (|\tilde{u}|^2 + 1)^{1/2}x_d/\lambda))d\tilde{u} & \text{if } \beta = 1; \\ ((1 + \lambda/((|\tilde{u}|^2 + 1)^{1/2}x_d))^{p-\beta} + (1 + (|\tilde{u}|^2 + 1)^{1/2}x_d/\lambda)^{\beta-1})d\tilde{u} & \text{if } \beta \in (1, 2) \end{cases} \\
& \leq c \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} \\
& \times \begin{cases} (x_d + \lambda/(|\tilde{u}|^2 + 1)^{1/2})^{p-\beta}d\tilde{u} & \text{if } \beta \in (0, 1); \\ ((x_d + \lambda/(|\tilde{u}|^2 + 1)^{1/2})^{p-\beta} + \log(1 + (|\tilde{u}|^2 + 1)^{1/2}/\lambda))d\tilde{u} & \text{if } \beta = 1; \\ ((x_d + \lambda/(|\tilde{u}|^2 + 1)^{1/2})^{p-\beta} + (1 + (|\tilde{u}|^2 + 1)^{1/2}/\lambda)^{\beta-1})d\tilde{u} & \text{if } \beta \in (1, 2) \end{cases} \\
& \leq c(\lambda) \begin{cases} \int_{\mathbb{R}^{d-1}} (|\tilde{u}|^2 + 1)^{-(d+\beta)/2}d\tilde{u} & \text{if } \beta \in (0, 1); \\ \int_{\mathbb{R}^{d-1}} \frac{\log(1+(|\tilde{u}|^2+1)^{1/2})}{(|\tilde{u}|^2+1)^{(d+\beta)/2}}d\tilde{u} & \text{if } \beta = 1; \\ \int_{\mathbb{R}^{d-1}} (|\tilde{u}|^2 + 1)^{-(d+1)/2}d\tilde{u} & \text{if } \beta \in (1, 2) \end{cases} = c(\lambda, \beta) < \infty.
\end{aligned}$$

If $p = \beta > 0$, then for small x_d ,

$$\begin{aligned}
|I_1| & \leq c \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} \times \\
& \begin{cases} \log(1 + \lambda/((|\tilde{u}|^2 + 1)^{1/2}x_d))d\tilde{u} & \text{if } \beta \in (0, 1); \\ [\log(1 + \lambda/((|\tilde{u}|^2 + 1)^{1/2}x_d)) + \log(1 + (|\tilde{u}|^2 + 1)^{1/2}x_d/\lambda)]d\tilde{u} & \text{if } \beta = 1; \\ [\log(1 + \lambda/((|\tilde{u}|^2 + 1)^{1/2}x_d)) + (1 + (|\tilde{u}|^2 + 1)^{1/2}x_d/\lambda)^{\beta-1}]d\tilde{u} & \text{if } \beta \in (1, 2) \end{cases} \\
& \leq c \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} \begin{cases} \log(1 + \lambda/x_d)d\tilde{u} & \text{if } \beta \in (0, 1); \\ [\log(1 + \lambda/x_d) + \log(1 + (|\tilde{u}|^2 + 1)^{1/2}/\lambda)]d\tilde{u} & \text{if } \beta = 1; \\ [\log(1 + \lambda/x_d) + (1 + (|\tilde{u}|^2 + 1)^{1/2}/\lambda)^{\beta-1}]d\tilde{u} & \text{if } \beta \in (1, 2) \end{cases} \\
& \leq c \log(1 + \lambda/x_d).
\end{aligned}$$

We now assume that $0 < p < \beta$. Note that by (3.19), (3.18) and simple algebra,

$$I_2 = \int_{(1-\frac{\lambda}{Mx_d})_+}^1 \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p} dw - \int_{(1-\frac{\lambda}{Mx_d})_+}^{\frac{Mx_d}{Mx_d+\lambda}} \frac{w^{\beta-1-p} - w^{\beta-1}}{(1-w)^{1+\beta}} dw.$$

Since $w \mapsto w^{\beta-1-p}$ is integrable near 0,

$$x_d^{p-\beta} I_1 \leq x_d^{p-\beta} \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\beta)/2}} \left(\int_0^1 \frac{(1-w^p)(1-w^{p-(\beta-1)})}{(1-w)^{1+\beta}} w^{\beta-1-p} dw \right) d\tilde{u}.$$

On the other hand, $-x_d^{p-\beta} I_1 \leq c(d)x_d^{p-\beta} I_{1,2}$, where

$$I_{1,2} := \int_0^\infty \frac{1}{(u^2 + 1)^{(d+\beta)/2}} \int_{(1-\frac{\lambda}{(u^2+1)^{1/2}x_d})_+}^{\frac{(u^2+1)^{1/2}x_d}{(u^2+1)^{1/2}x_d+\lambda}} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}} dw u^{d-2} du.$$

Note that

$$\sup_{v \geq 2\lambda} \int_{1-\lambda/v}^{v/(v+\lambda)} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}} dw \leq c \sup_{v \geq 2\lambda} \int_{1-\lambda/v}^{v/(v+\lambda)} (1-w)^{-\beta} dw$$

$$= c \sup_{v \geq 2\lambda} \int_{\lambda/(v+\lambda)}^{\lambda/v} t^{-\beta} dt \leq c \sup_{v \geq 2\lambda} (v + \lambda)^\beta \left(\frac{1}{v} - \frac{1}{v + \lambda} \right) \leq c \sup_{v \geq 2\lambda} v^{\beta-2} < \infty,$$

and, for $x_d < \lambda$,

$$\sup_{x_d \leq v < 2\lambda} \int_0^{v/(v+\lambda)} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}} dw \leq c \sup_{x_d \leq v < 2\lambda} \int_0^{2/3} w^{\beta-1-p} dw < \infty.$$

Thus for $x_d < \lambda$,

$$\begin{aligned} 0 < I_{1,2} &\leq \int_0^\infty \frac{\mathbf{1}_{(u^2+1)^{1/2}x_d < 2\lambda}}{(u^2+1)^{(d+\beta)/2}} \int_0^{\frac{(u^2+1)^{1/2}x_d}{(u^2+1)^{1/2}x_d+\lambda}} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}} dw u^{d-2} du \\ &\quad + \int_0^\infty \frac{\mathbf{1}_{(u^2+1)^{1/2}x_d \geq 2\lambda}}{(u^2+1)^{(d+\beta)/2}} \int_{(1-\frac{\lambda}{(u^2+1)^{1/2}x_d})_+}^{\frac{(u^2+1)^{1/2}x_d}{(u^2+1)^{1/2}x_d+\lambda}} \frac{w^{\beta-1-p}(1-w^p)}{(1-w)^{1+\beta}} dw u^{d-2} du \\ &\leq \int_0^\infty \frac{u^{d-2}}{(u^2+1)^{(d+\beta)/2}} du < \infty. \end{aligned}$$

□

Throughout the remainder of this subsection we assume that D is a bounded $C^{1,1}$ open subset of \mathbb{R}^d and $\alpha \in (0 \vee \beta, 2)$. We also assume that $b(x, y)$ is a symmetric Borel function on $D \times D$ such that $C_{b,1} := \sup_{x,y \in D} |b(x, y)| < \infty$ and the function

$$B(x, y) := \mathcal{A}(d, -\alpha) + |x - y|^{\alpha-\beta} b(x, y), \quad x, y \in D,$$

is bounded below by a positive constant, that is, $C_{b,2} \leq B(x, y)$ for some $C_{b,2} \in (0, \infty)$. Clearly, $B(x, y)$ is bounded above by $\mathcal{A}(d, -\alpha) + (\text{diam}(D))^{\alpha-\beta} C_{b,1}$.

We further assume that the first partials of $B(x, y)$ are bounded on $D \times D$. Note that, β and b can be negative, as long as the condition above is satisfied. Let $(\mathcal{E}^{(B)}, \overline{\mathcal{F}})$ be the Dirichlet form on $L^2(D, dx)$ defined by

$$\mathcal{E}^{(B)}(u, v) := \frac{1}{2} \int_D \int_D (u(x) - u(y))(v(x) - v(y)) \frac{B(x, y)}{|x - y|^{d+\alpha}} dx dy, \quad u, v \in \overline{\mathcal{F}}.$$

By [24], $(\mathcal{E}^{(B)}, \overline{\mathcal{F}})$ is a regular Dirichlet form on \overline{D} and its associated Hunt process $X^{(B)}$ is conservative and lives on \overline{D} . Moreover, since $B(x, y)$ is bounded on $D \times D$ between two strictly positive constants, the form $(\mathcal{E}^{(B)}, \overline{\mathcal{F}})$ satisfies the assumptions of [7, Remark 2.4], so we can freely use results of [7, Section 2]. Further, $X^{(B)}$ admits a strictly positive and jointly continuous transition density $p(t, x, y)$ with respect to the Lebesgue measure dx such that

$$C_0^{-1} \left[t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right] \leq p(t, x, y) \leq C_0 \left[t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right], \quad (3.20)$$

for $(t, x, y) \in (0, 1) \times D \times D$.

Let $L^{(B)}$ be the generator of $X^{(B)}$ in the L^2 sense. Similar to [48, Section 4], cf. also [41], we can show that $C_c^2(D)$ is contained in the domain of $L^{(B)}$ and give an explicit expression for $L^{(B)}f$ when $f \in C_c^2(D)$. Using these, one can check that the process $X^{(B)}$ satisfies Assumptions **A** and **U**.

If $m > 0$, by taking $\beta = \alpha - 2$ and $b(x, y) = \mathcal{A}(d, -\alpha)(\varphi(m^{1/\alpha}|x - y|) - 1)|x - y|^{-2}$ with

$$\varphi(r) := 2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}} ds,$$

we cover the reflected relativistic α -stable process X^m with weight $m > 0$ in \bar{D} . When $\alpha \in (1, 2)$, the killed process $X^{m,D}$ is the censored relativistic α -stable process in D . When $\alpha \in (0, 1]$, it follows from [7, Section 2] that, starting from inside D , the process X^m neither hits nor approaches ∂D at any finite time. Thus, the killed process $X^{m,D}$ is simply X^m restricted to D .

Recall that for $u : D \rightarrow [0, \infty)$,

$$L_\beta u(x) = \mathcal{A}(d, -\beta) \lim_{\varepsilon \downarrow 0} \int_{\{y \in D: \varepsilon < |y-x|\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\beta}}, \quad x \in D.$$

Let

$$L_{\beta,b} u(x) := \lim_{\varepsilon \downarrow 0} \int_{\{y \in D: \varepsilon < |y-x|\}} (u(y) - u(x)) \frac{b(x, y)}{|x-y|^{d+\beta}} dy, \quad x \in D.$$

Let $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$, $\kappa \in \mathcal{H}_\alpha(p)$. If $\beta \geq p$, then we always assume that, there exist $C_{b,3} > 0$ and $\beta_1 > \beta - p$ such that

$$|b(x, y) - b(x, x)| \leq C_{b,3} |x - y|^{\beta_1}, \quad x, y \in D. \quad (3.21)$$

Note that, under (3.21), for any bounded Borel function u satisfying $|u(x) - u(y)| \leq c|x - y|^p$ on D ,

$$\begin{aligned} |L_{\beta,b} u(x)| &\leq \int_{\{y \in D: \varepsilon < |y-x|\}} |u(y) - u(x)| \frac{|b(x, y) - b(x, x)|}{|x-y|^{d+\beta}} dy + \frac{|b(x, x)|}{\mathcal{A}(d, -\beta)} |L_\beta u(x)| \\ &\leq c_1 + c_2 |L_\beta u(x)|. \end{aligned} \quad (3.22)$$

Recall that for an open set D and $q \geq 0$, $h_q(x) = \delta_D(x)^q$.

Lemma 3.6. *Let D be a bounded $C^{1,1}$ open set with characteristics (R_2, Λ) . For any $q \geq p$, there exist constants $c_1 > 0$ and $c_2 \in (0, (R_2 \wedge 1)/4)$ depending only on $p, q, d, \beta, R_2, \Lambda, \text{diam}(D), C_{b,1}, C_{b,2}, C_{b,3}, \beta_1$ such that for every $x \in D$ with $0 < \delta_D(x) < c_2$, the following inequalities hold:*

$$|L_{\beta,b} h_q(x)| \leq c_1 \begin{cases} 1 & \text{if } q > \beta; \\ |\log \delta_D(x)| & \text{if } q = \beta; \\ \delta_D(x)^{q-\beta} & \text{if } q < \beta. \end{cases}$$

Proof. Without loss of generality we assume $\text{diam}(D) \leq 1$ and let $x \in D$ with $\delta_D(x) < R_2/4$. Choose a point $z \in \partial D$ such that $\delta_D(x) = |x - z|$. Then, there exists a $C^{1,1}$ function

$\Gamma : \mathbb{R}^{d-1} \mapsto \mathbb{R}$ such that $\Gamma(z) = \nabla\Gamma(z) = 0$ and an orthonormal coordinate system CS_z with origin at z such that

$$D \cap B(z, R_2) = \{y = (\tilde{y}, y_d) \text{ in } CS_z : y_d > \Gamma(\tilde{y})\} \cap B(z, R_2),$$

and $z = 0$ and $x = (\tilde{x}, x_d) = (0, x_d)$ in CS_z . Define $w_q(y) := (y_d)^q$ for $y \in \mathbb{R}_+^d$ and $w_q(y) := 0$, otherwise. For any open subset $U \subset \mathbb{R}^d$, define $\widehat{\kappa}_U(x) := \mathcal{A}(d, -\beta) \int_{U \cap B(x, 1)} |y - x|^{-d-\beta} dy$. Since $h_q(x) = w_q(x) = x_d^q$, using (3.22) we have

$$\begin{aligned} L_\beta h_q(x) &= \mathcal{A}(d, -\beta) \lim_{\epsilon \downarrow 0} \left[\int_{1 > |y-x| > \epsilon} \frac{h_q(y) - h_q(x)}{|y-x|^{d+\beta}} dy + \widehat{\kappa}_D(x) h_q(x) \right] \\ &= \mathcal{A}(d, -\beta) \lim_{\epsilon \downarrow 0} \left[\int_{1 > |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\beta}} dy + \int_{1 > |y-x| > \epsilon} \frac{w_q(y) - w_q(x)}{|y-x|^{d+\beta}} dy + \widehat{\kappa}_D(x) w_q(x) \right] \\ &= L_{d,1}^\beta w_q(x) + \mathcal{A}(d, -\beta) \lim_{\epsilon \downarrow 0} \left[\int_{1 > |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\beta}} dy + (\widehat{\kappa}_D(x) - \widehat{\kappa}_{\mathbb{R}_+^d}(x)) w_q(x) \right]. \end{aligned}$$

By a similar calculation as in [7, Lemma 5.6], for any $0 < \beta < 2$, we get

$$|(\widehat{\kappa}_D(x) - \widehat{\kappa}_{\mathbb{R}_+^d}(x)) w_q(x)| \leq c x_d^q (x_d^{1-\beta} + |\log x_d|) \leq c.$$

Therefore, it remains to bound $I_\epsilon := \lim_{\epsilon \downarrow 0} \int_{1 > |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\beta}} dy$. When $q < \beta$, by the proof of Lemma 3.1,

$$\sup_{\epsilon < 1/2} |I_\epsilon| \leq \sup_{\epsilon < 1/2} \left| \int_{\mathbb{R}^d, |y-x| > \epsilon} \frac{h_q(y) - w_q(y)}{|y-x|^{d+\beta}} dy \right| \leq c(1 + \log |x_d|).$$

When $q \geq \beta$, by [23, (3.13)], we get $\sup_{\epsilon < 1/2} |I_\epsilon| \leq c$. The lemma now follows from these bounds, Lemma 3.5 and (3.22). \square

Let $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$, $\kappa \in \mathcal{H}_\alpha(p)$, and define

$$\widetilde{L}f(x) := L_\alpha f(x) + L_{\beta,b} f(x) - \kappa(x) f(x) = Lf(x) + L_{\beta,b} f(x).$$

Combining Lemmas 3.1 and 3.6, we get the following lemma.

Lemma 3.7. *Let $0 < p \leq q < \alpha$ and $\beta < \alpha$ and define*

$$h_q(x) := \delta_D(x)^q.$$

Then there exist $c_1 > 0$ and $c_2 \in (0, (R_2 \wedge 1)/4)$ depending only on $R_2, p, q, d, \alpha, \beta, \Lambda, C_2, \eta, C_{b,1}, C_{b,2}, C_{b,3}, \beta_1$ such that the following inequalities hold:

(i) *If $q > p$,*

$$c_1^{-1} \delta_D(x)^{q-\alpha} \leq \widetilde{L}h_q(x) \leq c_1 \delta_D(x)^{q-\alpha}$$

for every $x \in D$ with $0 < \delta_D(x) < c_2$.

(ii) *If $q = p$,*

$$|\widetilde{L}h_p(x)| \leq c_1 (\delta_D(x)^{p-(\beta \vee \eta)} + |\log \delta_D(x)|)$$

for every $x \in D$ with $0 < \delta_D(x) < c_2$.

Recall that $X^{(B),D}$ denote the process $X^{(B)}$ killed upon exiting D . Note that the operator \tilde{L} coincides with the restriction to $C_c^2(D)$ of the generator of the Feynman-Kac semigroup of $X^{(B),D}$ via the multiplicative functional $e^{-\int_0^t \kappa(X_s^{(B),D}) ds}$ in $C_0(D)$. We now follow the argument of the previous subsection (choosing $q \in (p, (p - (\eta \vee \beta) + \alpha) \wedge \alpha)$) and can conclude the following.

Theorem 3.8. *Suppose that D is a bounded $C^{1,1}$ open set in \mathbb{R}^d , $d \geq 2$, with characteristics (R_2, Λ) . For all $T > 0$, $p \in [\alpha - 1, \alpha) \cap (0, \alpha)$, $\beta < \alpha$ and $\eta \in [0, \alpha)$, there exists $c = c(C_1, C_2, p, \alpha, \beta, d, \eta, b, \text{diam}(D), T, C_{b,1}, C_{b,2}, C_{b,3}, \beta_1) \geq 1$ such that for all $\kappa \in \mathcal{H}_\alpha(p)$, the transition density $q^D(t, x, y)$ of the Hunt process Y on D corresponding to the Feynman-Kac semigroup of $X^{(B),D}$ via the multiplicative functional $e^{-\int_0^t \kappa(X_s^{(B),D}) ds}$ satisfies that*

$$\begin{aligned} & c^{-1} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right] \\ & \leq q^D(t, x, y) \leq c \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right] \end{aligned}$$

for $(t, x, y) \in (0, T] \times D \times D$.

We remark here that Theorem 3.8 recovers [22, Theorem 4.8]. Let κ_D^m be the killing function of the killed relativistic α -stable process $Z^{m,D}$ in D . It follows from [7, pp.94–95] that the killed relativistic α -stable process $Z^{m,D}$ can be obtained from $X^{m,D}$ via a Feynman-Kac perturbation of the form $e^{-\int_0^t \kappa_D^m(X_s^{m,D}) ds}$. It follows [27, p. 278] that $0 \leq \kappa_D(x) - \kappa_D^m(x) \leq c\delta_D(x)^{2-\alpha}$ for all $x \in D$. Combining this with (3.11), we get

$$|(\kappa_D^m(x) - \kappa_{\mathbb{R}_+^d}(x))w_q(x)| \leq cx_d^q(x_d^{1-\alpha} + |\log x_d|) \leq c.$$

Now by the same argument as in Remark 3.4, we see that Theorem 3.8 recovers the main result of [20] for bounded $C^{1,1}$ open set D .

3.3 $\mathbb{R}^d \setminus \{0\}$

In this subsection we assume that $\mathfrak{X} = \mathbb{R}^d$, $d \geq 2$, X is an isotropic α -stable process on \mathbb{R}^d and $D = \mathbb{R}^d \setminus \{0\}$. Obviously, D is a $(1/2)$ -fat open set with characteristics $(\infty, 1/2)$ and X satisfies Assumptions **A** and **U**. Since X does not hit $\{0\}$, the killed process X^D is simply the restriction of X to D .

Recall that $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}$. Let $p \in (0, \alpha)$ and define

$$H(s) = 2\pi \frac{\pi^{\frac{d-3}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^\pi \sin^{d-2}\theta \frac{(\sqrt{s^2 - \sin^2\theta} + \cos\theta)^{1+\alpha}}{\sqrt{s^2 - \sin^2\theta}} d\theta, \quad s \geq 1,$$

and

$$\tilde{C}(\alpha, d, p) := \mathcal{A}(d, -\alpha) \int_1^{+\infty} (s^p - 1)(1 - s^{-d+\alpha-p})s(s^2 - 1)^{-1-\alpha} H(s) ds.$$

Note that $p \rightarrow \tilde{C}(\alpha, d, p)$ is strictly increasing on $(0, \alpha)$. The function $H(s)$ is positive and continuous on $[1, +\infty)$ with $H(s) \asymp s^\alpha$ for large s and

$$s(s^2 - 1)^{-1-\alpha} H(s) \asymp (s - 1)^{-1-\alpha}, \quad s \geq 1,$$

(see the paragraph after [31, Theorem 1.1]). Thus

$$\lim_{p \downarrow 0} \tilde{C}(\alpha, d, p) = 0 \quad \text{and} \quad \lim_{p \uparrow \alpha} \tilde{C}(\alpha, d, p) = \infty. \quad (3.23)$$

Applying [31, Theorem 1.1] to $u_p := |x|^p$, we get that

$$-(-\Delta)^{\alpha/2} u_p(x) = \tilde{C}(\alpha, d, p) |x|^{p-\alpha}, \quad |x| > 0, x \in \mathbb{R}^d. \quad (3.24)$$

Let \mathcal{G}_α be the collection of non-negative functions on D such that for each $\kappa \in \mathcal{G}_\alpha$ there exist constants $C_1 > 0$, $C_2 \geq 0$ and $\eta \in [0, \alpha)$ such that $\kappa(x) \leq C_2$ for all x with $|x| \geq 1$ and

$$|\kappa(x) - C_1 |x|^{-\alpha}| \leq C_2 |x|^{-\eta}, \quad (3.25)$$

for all $x \in D$ with $|x| < 1$. By (3.23) we can find a unique $p \in (0, \alpha)$ such that $C_1 = \tilde{C}(\alpha, d, p)$. Define

$$\mathcal{G}_\alpha(p) := \{\kappa \in \mathcal{G}_\alpha : \text{the constant } C_1 \text{ in (3.25) is } \tilde{C}(\alpha, d, p)\}. \quad (3.26)$$

Note that $\mathcal{G}_\alpha = \cup_{0 < p < \alpha} \mathcal{G}_\alpha(p)$. We fix a $\kappa \in \mathcal{G}_\alpha(p)$ and let Y be a Hunt process on D corresponding to the Feynman-Kac semigroup of X^D via the multiplicative functional $e^{-\int_0^t \kappa(X_s^D) ds}$, that is,

$$\mathbb{E}_x [f(Y_t)] = \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_s^D) ds} f(X_t^D) \right], \quad t \geq 0, x \in D.$$

Since, by Example 2.17, $\kappa(x) dx \in \mathbf{K}_1(D)$, it follows from Theorem 2.22 that Y has a transition density $q^D(t, x, y)$ with the following estimate

$$q^D(t, x, y) \asymp \mathbb{P}_x(\zeta > t) \mathbb{P}_y(\zeta > t) \left[t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right], \quad (3.27)$$

for $(t, x, y) \in (0, 1) \times D \times D$, where ζ is the lifetime of Y . Moreover, when $C_2 = 0$, $\kappa(x) dx \in \mathbf{K}_\infty(D)$ by Example 2.18. Thus, by Theorem 2.23, (3.27) holds for all $t > 0$.

Define

$$Lf(x) := -(-\Delta)^{\alpha/2} f(x) - \kappa(x) f(x).$$

Fix a $q \in (p, \alpha)$ such that $q < p - \eta + \alpha$ and let $A = \tilde{C}(\alpha, d, q) - \tilde{C}(\alpha, d, p) > 0$. Define

$$v_1(x) := u_p(x) + u_q(x), \quad v_2(x) := u_p(x) - \frac{1}{2} u_q(x).$$

Since, for $|x| < C_2^{-1}$, in view of (3.24) and (3.25),

$$\begin{aligned} Lv_1(x) &\geq A|x|^{q-\alpha} - |x|^p |\tilde{C}(d, \alpha, p)| |x|^{-\alpha} - \kappa(x) - |x|^q |\tilde{C}(d, \alpha, p)| |x|^{-\alpha} - \kappa(x) \\ &\geq A|x|^{q-\alpha} - 2C_2(|x|^{p-\eta} + |x|^{q-\eta}) \end{aligned}$$

and

$$Lv_2(x) \leq -2^{-1} A|x|^{q-\alpha} + |x|^p |\tilde{C}(d, \alpha, p)| |x|^{-\alpha} - \kappa(x) + 2^{-1} |x|^q |\tilde{C}(d, \alpha, p)| |x|^{-\alpha} - \kappa(x)$$

$$\leq -2^{-1}A|x|^{q-\alpha} + (3/2)C_2(|x|^{p-\eta} + |x|^{q-\eta}),$$

there exists $c_1 > 0$ such that $Lv_1(x) \geq 0$ and $Lv_2(x) \leq 0$ whenever $0 < |x| < c_1$. Pick any $(t, x) \in (0, 1) \times D$ and set $r = r(t) = c_1 t^{1/\alpha}$ for $t < 1$. Now we can follow the argument before the statement of Theorem 3.2 and get $\mathbb{P}_x(\zeta > t) \asymp (1 \wedge |x|/r)^p$ for $t < 1$.

Moreover, if $\kappa(x) = \tilde{C}(\alpha, d, p)|x|^{-\alpha}$, we can simply take $v_1(x) = v_2(x) = u_p(x)$ and $r(t) = t^{1/\alpha}$ for all $t > 0$ and get $\mathbb{P}_x(\zeta > t) \asymp (1 \wedge |x|/r(t))^p$ for all $t > 0$.

Therefore, we conclude that

Theorem 3.9. *For all positive $T > 0$, $p \in (0, \alpha)$ and $\eta \in [0, \alpha)$, there exists a $c = c(C_1, C_2, p, \alpha, d, \eta, T) \geq 1$ such that for all $\kappa \in \mathcal{G}_\alpha(p)$, the transition density $q(t, x, y)$ of Y , the Hunt process on $\mathbb{R}^d \setminus \{0\}$ associated with the Feynman-Kac semigroup of the isotropic α -stable process Z via the multiplicative functional $e^{-\int_0^t \kappa(Z_s) ds}$, satisfies that*

$$\begin{aligned} c^{-1} \left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right] \\ \leq q(t, x, y) \leq c \left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right], \end{aligned}$$

for $(t, x, y) \in (0, T) \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$. Moreover, if $\kappa(x) = \tilde{C}(\alpha, d, p)|x|^{-\alpha}$, then the above estimates holds for all $t > 0$.

The last claim in Theorem 3.9 can be proved using the scaling property and the finite time estimates in Theorem 3.9. This was proved independently in [36] using a different method.

Let $\alpha \in (1, 2)$ and g be an \mathbb{R}^d -valued C^1 function with $\|g\|_\infty + \|\nabla g\|_\infty < \infty$. Let \tilde{X}^g be an α -stable process with drift g , that is, a non-symmetric Hunt process with generator $-(-\Delta)^{\alpha/2}f(x) + g \cdot \nabla f(x)$, see [15]. Let X^g be the Hunt process obtained from \tilde{X}^g by killing with rate $\|\text{div } g\|$. The generator of X^g is $-(-\Delta)^{\alpha/2}f(x) + g \cdot \nabla f(x) - \|\text{div } g\|_\infty f(x)$. By [15], the transition density $p(t, x, y)$ of X^g satisfies

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}, \quad (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.28)$$

The dual of $-(-\Delta)^{\alpha/2}f(x) + g \cdot \nabla f(x) - \|\text{div } g\|_\infty f(x)$ is $-(-\Delta)^{\alpha/2}f(x) - g \cdot \nabla f(x) - \text{div } g(x)f(x) - \|\text{div } g\|_\infty f(x)$, which is the generator of a Hunt process \hat{X}^g which can be obtained from an α -stable process with drift via the killing potential $-\text{div } g(x) - \|\text{div } g\|_\infty$. It is easy to check that X^g and \hat{X}^g are strong duals of each other with respect to the Lebesgue measure. It is also easy to check that X^g and \hat{X}^g satisfy the sector condition, thus, by [33, Theorem 4.17], all semipolar sets are polar. Moreover, since $\alpha \in (1, 2)$, Assumption **U** holds true.

Fix a $\kappa \in \mathcal{G}_\alpha(p)$ and let Y^g be a Hunt process on D corresponding to the Feynman-Kac semigroup of $X^{g,D}$ via the multiplicative functional $e^{-\int_0^t \kappa(X_s^{g,D}) ds}$, that is,

$$\mathbb{E}_x [f(Y_t^g)] = \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_s^{g,D}) ds} f(X_t^{g,D}) \right], \quad t \geq 0, x \in D.$$

Since $\kappa(x)dx \in \mathbf{K}_1(D)$ by (3.28), it follows from (3.28) and Theorem 2.22 that Y^g has a transition density $q^g(t, x, y)$ with the following estimate

$$q^g(t, x, y) \asymp \mathbb{P}_x(\zeta > t)\mathbb{P}_y(\zeta > t) \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right], \quad (3.29)$$

for $(t, x, y) \in (0, 1] \times D \times D$, where ζ is the lifetime of Y^g . Note that with $u_p = |x|^p$, we get that

$$|g \cdot \nabla u_p(x)| + \|\operatorname{div} g\|_\infty |u_p(x)| \leq \tilde{C}(\alpha, d, p) |x|^{p-1}, \quad 0 < |x| < 1. \quad (3.30)$$

From (3.24), (3.30) and the assumption $\alpha \in (1, 2)$, we see that terms $g \cdot \nabla f(x) - \|\operatorname{div} g\|_\infty f(x)$ and $-g \cdot \nabla f(x) - \operatorname{div} g(x)f(x) - \|\operatorname{div} g\|_\infty f(x)$ can be treated as lower order terms. Thus, using (3.30) and the assumption $\alpha \in (1, 2)$, by repeating the argument of the first part of this subsection, we can easily get the following result.

Theorem 3.10. *Suppose that $\alpha \in (1, 2)$. For all positive $T > 0$, $p \in (0, \alpha)$ and $\eta \in [0, \alpha)$, there exists a $c = c(C_1, C_2, p, \|g\|_\infty, \alpha, d, \eta, T, \|\nabla g\|_\infty) \geq 1$ such that for all $\kappa \in \mathcal{G}_\alpha(p)$, the transition semigroup $q^g(t, x, y)$ of Y^g satisfies that*

$$\begin{aligned} & c^{-1} \left(1 \wedge \frac{|x|}{t^{1/\alpha}} \right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}} \right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right] \\ & \leq q^g(t, x, y) \leq c \left(1 \wedge \frac{|x|}{t^{1/\alpha}} \right)^p \left(1 \wedge \frac{|y|}{t^{1/\alpha}} \right)^p \left[t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right], \end{aligned}$$

for $(t, x, y) \in (0, T) \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})$.

Note that, Theorem 3.10 also holds for the fundamental solution to $\partial_t = -(-\Delta)^{\alpha/2} + g \cdot \nabla - \kappa(x)$.

4 Appendix: Continuous additive functionals for killed non-symmetric processes

We keep the assumptions and the notations in Sections 2.1–2.3. In this section, D is an open subset of \mathfrak{X} and U is a relatively compact subset of D .

Lemma 4.1. *If $h \in \mathcal{D}(\widehat{\mathcal{L}})$ is non-negative, bounded and has compact support contained in U , then for any $t \geq 0$,*

$$\limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_U \widehat{P}_t^U h(x) \mathbb{P}_x(\tau_U^X \leq \epsilon) m(dx) < \infty.$$

Proof. Noticing $h(\widehat{X}_{\tau_{\widehat{U}}^X}) = 0$, we get

$$\widehat{P}_t^U h(x) = \widehat{\mathbb{E}}_x[h(\widehat{X}_t) 1_{t < \widehat{\tau}_U^X}] = \widehat{\mathbb{E}}_x h(\widehat{X}_{t \wedge \widehat{\tau}_U^X}) = h(x) + \widehat{\mathbb{E}}_x \int_0^{t \wedge \widehat{\tau}_U^X} \widehat{\mathcal{L}} h(\widehat{X}_s) ds.$$

Using this and the duality, we have

$$\begin{aligned}
& \int_U \widehat{P}_t^U h(x) \mathbb{P}_x(\tau_U^X \leq \epsilon) m(dx) = \int_U \widehat{P}_t^U h(x) (1 - P_\epsilon^U \mathbf{1}(x)) m(dx) \\
& = \int_U (\widehat{P}_t^U h(x) - \widehat{P}_{t+\epsilon}^U h(x)) m(dx) = - \int_U \widehat{\mathbb{E}}_x \int_t^{t+\epsilon} \widehat{\mathcal{L}}h(\widehat{X}_s) \mathbf{1}_{s < \widehat{\tau}_U^X} ds m(dx) \\
& \leq \epsilon \left(\sup_{x \in U} |\widehat{\mathcal{L}}h(x)| \right) m(U),
\end{aligned}$$

from which the conclusion follows immediately. \square

Lemma 4.2. *Let $\mu \in \mathbf{K}_T(D)$ for some $T > 0$. If A is the continuous additive functional of X^D associated with μ , $h \in \mathcal{D}(\widehat{\mathcal{L}})$ is non-negative, bounded and has compact support contained in U , then for any bounded Borel function f on U and $t \geq 0$,*

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_U \widehat{P}_t^U h(x) \left(\mathbb{E}_x \int_0^\epsilon \mathbf{1}_{\tau_U^X \leq s} f(X_s^D) dA_s \right) m(dx) = 0.$$

Proof. Since by the strong Markov property

$$\mathbb{E}_x \int_0^\epsilon \mathbf{1}_{\tau_U^X \leq s} f(X_s^D) dA_s = \mathbb{E}_x \left[\mathbf{1}_{\tau_U^X < \epsilon} \mathbb{E}_{X_{\tau_U^X}^D} \int_0^{\epsilon - \tau_U^X} f(X_s^D) dA_s \right],$$

we have

$$\left| \mathbb{E}_x \int_0^\epsilon \mathbf{1}_{\tau_U^X \leq s} f(X_s^D) dA_s \right| \leq \left(\sup_{y \in \mathfrak{X}} \mathbb{E}_y \int_0^\epsilon |f(X_s^D)| dA_s \right) \mathbb{P}_x(\tau_U^X \leq \epsilon).$$

The assertion now follows from Lemma 4.1 and condition (2) in Definition 2.12. \square

Proposition 4.3. *Let $\mu \in \mathbf{K}_T(D)$ for some $T > 0$. If A is the continuous additive functional of X associated with μ , then $(A_{t \wedge \tau_U^X})$ is the continuous additive functional of X^U associated with μ_U .*

Proof. Let $A_t^U := A_{t \wedge \tau_U^X}$. Then A^U is a continuous additive functional of X^U . Let $h \in \mathcal{D}(\mathcal{L})$ be non-negative, bounded and have compact support contained in U , and let f be a bounded Borel function supported in U . Define

$$g_t := \int_U h(x) \mathbb{E}_x \int_0^t f(X_s^U) dA_s^U m(dx).$$

Since

$$g_{t+\epsilon} - g_t = \int_U h(x) \mathbb{E}_x \int_t^{t+\epsilon} f(X_s^U) dA_s^U m(dx) = \int_U h(x) \mathbb{E}_x \int_0^\epsilon f(X_{s+t}^U) dA_{s+t}^U m(dx)$$

$$= \int_U h(x) P_t^U (\mathbb{E} \int_0^\epsilon f(X_s^U) dA_s)(x) m(dx) = \int_U \widehat{P}_t^U h(x) \mathbb{E}_x \int_0^\epsilon \mathbf{1}_{\tau_U^X > s} f(X_s^U) dA_s m(dx),$$

it follows from Lemma 4.2 that

$$\lim_{\epsilon \rightarrow 0} \frac{g_{t+\epsilon} - g_t}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_U \widehat{P}_t^U h(x) \mathbb{E}_x \int_0^\epsilon f(X_s^D) dA_s m(dx) = \int_U \widehat{P}_t^U h(x) f(x) \mu(dx)$$

which implies

$$\int_U h(x) \mathbb{E}_x \int_0^t f(X_s^U) dA_s^U m(dx) = \int_0^t \int_U \widehat{P}_s^U h(x) f(x) \mu(dx) ds = \int_U h(x) \int_0^t P_s^U f(x) ds \mu(dx).$$

Using the dominated convergence theorem and the monotone convergence theorem, one can show that the equality above is valid for all bounded non-negative Borel functions h and f supported in U . Therefore,

$$\int_U f(x) \mu(dx) = \lim_{t \downarrow 0} \mathbb{E}_{m_U} \left[\frac{1}{t} \int_0^t f(X_s^U) dA_s^U \right].$$

□

Acknowledgments: We thank the referee for the very helpful comments.

References

- [1] M. Aizenman, B. Simon, Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure Appl. Math.*, **35** (1982), 209–273.
- [2] P. Baras, J. A. Goldstein, The heat equation with a singular potential. *Trans. Amer. Math. Soc.*, **284** (1984) 121–139.
- [3] G. Barbatis, S. Filippas, A. Tertikas, Critical heat kernel estimates for Schrödinger operators via Hardy–Sobolev inequalities. *J. Funct. Anal.*, **208** (2004), 1–30.
- [4] M. T. Barlow, R. F. Bass, Z.-Q. Chen, M. Kassmann, Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.* **361** (2009), 1963–1999.
- [5] Ph. Blanchard, Z.-M. Ma, *Semigroup of Schrödinger operators with potentials given by Radon measures*. Stochastic processes, physics and geometry (Ascona and Locarno, 1988), World Sci. Publ., Teaneck, NJ, 1990, 160–195.
- [6] R. M. Blumenthal, R. K. Gettoor, *Markov processes and potential theory*. Academic Press, New York, 1968.
- [7] K. Bogdan, K. Burdzy, Z.-Q. Chen, Censored stable processes. *Probab. Theory Related Fields*, **127** (2003), 89–152.
- [8] K. Bogdan, B. Dyda, The best constant in a fractional Hardy inequality. *Math. Nachr.* **284** (2011), 629–638.
- [9] K. Bogdan, B. Dyda, P. Kim, Hardy inequalities and non-explosion results for semigroups. *Potential Anal.* **44**(2) (2016) 229–247.

- [10] K. Bogdan, T. Grzywny, Heat kernel of fractional Laplacian in cones. *Coll. Math.* **118** (2010), 365–377.
- [11] K. Bogdan, T. Grzywny, M. Ryznar, Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.* **38** (2010), no. 5, 1901–1923.
- [12] K. Bogdan, T. Grzywny, T. Jakubowski, D. Pilarczyk, Fractional Laplacian with Hardy potential. *Comm. Partial Differential Equations*, **44** (2019), 20–50.
- [13] K. Bogdan, T. Grzywny, M. Ryznar, Density and tails of unimodal convolution semigroups. *J. Funct. Anal.* **266** (2014), 3543–3571.
- [14] K. Bogdan, W. Hansen, T. Jakubowski, Time-dependent Schrödinger perturbations of transition densities. *Studia Math.* **189** (2008), 235–254.
- [15] K. Bogdan, T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys.* **271** (2007), 179–198.
- [16] K. Bogdan, T. Kumagai, M. Kwaśnicki, Boundary Harnack inequality for Markov processes with jumps. *Trans. Amer. Math. Soc.*, **367** (2014), 477–517.
- [17] Z.-Q. Chen, P. Kim, T. Kumagai, On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. *Acta Math. Sin. (Engl. Ser.)*, **25** (2009), 1067–1086.
- [18] Z.-Q. Chen, P. Kim, R. Song, Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc. (JEMS)*, **12** (2010), 1307–1329.
- [19] Z.-Q. Chen, P. Kim, R. Song, Two-sided heat kernel estimates for censored stable-like processes. *Probab. Theory Related Fields*, **146** (2010), 361–399.
- [20] Z.-Q. Chen, P. Kim, R. Song, Sharp heat kernel estimates for relativistic stable processes in open sets. *Ann. Probab.* **40** (2012), 213–244.
- [21] Z.-Q. Chen, P. Kim, R. Song, Dirichlet heat kernel estimates for rotationally symmetric Lévy processes. *Proc. London Math. Soc.* **109** (2014), 90–120.
- [22] Z.-Q. Chen, P. Kim, R. Song, Stability of Dirichlet heat kernel estimates for non-local operators under Feynman-Kac perturbation. *Trans. Amer. Math. Soc.*, **367** (2015), 5237–5270.
- [23] Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček, Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$. *Trans. Amer. Math. Soc.*, **364** (2012), 4169–4205.
- [24] Z.-Q. Chen, T. Kumagai, Heat kernel estimates for stable-like processes on d -sets. *Stochastic Process. Appl.*, **108** (2003), 27–62.
- [25] Z.-Q. Chen, T. Kumagai, J. Wang, Stability of heat kernel estimates for symmetric jump processes on metric measure spaces, To appear in *Memoirs Amer. Math. Soc.*, arXiv:1604.04035.
- [26] Z.-Q. Chen, T. Kumagai, J. Wang, Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms. To appear in *J. European Math. Soc.*, arXiv:1609.07594.
- [27] Z.-Q. Chen, R. Song, Drift transforms and Green function estimates for discontinuous processes. *J. Funct. Anal.* **201** (2003), 262–281.
- [28] Z.-Q. Chen, R. Song, Hardy inequality for censored stable processes. *Tohoku Math. J.* **55** (2003), 439–450.
- [29] K. L. Chung, J. B. Walsh, *Markov processes, Brownian motion, and time symmetry*. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 249. Springer, New York, 2005.

- [30] K. L. Chung, Z. X. Zhao, *From Brownian motion to Schrödinger's equation*. Springer-Verlag, Berlin, 1995.
- [31] F. Ferrari, I. E. Verbitsky, Radial fractional Laplace operators and Hessian inequalities. *J. Differential Equations*, **253** (2012), 244–272.
- [32] S. Filippas, L. Moschini, A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains. *Commun. Math. Physics*, **273** (2007), 237–281.
- [33] P. J. Fitzsimmons, Markov processes and nonsymmetric Dirichlet forms without regularity. *J. Funct. Anal.* **85** (1989), 287–306.
- [34] P. J. Fitzsimmons, R. K. Gettoor, Smooth measures and continuous additive functionals of right Markov processes. In *Itô's stochastic calculus and probability theory*, 31–49, Springer, Tokyo, 1996.
- [35] K. Ishige, Y. Kabeya, E. M. Ouhabaz, The heat kernel of a Schrödinger operator with inverse square potential. *Proc. London Math. Soc.*, **115** (2017), 381–410.
- [36] T. Jakubowski, J. Wang, Heat kernel estimates of fractional Schrödinger operators with negative Hardy potential. To appear in *Potential Anal.* doi.org/10.1007/s11118-019-09795-7.
- [37] P. Kim, T. Kumagai, J. Wang, Laws of the iterated logarithm for symmetric jump processes. *Bernoulli*, **23** (2017), 2330–2379.
- [38] P. Kim, A. Mimica, Estimates of Dirichlet heat kernels for subordinate Brownian motions. *Elect. J. of Probab.* **23** (2018), No. 64, 1–45.
- [39] P. Kim, R. Song, Estimates on Green functions and Schrödinger-type equations for non-symmetric diffusions with measure-valued drifts. *J. Math. Anal. Appl.* **332** (2007), 57–80.
- [40] P. Kim, R. Song, Z. Vondraček, Potential theory of subordinate killed Brownian motion. *Trans. Amer. Math. Soc.* **371** (2019), 3417–3450.
- [41] T. Komatsu, Continuity estimates for solutions of parabolic equations associated with jump type Dirichlet forms. *Osaka J. Math.* **25** (1988), 697–728.
- [42] P. D. Milman, Yu. A. Semenov, Heat kernel bounds and desingularizing weights. *J. Funct. Anal.* **202** (2003) 1–24.
- [43] P. D. Milman, Yu. A. Semenov, Global heat kernel bounds via desingularizing weights. *J. Funct. Anal.* **212** (2004) 373–398.
- [44] L. Moschini, A. Tesei, Parabolic Harnack inequality for the heat equation with inverse-square potential. *Forum Math.* **19** (2007), 407–427.
- [45] D. Revuz, Mesures associées aux fonctionnelles additives de Markov. I. *Trans. Amer. Math. Soc.* **148** (1970), 501–531.
- [46] B. Simon, Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 447–526.
- [47] R. Song, Two-sided estimates on the density of the Feynman-Kac semigroups of stable-like processes. *Electron. J. Probab.*, **11** (2006), 146–161.
- [48] R. Song, Z. Vondraček, Harnack inequality for some classes of Markov processes. *Math. Z.* **246** (2004), 177–202.
- [49] N. Th. Varopoulos, Gaussian estimates in Lipschitz domains. *Canad. J. Math.* **55** (2003), 401–431.

- [50] C. Wang, On estimates of the density of Feynman-Kac semigroups of α -stable-like processes. *J. Math. Anal. Appl.* **348** (2008), 938–970.
- [51] J. Ying, Bivariate Revuz measures and the Feynman-Kac formula. *Ann. Inst. H. Poincaré Probab. Statist.*, **32** (1996), 251–287.

Soobin Cho

Department of Mathematical Sciences,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 08826, Republic of Korea
E-mail: soobin15@snu.ac.kr

Panki Kim

Department of Mathematical Sciences and Research Institute of Mathematics,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 08826, Republic of Korea
E-mail: pkim@snu.ac.kr

Renming Song

Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
E-mail: rsong@math.uiuc.edu

Zoran Vondraček

Department of Mathematics, University of Zagreb, Zagreb, Croatia,
Email: vondra@math.hr