# On the monotonicity of a function related to the local time of a symmetric Lévy process

Renming Song *
Department of Mathematics
University of Illinois, Urbana, IL 61801
Email: rsong@math.uiuc.edu
and
Zoran Vondraček †
Department of Mathematics
University of Zagreb, Zagreb, Croatia
Email: vondra@math.hr

**Abstract**

Let $\psi$ be the characteristic exponent of a symmetric Lévy process $X$. The function

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\psi(\lambda)} \, d\lambda$$

appears in various studies on the local time of $X$. We study monotonicity properties of the function $h$. In case when $X$ is a subordinate Brownian motion, we show that $x \mapsto h(\sqrt{x})$ is a Bernstein function.

**AMS 2000 Mathematics Subject Classification**: Primary 60J25, Secondary 60J55.
**Keywords and phrases:** Lévy processes, subordinate Brownian motion, local time, Bernstein function

---

*The research of this author is supported in part by a joint US-Croatia grant INT 0302167.

†The research of this author is supported in part by MZOS grant 0037107 of the Republic of Croatia and in part by a joint US-Croatia grant INT 0302167.

# 1 Introduction

Let $X$ be a symmetric Lévy process in $\mathbb{R}$ with the characteristic exponent $\psi$, i.e.,

$$\mathbb{E}e^{i\lambda X_t} = e^{-t\psi(\lambda)}.$$

Throughout this paper we assume that the point 0 is regular for itself, and that the characteristic exponent $\psi$ satisfies

$$\int_0^\infty \frac{1}{1+\psi(\lambda)} < \infty\,. \tag{1}$$

These two conditions guarantee that the process $X$ admits a local time $L(0,t)$ at zero. Let $T_x = \inf\{s > 0 : X_s = x\}$ be the hitting time to $x \in \mathbb{R}$, and let

$$h(x) := \mathbb{E}(L(0, T_x))\,.$$

Then by Lemma 11 in Chapter 5 of [2]

$$h(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda x)}{\psi(\lambda)}\,d\lambda\,. \tag{2}$$

This function appears often in studies of the local time of Lévy processes. For instance, a monotone rearrangement of this function was used in [1] to formulate necessary and sufficient conditions for the joint continuity of the local time. In his study on the most visited sites of $X$ [5], M. B. Marcus assumed that the function $h$ is strictly increasing on $[0,\infty)$. This assumption on $h$ does not seem easy to check. In Section 5 of [5], Marcus showed that the so called stable mixtures satisfy the assumption.

The purpose of this note is to better understand the monotonicity of the function $h$, and to provide more examples of strictly increasing $h$. We also show that for subordinate Brownian motions, $x \mapsto h(\sqrt{x})$ is, in fact, a Bernstein function. Under a reasonable additional assumption, it is even a complete Bernstein function.

We start with a simple sufficient condition for $h$ to be increasing. To this end we first rewrite the function $h$ in a different way. It follows from Theorem 16 and Theorem 19 in Chapter 2 of [2] that under the assumptions stated in the first paragraph, the $q$-potential measure $U^q$ of $X$ has a density $u^q$ which is bounded and continuous. From the proof of Lemma 11 in Chapter 5 of [2] we see that $h$ defined by (2) may be written as

$$h(x) = 2\lim_{q\downarrow 0}(u^q(0) - u^q(x)), \quad x \in \mathbb{R}. \tag{3}$$

Thus if we know that for any $q > 0$, the function $u^q$ is decreasing in $[0,\infty)$, then the equation above immediately gives us that $h$ is increasing in $[0,\infty)$. Using this fact and Theorem 54.2 of [9] we immediately get the following

**Proposition 1.1** *If the Lévy measure $\nu$ of the process $X$ is given by*

$$\nu(dx) = n(x)dx$$

*for some even function $n$ which is decreasing in $(0, \infty)$, then $h$ is increasing in $[0, \infty)$.*

**Proof.** It follows from Theorem 54.2 of [9] that when the Lévy measure $\nu$ of the process $X$ is given by

$$\nu(dx) = n(x)dx$$

for some even function $n$ which is decreasing in $(0, \infty)$, the distribution of $X_t$ is unimodal with mode 0 for every $t > 0$. This implies that, for any $q > 0$, $u^q$ is a decreasing function in $[0, \infty)$. Therefore $h$ is increasing in $[0, \infty)$. $\square$

# 2 Subordinate Brownian motion

In this section we first make a comment that a subordinate Brownian motion satisfies condition of Proposition 1.1, and then prove that a much stronger result than Proposition 1.1 holds in this case. Let us begin by recalling relevant definitions.

Let $T = (T_t : t \geq 0)$ be a subordinator with Laplace exponent $f$, that is,

$$\mathbb{E}e^{-\lambda T_t} = e^{-tf(\lambda)},$$

and let $B = (B_t : t \geq 0)$ be a Brownian motion with generator $\frac{d^2}{dx^2}$. If $B$ and $T$ are independent, then the process $X_t := B(T_t)$ is called a subordinate Brownian motion with subordinator $T$. It is well known that the characteristic exponent of this subordinate Brownian motion satisfies $\psi(\lambda) = f(\lambda^2)$, that is,

$$\mathbb{E}e^{i\lambda X_t} = e^{-tf(\lambda^2)}.$$

We still assume that (1) holds. This implies that $\lim_{\lambda\to\infty} f(\lambda) = \infty$, which means that $T$ is not a compound Poisson process. It is well known that the Lévy measure of subordinate Brownian motion has the density $n$ given by

$$n(x) = \int_0^\infty \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\, \mu(dt)\,,$$

where $\mu$ is the Lévy measure of the subordinator. Clearly, $n$ is even and decreasing on $(0, \infty)$, hence the assumption of Proposition 1.1 is satisfied.

The Laplace exponent $f$ is a Bernstein function, that is, $f \in C^\infty(0, \infty)$, and satisfies $(-1)^n D^n f \leq 0$ for every $n \in \mathbb{N}$. Note that a nonconstant Bernstein function is strictly increasing. We will also need the notion of a complete Bernstein function: A function $f : (0,\infty) \to [0,\infty)$ is called a complete Bernstein function if there exists a Bernstein function $g$ such that

$$f(\lambda) = \lambda^2 \mathcal{L} g(\lambda), \quad \lambda > 0,$$

where $\mathcal{L}$ stands for the Laplace transform. Complete Bernstein function is a Bernstein function. The family of all complete Bernstein functions is a convex cone containing positive constants and it is closed under compositions. For more on complete Bernstein functions see [4].

Let $V$ be the potential measure of $T$, that is,

$$V(A) = \mathbb{E} \int_0^\infty 1_{\{T_t \in A\}} dt\,.$$

Then it is well known that

$$\frac{1}{f(\lambda)} = \int_0^\infty e^{-\lambda t} dV(t), \quad \lambda > 0. \tag{4}$$

**Proposition 2.1** *The function $\phi : [0,\infty) \to [0,\infty)$ defined by*

$$\phi(x) := h(\sqrt{x}) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda\sqrt{x})}{f(\lambda^2)} d\lambda \tag{5}$$

*is a Bernstein function.*

**Proof.** Using (4) we get

$$\begin{aligned} h(x) &= \frac{2}{\pi} \int_0^\infty (1 - \cos(\lambda x)) \int_0^\infty e^{-\lambda^2 t} dV(t)\, d\lambda \\ &= \frac{2}{\pi} \int_0^\infty dV(t) \int_0^\infty (1 - \cos(\lambda x)) e^{-\lambda^2 t}\, d\lambda \\ &= \frac{1}{\pi} \int_0^\infty dV(t) \int_0^\infty q^{-1/2} (1 - \cos(x\sqrt{q})) e^{-qt} dq \\ &= \frac{1}{\pi} \int_0^\infty dV(t) \left( \int_0^\infty q^{-1/2} e^{-qt} dq - \int_0^\infty q^{-1/2} \cos(x\sqrt{q}) e^{-qt} dq \right) \end{aligned} \tag{6}$$

Using formula (67) on page 158 of [3] we see that

$$\int_0^\infty q^{-1/2} \cos(x\sqrt{q}) e^{-qt} dq = \sqrt{\frac{\pi}{t}} e^{-\frac{x^2}{4t}}, \tag{7}$$

thus we have

$$h(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} (1 - e^{-\frac{x^2}{4t}})\, dV(t).$$

Consequently we have

$$\phi(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} (1 - e^{-\frac{x}{4t}})\, dV(t).$$

Let $\tilde{V}$ be the image measure of $V$ with respect to the mapping $t \mapsto 1/4t$. Then

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{4t}(1 - e^{-xt})\, d\tilde{V}(t) \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty (1 - e^{-xt}) t^{1/2}\, d\tilde{V}(t) \end{aligned} \tag{8}$$

It is straightforward to check that the measure $t^{1/2}\, d\tilde{V}(t)$ is a Lévy measure, thus proving that $\phi$ is a Bernstein function. □

**Remark 2.2** *Let $p(t,x) = (1/\sqrt{4\pi t}) \exp\{-x^2/4t\}$ be the transition density of Brownian motion $B$. By use of (6) and (7), we may rewrite the formula for the function $h$ in the following explicit form:*

$$h(x) = 2 \int_0^\infty (p(t,0) - p(t,x))\, dV(t)\,. \tag{9}$$

*This formula should be compared with the formula for the compensated potential density of Brownian motion.*

*In the same spirit as above we can show that, for any $q > 0$, the $q$-potential density $u^q(x)$ of the subordinate process $X$ is strictly decreasing on $[0, \infty)$. Indeed, let $q > 0$, and let $V^q$ denote the potential measure of the subordinator $T$ killed at an independent exponential time with parameter $q$. Then*

$$u^q(x) = \int_0^\infty p(t,x)\, dV^q(t).$$

*This formula clearly proves that $x \to u^q(x)$ is strictly decreasing on $[0, \infty)$.*

Proposition 2.1 can be strengthened as follows.

**Proposition 2.3** *Suppose that $T$ is a subordinator with Laplace exponent $f$ such that*

$$\frac{x}{f(x)} = x^2 \mathcal{L} g(x)$$

*for some Bernstein function $g$. If $g$ is given by*

$$g(x) = \int_0^\infty (1 - e^{-tx}) \rho(t) dt \tag{10}$$

*for some Lévy density $\rho$ such that $t\rho(t)$ is decreasing on $(0, \infty)$, then the function $\phi$ defined in (5) is a complete Bernstein function.*

**Proof.** Since $T$ corresponds to a complete Bernstein function and is not a compound Poisson process, we know from [8] that

$$dV(t) = v(t)\,dt$$

where $v$ is a locally integrable decreasing function on $(0,\infty)$. The formula 8 tells us that under the present assumption we have

$$\phi(x) = \frac{2}{\pi}\int_0^\infty (1-e^{-qx})\frac{1}{q^{3/2}}v(\frac{1}{4q})dq.$$

To show that $\phi$ is a complete Bernstein function, it suffices to show that the function

$$\frac{2}{\pi}\frac{1}{q^{3/2}}v(\frac{1}{4q})$$

is the Laplace transform of some positive function. Assumption (10) implies that

$$g'(x) = \int_0^\infty e^{-tx}t\rho(t)dt.$$

From the proof of Theorem 2.3 in [8] we know that $v(x) = g'(x)$ and so $v$ is the Laplace transform of the function $t\rho(t)$. Therefore by formula (30) of [3] we know that the function

$$\pi^{1/2}\lambda^{-3/2}v(\lambda^{-1})$$

is the Laplace transform of the function

$$\int_0^\infty \sin(2s^{1/2}t^{1/2})s^{1/2}\rho(s)ds.$$

The function above can be rewritten as

$$2\int_0^\infty \sin(2t^{1/2}r)r^2\rho(r^2)dr.$$

Now using the assumption that $t\rho(t)$ is decreasing we can easily show that the function about is positive. Using this and properties of the Laplace transform it follows that the function

$$\frac{2}{\pi}\frac{1}{q^{3/2}}v(\frac{1}{4q})$$

is the Laplace transform of some positive function. □

# 3 Examples

We first recall an example from [5] and show that it fits into the setting of Section 2. Take the Laplace exponent

$$f(\lambda) = \int_{1/2}^{1} \lambda^s \, d\xi(s), \quad \lambda > 0$$

with $\xi$ a finite measure on $(\frac{1}{2}, 1]$. Then

$$\psi(\lambda) = f(\lambda^2),$$

is a function of the type given in (5.1) of [5]. The function $h$ is strictly increasing on $[0, \infty)$, and $x \mapsto h(\sqrt{x})$ is a Bernstein function.

The above example belongs to the class of stable mixtures studied by Marcus and Rosen in [6] and [7]. We give now several examples of different type.

**Example 3.1** Let

$$f(\lambda) = (\lambda^\alpha + 1)^\beta - 1$$

for $0 < \alpha \leq 1$ and $0 < \beta < 1$. Being a composition of complete Bernstein function, $f$ itself is a Bernstein function. When $\alpha = 1$, the corresponding subordinator is a relativistic $\beta$-stable subordinator. In order for (1) to be satisfied, we assume that $\alpha\beta > 1/2$. By Proposition 2.1, the function

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{f(\lambda^2)} d\lambda$$

is strictly increasing. Moreover, the characteristic exponent $\psi(\lambda) = f(\lambda^2)$ is regularly varying at 0 with index $2\alpha$. In [5], Marcus assumed another condition, namely that $\psi$ is regularly varying at zero with index $\alpha \in (1, 2]$. Hence, by assuming $1/2 < \alpha \leq 1$, we see that $\psi$ is regularly varying with index $\alpha \in (1, 2)$. On the other hand, since $\psi$ is regularly varying at infinity with index $2\alpha\beta < 2\alpha$, it cannot be a stable mixture (see [7], Lemma 7.1).

We describe now a class of examples that satisfy the assumptions of Proposition 2.3. Let $\mu$ be a Lévy measure on $(0, \infty)$, i.e.,

$$\int_0^\infty (1 \wedge t) \, \mu(dt) < \infty \, .$$

Define $g : (0, \infty) \to (0, \infty)$ by

$$g(x) := \int_0^\infty (1 - e^{-tx}) \mu(dt) \, .$$

Clearly, $g$ is a Bernstein function. An easy calculation shows that

$$\mathcal{L}g(\lambda) = \int_0^\infty \frac{1}{\lambda(\lambda + t)} \, t \, \mu(dt) \, .$$

Therefore

$$\lambda^2 \mathcal{L}g(\lambda) = \lambda \int_0^\infty \frac{1}{\lambda + t}\, t\, \mu(dt)\,.$$

Define $k : (0,\infty) \to (0,\infty)$ by

$$k(\lambda) := \lambda \int_0^\infty \frac{1}{\lambda + t}\, t\, \mu(dt)\,.$$

Then

$$\frac{k(\lambda)}{\lambda} = \int_0^\infty \frac{1}{\lambda + t}\, t\, \mu(dt)\,,$$

is a Stieltjes function. By Theorem 3.9.29 in [4], $k$ is a complete Bernstein function. Define $f : (0,\infty) \to (0,\infty)$ by $f(\lambda) := \lambda / k(\lambda)$. By the same theorem, $f$ is a complete Bernstein function. But,

$$\frac{\lambda}{f(\lambda)} = k(\lambda) = \lambda^2 \mathcal{L}g(\lambda)$$

for $g$ of the form in Proposition 2.3. This shows that for any Lévy measure $\mu$ and $g$ defined as above, the function $f(\lambda)$ defined by $\lambda / f(\lambda) := \lambda^2 \mathcal{L}g(\lambda)$ is a complete Bernstein function.

Suppose, additionally, that $\mu(dt) = \rho(t)\, dt$ where $\rho : (0,\infty) \to (0,\infty)$ is such that $t\rho(t)$ is decreasing. By Proposition 2.3, the corresponding $\phi$ is a complete Bernstein function.

**Example 3.2** Let $\xi$ be a finite measure on $(1,2)$ with compact support. Define

$$\rho(t) = \int_1^2 t^{-\beta}\, \xi(d\beta).$$

Clearly, $t\rho(t)$ is decreasing. Since

$$\int_0^\infty \frac{t^{1-\beta}}{t+x}\, dt = \left(-\frac{\pi}{\sin \beta\pi}\right) x^{1-\beta}\,,$$

it follows that

$$\begin{aligned} \int_0^\infty \frac{1}{t+\lambda} t\rho(t)\, dt &= \int_1^2 \int_0^\infty \frac{t^{1-\beta}}{t+\lambda}\, dt\, \xi(d\beta) \\ &= \int_1^2 \left(-\frac{\pi}{\sin \beta\pi}\right) \lambda^{1-\beta}\, \xi(d\beta)\,. \end{aligned}$$

Therefore,

$$k(\lambda) = \lambda \int_0^\infty \frac{1}{t+\lambda} t\rho(t)\, dt = \int_1^2 \left(-\frac{\pi}{\sin \beta\pi}\right) \lambda^{2-\beta}\, \xi(d\beta)$$

and

$$f(\lambda) = \frac{\lambda}{k(\lambda)} = \frac{\lambda}{\int_1^2 \left(-\frac{\pi}{\sin \beta\pi}\right) \lambda^{2-\beta}\, \xi(d\beta)}\,.$$

The corresponding $\psi(\lambda) = f(\lambda^2)$ is of the form

$$\psi(\lambda) = \frac{\lambda^2}{\int_1^2 \left(-\frac{\pi}{\sin \beta\pi}\right) \lambda^{4-2\beta}\, \xi(d\beta)}\,.$$

Moreover, if the support of the measure $\xi$ is contained in $(3/2, 2)$, then $\psi$ is regularly varying at zero with index $\alpha \in (1, 2)$.

# References


[1] Barlow, M.T. (1988), Necessary and sufficient conditions for the continuity of local time of Lévy processes, *Ann. Probab.* **16**, 1389–1427.

[2] Bertoin, J. (1996), *Lévy Processes* (Cambridge University Press, Cambridge).

[3] Erdèlyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1954) *Tables of integral transforms. Vol. I. Based, in part, on notes left by Harry Bateman* (McGraw-Hill, New York).

[4] Jacob, N. (2001), *Pseudo Differential Operators and Markov Processes*, Vol.1 (Imperial College Press, London).

[5] Marcus, M.B. (2001), The most visited sites of certain Lévy processes. *J. Theoret. Probab.* **14**, 867–885.

[6] Marcus, M.B. and Rosen, J. (1993), $\phi$-variation of the local times of symmetric Lévy processes and stationary Gaussian processes, in *Seminar on Stochastic Processes, 1992*, Vol. 33, *Progress in Probability*, (Birkhäuser, Boston) pp.209–220.

[7] Marcus, M.B. and Rosen, J. (1999), Renormalized self-intersection local times and Wick power chaos processes, *Memoirs of the Amer. Math. Soc. No. 675*, **142**.

[8] Rao, M., Song, R. and Vondraček, Z. (2005), Green function estimates and Harnack inequality for subordinate Brownian motions, *Potential Analysis*, to appear.

[9] Sato, K.-I. (1999), *Lévy processes and infinitely divisible distributions*, (Cambridge University Press, Cambridge).