Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$

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Abstract

For $d \geq 1$ and $\alpha \in (0, 2)$, consider the family of pseudo differential operators $\{\Delta + b\Delta^{\alpha/2}; b \in [0, 1]\}$ on $\mathbb{R}^d$ that evolves continuously from Δ to $\Delta + \Delta^{\alpha/2}$. In this paper, we establish a uniform boundary Harnack principle (BHP) with explicit boundary decay rate for nonnegative functions which are harmonic with respect to $\Delta + b\Delta^{\alpha/2}$ (or equivalently, the sum of a Brownian motion and an independent symmetric α-stable process with constant multiple $b^{1/\alpha}$) in $C^{1,1}$ open sets. Here a "uniform" BHP means that the comparing constant in the BHP is independent of $b \in [0, 1]$. Along the way, a uniform Carleson type estimate is established for nonnegative functions which are harmonic with respect to $\Delta + b\Delta^{\alpha/2}$ in Lipschitz open sets. Our method employs a combination of probabilistic and analytic techniques.

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1 Introduction

Discontinuous Markov processes have been receiving intensive study recently due to their importance both in theory and in applications. Many physical and economic systems could be and in fact have been successfully modeled by discontinuous Markov processes (or jump diffusions as some authors call them); see for example, [29, 34, 36] and the references therein. The infinitesimal generator of a discontinuous Markov process in $\mathbb{R}^d$ is no longer a differential operator but rather a non-local (or integro-differential) operator. For instance, the infinitesimal generator of a rotationally symmetric α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$ is a fractional Laplacian operator $c\,\Delta^{\alpha/2} := -c\,(-\Delta)^{\alpha/2}$.

Discontinuous Markov processes include the very important Lévy processes as special cases and they are of intrinsic importance in probability theory. Integro-differential operators are very important in the theory of partial differential equations. Most of the recent study concentrates

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on discontinuous Markov processes, like the rotationally symmetric α-stable processes, that do not have a diffusion component. For a summary of some of these recent results from the probability literature, one can see [10, 15] and the references therein. We refer the readers to [12, 13, 14] for a sample of recent progresses in the PDE literature.

However, in many situations, like in finance and control theory, one needs Markov processes that have both a diffusion component and a jump component, see for instance, [28, 35, 36]. The fact that such a process X has both diffusion and jump components is the source of many difficulties in investigating the potential theory of the process X. The main difficulty in studying X stems from the fact that it runs on two different scales: on the small scale the diffusion part dominates, while on the large scale the jumps take over. Another difficulty is encountered when looking at the exit of X from an open set: for diffusions, the exit is through the boundary, while for the pure jump processes, typically the exit happens by jumping out from the open set. For the process X, both cases will occur which makes the process X much more difficult to study.

Despite these difficulties, in the last few years significant progress has been made in understanding the potential theory of such processes. Green function estimates (for the whole space) and the Harnack inequality for a class of processes with both continuous and jump components were established in [37, 38]. The parabolic Harnack inequality and heat kernel estimates were studied in [39] for Lévy processes in $\mathbb{R}^d$ that are independent sums of Brownian motions and symmetric stable processes, and in [21] for much more general symmetric diffusions with jumps. Moreover, a priori Hölder estimate is established in [21] for bounded parabolic functions. For earlier results on second order integro-differential operators, one can see [25] and the references therein.

The boundary Harnack principle (BHP) is a result about the ratio of positive harmonic functions. We say that the BHP holds for an open set $D \subset \mathbb{R}^d$ if there exist positive constants R_0 and C depending on D with the property that for any $Q \in \partial D$, $r \in (0, R_0]$, and any positive harmonic functions u and v in $D \cap B(Q, r)$ that vanish continuously on $\partial D \cap B(Q, r)$, we have

$$\frac{u(x)}{v(x)} \le C \frac{u(y)}{v(y)} \qquad \text{for all } x, y \in D \cap B(Q, r/2). \tag{1.1}$$

The BHP for Brownian motion (or, equivalently, for the Laplacian) is a fundamental result in analysis and PDE. It was independently established for Lipschitz domains in the late 1970's by Ancona, Dahlberg and Wu ([1, 23, 43]). Later, Bass and Burdzy developed a probabilistic method in [5] to prove the BHP and extended the BHP to more general domains (see also [4]). When D is a Lipschitz domain and $x_0 \in D$ fixed, the Green function $G_D(x, x_0)$ in D is harmonic in $D \setminus \{x_0\}$ and vanishes continuously on ∂D hence can be taken as $v(x)$ in (1.1). When D is a bounded $C^{1,1}$ domain, it can be shown that $G_D(x, x_0)$ is comparable on $D \setminus B(x_0, \varepsilon)$ to the Euclidean distance function $\delta_D(x)$ between x and D^c, where ε is sufficiently small so that $B(x_0, 2\varepsilon) \subset D$. In this case, one can equivalently express the BHP by using $v(x) = \delta_D(x)$ in (1.1) although this latter function is not harmonic. Therefore, when D is a $C^{1,1}$ domain, the BHP (1.1) can be strengthened to the following version that gives the explicit boundary decay rate of non-negative harmonic functions u that vanish on the boundary:

$$\frac{u(x)}{\delta_D(x)} \le C \frac{u(y)}{\delta_D(y)} \qquad \text{for all } x, y \in D \cap B(Q, r/2)\,. \tag{1.2}$$

Observe that (1.2) clearly implies (1.1) but with C^2 in place of the C there (see Remark 1.5 below).

The BHP plays a vital role in the study of potential theory of Brownian motion and Dirichlet Laplacian in domains. For example, the BHP can be used to show that the Martin boundary can be identified with the Euclidean boundary for a large class of domains and to study the non-tangential limit of non-negative harmonic functions near the boundary (see [2] for an analytic approach and [3] for a probabilistic approach). In fact, the BHP has also be established for a large class of diffusions (or, equivalently, for second order elliptic equations), see [11, 24].

The study of the BHP for discontinuous Markov processes and integro-differential operators is quite recent. It was first established for rotationally symmetric stable processes in bounded Lipschitz domains in [7] and then extended to more general open sets in [41]. Subsequently in [9, 42], the BHP is extended to symmetric (but not necessarily rotationally invariant) stable processes. Recently, the BHP has been extended in [32] to a large class of pure jump Lévy processes that can be obtained from Brownian motion through subordination. Very recently, the BHP for some one-dimensional Lévy processes with both continuous and jump components was studied in [33]. However the BHP for processes on $\mathbb{R}^d$ in dimension two and higher that have both diffusion and jump components has been completely open until now. Note that the fact that a pure jump process may (and typically does) exit an open set by jumping out of it stipulates that, in the BHP for such processes, the nonnegative harmonic functions vanish continuously on $D^c \cap B(Q, r)$.

The principal goal of this paper is to establish the BHP for nonnegative functions which are harmonic with respect to the independent sum of a Brownian motion and a symmetric stable process in $C^{1,1}$ open sets in $\mathbb{R}^d$ for every $d \geq 1$. The process X studied in this paper, although quite specific, serves as a test case for more general processes with both continuous and jump parts. The study of this test case will hopefully shed new light on the understanding of the boundary behavior of nonnegative harmonic functions of general Markov processes.

Intuitively, the independent sum X of a Brownian motion and a symmetric stable process can be thought roughly as some sort of "perturbation" of Brownian motion. Thus some people might expect the BHP for X could be established through some general perturbation technique. However, this kind of approach may not always work. See Remark 1.6 below for details.

Let us now describe the main result of this paper more precisely and at the same time fix the notations. A (rotationally) symmetric α-stable process $Y = (Y_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ in $\mathbb{R}^d$ is a Lévy process such that

$$\mathbb{E}_x \left[e^{i\xi\cdot(Y_t - Y_0)} \right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

The infinitesimal generator of a symmetric α-stable process Y in $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2} u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y\in\mathbb{R}^d:\, |y-x|>\varepsilon\}} (u(y) - u(x)) \frac{\mathcal{A}(d, \alpha)}{|x-y|^{d+\alpha}}\, dy \tag{1.3}$$

where $\mathcal{A}(d, \alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$. Here Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$ for every $\lambda > 0$.

Suppose X^0 is a Brownian motion in $\mathbb{R}^d$ with generator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, and Y is a symmetric α-stable process in $\mathbb{R}^d$. Both X^0 and Y satisfy a self-similarity, which will be used several times in this paper. That is, for every $\lambda > 0$, $\{\lambda^{-1/2}(X^0_{\lambda t} - X^0_0), t \geq 0\}$ and $\{\lambda^{-1/\alpha}(Y_{\lambda t} - Y_0), t \geq 0\}$ have the same distributions as that of $\{X^0_t - X^0_0, t \geq 0\}$ and $\{Y_t - Y_0, t \geq 0\}$, respectively. Assume that X^0

and Y are independent. For any $a > 0$, we define X^a by $X^a_t := X^0_t + aY_t$. We will call the process X^a the independent sum of the Brownian motion X^0 and the symmetric α-stable process Y with weight $a > 0$. The infinitesimal generator of X^a is $\Delta + a^\alpha \Delta^{\alpha/2}$. For every open subset $D \subset \mathbb{R}^d$, we denote by $X^{a,D}$ the subprocess of X^a killed upon leaving D. The infinitesimal generator of $X^{a,D}$ is $(\Delta + a^\alpha \Delta^{\alpha/2})|_D$. It is known (see [39]) that $X^{a,D}$ has a continuous transition density $p^a_D(t, x, y)$ with respect to the Lebesgue measure. We will use $p^a(t, x, y)$ to denote the transition density of X^a (or equivalently, the heat kernel of $\Delta + a^\alpha \Delta^{\alpha/2}$). The quadratic form $(\mathcal{E}, \mathcal{F})$ associated with the generator $\Delta + a^\alpha \Delta^{\alpha/2}$ of X^a is given by

$$\mathcal{F} = W^{1,2}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d; dx) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^d; dx) \text{ for every } 1 \le i \le d \right\}$$

and for $u, v \in \mathcal{F}$,

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x)\, dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) \frac{\mathcal{A}(d, \alpha)\, a^\alpha}{|x - y|^{d+\alpha}} dxdy.$$

In probability theory, the quadratic form $(\mathcal{E}, W^{1,2}(\mathbb{R}^d))$ is called the Dirichlet form of X^a. A statement is said to hold quasi-everywhere (q.e. in abbreviation) if there is a set N having zero capacity with respect to $(\mathcal{E}_1, W^{1,2}(\mathbb{R}^d))$ such that the statement holds everywhere outside N. Here $\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u(x)^2 dx$. The function $J^a(x, y) := a^\alpha \mathcal{A}(d, \alpha)|x - y|^{-(d+\alpha)}$ is the Lévy intensity of X^a. It determines a Lévy system for X^a, which describes the jumps of the process X^a: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X^a) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[\sum_{s \le T} f(s, X^a_{s-}, X^a_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^a_s, y) J^a(X^a_s, y) dy \right) ds \right]. \tag{1.4}$$

(See, for example, [19, Proof of Lemma 4.7] and [20, Appendix A].)

The purpose of this paper is to establish the scale invariant version of the BHP in Theorem 1.4. To state this theorem, we first recall that an open set D in $\mathbb{R}^d$ (when $d \ge 2$) is said to be $C^{1,1}$ if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(y)| \le \Lambda|x - y|$, and an orthonormal coordinate system CS_Q: $y = (y_1, \cdots, y_{d-1}, y_d) =: (\widetilde{y}, y_d)$ with its origin at Q such that

$$B(Q, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\widetilde{y})\}.$$

The pair (R, Λ) is called the characteristics of the $C^{1,1}$ open set D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set can be unbounded and disconnected.

For any $x \in D$, let $\delta_D(x)$ denote the distance between x and ∂D. It is well known that any $C^{1,1}$ open set D satisfies the uniform interior ball condition: there exists $\widetilde{R} \le R$ such that for every $x \in D$ with $\delta_D(x) < \widetilde{R}$, there is $Q_x \in \partial D$ so that $|x - Q_x| = \delta_D(x)$ and that $B(\widetilde{x}, \widetilde{R}) \subset D$ for $\widetilde{x} = Q_x + \widetilde{R}(x - Q_x)/|x - Q_x|$. Without loss of generality, throughout this paper, we assume that the characteristics (R, Λ) of a $C^{1,1}$ open set satisfies $R = \widetilde{R} \le 1$ and $\Lambda \ge 1$.

For any open set $D \subset \mathbb{R}^d$, $\tau^a_D := \inf\{t > 0 : X^a_t \notin D\}$ denotes the first exit time from D by X^a.

Definition 1.1 A real-valued function u defined on $\mathbb{R}^d$ is said to be harmonic in $D \subset \mathbb{R}^d$ with respect to X^a if for every open set B whose closure is a compact subset of D,

$$\mathbb{E}_x\left[\left|u(X^a_{\tau^a_B})\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X^a_{\tau^a_B})\right] \qquad \text{for q.e. } x \in B. \tag{1.5}$$

Note that by using the Lévy system of X^a, we have

$$\begin{aligned}\mathbb{E}_x\left[\left|u(X^a_{\tau^a_B})\right|\right] &\geq \mathbb{E}_x\left[\left|u(X^a_{\tau^a_B})\right|;\, X^a_{\tau_B} \in \mathbb{R}^d \setminus \overline{B}\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_B}\left(\int_{\mathbb{R}^d\setminus\overline{B}} |u(y)| \frac{\mathcal{A}(d,\alpha)\, a^\alpha}{|X^a_s - y|^{d+\alpha}}\, dy\right) ds\right].\end{aligned}$$

Hence if u is a harmonic function in D with respect to X^a, then $u(y)(1 \wedge |y|^{-(d+\alpha)})$ is integrable on B^c for any relatively compact open subset B with $\overline{B} \subset D$. It follows from Theorems 1.2 and 1.3 of [21] that all harmonic functions in D with respect to X^a are continuous on D, since every harmonic function in D with respect to X^a can be approximated locally uniformly in D by functions that are bounded on $\mathbb{R}^d$ and harmonic with respect to X^a in relatively compact open subsets of D. Therefore, for any harmonic function u in D, (1.5) holds for *every* point $x \in D$. The above also implies that any harmonic function u in D with respect to X^a is locally bounded in D with $\int_{\mathbb{R}^d} |u(y)|(1 \wedge |y|^{-(d+\alpha)})dy < \infty$. A function u is said to be in $W^{1,2}_{\text{loc}}(D)$ if for every relatively compact subset B with $\overline{B} \subset D$, there is a function $f \in W^{1,2}(\mathbb{R}^d)$ such that $u = f$ a.e. on B. The following analytic characterization of a function u being harmonic in D with respect to X^a follows immediately from Example 2.14 in [16].

Proposition 1.2 *Let D be an open subset of $\mathbb{R}^d$. Then the following are equivalent.*

(i) *u is harmonic in D with respect to X^a;*

(ii) *u is locally bounded in D, $\int_{\mathbb{R}^d} |u(y)|(1 \wedge |y|^{-(d+\alpha)})dy < \infty$, $u \in W^{1,2}_{\text{loc}}(D)$ and $(\Delta + a^\alpha \Delta^{\alpha/2})u = 0$ in D in the distributional sense: for every $\phi \in C^\infty_c(D)$*

$$\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla \phi(x)\, dx + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (u(x) - u(y))(\phi(x) - \phi(y)) \frac{\mathcal{A}(d,\alpha)\, a^\alpha}{|x-y|^{d+\alpha}}\, dxdy = 0.$$

The following uniform Harnack principle will be used to prove the main result of this paper. Its proof will be given in Section 4 below.

Proposition 1.3 (Harnack principle) *Suppose that $M > 0$. There exists a constant $C_0 = C_0(\alpha, M) > 0$ such that for any $r \in (0, 1]$, $a \in [0, M]$, $x_0 \in \mathbb{R}^d$ and any function u which is nonnegative in $\mathbb{R}^d$ and harmonic in $B(x_0, r)$ with respect to X^a we have*

$$u(x) \leq C_0 u(y) \qquad \textit{for all } x, y \in B(x_0, r/2).$$

Let $Q \in \partial D$. We will say that a function $u : \mathbb{R}^d \to \mathbb{R}$ vanishes continuously on $D^c \cap B(Q, r)$ if $u = 0$ on $D^c \cap B(Q, r)$ and u is continuous at every point of $\partial D \cap B(Q, r)$. The following is the main result of this paper.

Theorem 1.4 *Suppose that $M > 0$. For any $C^{1,1}$ open set D in $\mathbb{R}^d$ with the characteristics (R, Λ), there exists a positive constant $C = C(\alpha, d, \Lambda, R, M)$ such that for $a \in [0, M]$, $r \in (0, R]$, $Q \in \partial D$ and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$\frac{u(x)}{\delta_D(x)} \le C \frac{u(y)}{\delta_D(y)} \qquad \text{for every } x, y \in D \cap B(Q, r/2). \tag{1.6}$$

Remark 1.5 As we mentioned earlier, this is a strengthened version of the BHP. Interchanging the role of x and y, we have from (1.6) that

$$C^{-1} \frac{u(y)}{\delta_D(y)} \le \frac{u(x)}{\delta_D(x)} \le C \frac{u(y)}{\delta_D(y)} \qquad \text{for every } x, y \in D \cap B(Q, r/2).$$

Hence for any two positive functions u and v that satisfy the condition of Theorem 1.4, by taking the quotient of the last display for u and v, we deduce that

$$C^{-2} \frac{u(y)}{v(y)} \le \frac{u(x)}{v(x)} \le C^2 \frac{u(y)}{v(y)} \qquad \text{for every } x, y \in D \cap B(Q, r/2), \tag{1.7}$$

which gives the usual form of the BHP. While (1.6) clearly no longer holds for Lipschitz domains, we expect that (1.7) is true for Lipschitz domains. □

When a changes from 0 to M, $\Delta + a^\alpha \Delta^{\alpha/2}$ changes continuously from Δ to $\Delta + M^\alpha \Delta^{\alpha/2}$. So Theorem 1.4 says that the BHP holds uniformly for the family $\{\Delta + a^\alpha \Delta^{\alpha/2},\ a \in [0, M]\}$ of pseudo differential operators in the sense that the constant C in (1.6) can be chosen to be independent of $a \in [0, M]$. Note that $a = 0$ corresponds to the classical case of the BHP for the Laplacian. We will therefore in the rest of the paper assume that $a \in (0, M]$.

As far as we know, this is the first time that a BHP has been established for non-local integro-differential operators that have second order differential operator components in dimension two and higher. Unlike (1.1) and the paragraph following it, in this paper we are concerned with the above BHP for $C^{1,1}$ open sets only. The main focus and goal of this paper is to get the explicit decay rate of harmonic functions near the boundary of D as in (1.6) and to show that the BHP is *uniform* in $a \in [0, M]$. We emphasize that (1.6) is not true in Lipschitz domains even in the classical case of the BHP for the Laplacian. However, a uniform Carleson type estimate is shown to hold for Lipschitz open sets in Theorem 4.3. The BHP of above type is very useful in studying other fine properties of the process. For example, it has been used in [18] to derive derive sharp two-sided Green function estimates of X^a in $C^{1,1}$ open sets. Very recently, it has been used in [17] to obtain sharp two-sided heat kernel estimates for X^a in $C^{1,1}$ open sets.

For $a > 0$, X^a and $X := X^1$ are in fact related by a scaling. More precisely, for $a \in (0, M]$, X^a has the same distribution as $\lambda X_{\lambda^{-2}t}$, where $\lambda = a^{\alpha/(\alpha-2)} \ge M^{\alpha/(\alpha-2)}$. Consequently, if u is harmonic in an open set U with respect to X^a, then $v(x) := u(\lambda x)$ is harmonic in $\lambda^{-1} U$ with respect to X. Hence the uniform Harnack inequality of Proposition 1.3 follows from the Harnack inequality for X. The latter is known, see Theorem 6.7 of [21] or Theorem 4.5 of [39]. However the uniform BHP of Theorem 1.4 can not be obtained by such a scaling argument from the BHP of X. This is because for a $C^{1,1}$ open set D with the characteristics (R, Λ), $\lambda^{-1}D$ is, in general, a $C^{1,1}$ open set with $C^{1,1}$ characteristics $(R/\lambda,\ \lambda\Lambda)$, which tends to $(0, \infty)$ as $\lambda \to \infty$.

For each fixed $\alpha_0 \in (0, 2)$, when α changes from α_0 to 2, the operator $\Delta + a^\alpha \Delta^{\alpha/2}$ evolves continuously from $\Delta + a^{\alpha_0} \Delta^{\alpha_0/2}$ to $(1+a^2)\Delta$. So in view of Theorem 1.4, it is reasonable to expect that one can get the BHP for $\Delta + a^\alpha \Delta^{\alpha/2}$ uniformly both in $a \in (0, M]$ and in $\alpha \in [\alpha_0, 2)$. We believe this is the case and that it can be achieved by carefully keeping track of all the comparison constants in the arguments of this paper. However in order to keep our exposition as transparent as possible, we are content with establishing the result stated in Theorem 1.4 and leave the details of the proof for the last claim to interested readers.

Our method of establishing the above BHP is different from those in [7, 41] for symmetric stable processes and in [32] for more general subordinate Brownian motions. The reason that the approaches in [7, 41, 32] do not work well in our setting lies exactly in the fact that X^a leaves open set D by jumping out across the boundary ∂D as well as by continuously exiting D through the boundary of D. To circumvent this difficulty, in this paper we adopt the ideas from [8] for the BHP of censored stable processes, which were further refined in [27]. That is, we use suitably chosen subharmonic and superharmonic functions of the process X^a (or equivalently, of $\Delta + a^\alpha \Delta^{\alpha/2}$) to derive some exit distribution estimates that are needed to establish the BHP. However, had we done it in this way directly, we would only get the BHP for $\Delta + a^\alpha \Delta^{\alpha/2}$ with $\alpha \in (1, 2)$. The reason is that, when $D = \mathbb{H}^d_+ := \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_1 > 0\}$, we need to consider testing functions $w_p(x) = (x_1 \vee 0)^p$ for $p > 1$. But for w_p to be $\Delta^{\alpha/2}$-differentiable in $\mathbb{H}^d_+$, see (1.3), one requires $p < \alpha$, which would be impossible when $\alpha \in (0, 1]$. To overcome this difficulty, for each $\lambda > 0$, we consider the finite range (or truncated) symmetric α-stable process $\widehat{Y}^\lambda$ obtained from Y by suppressing all its jumps of size larger than λ. The infinitesimal generator of $\widehat{Y}^\lambda$ is

$$\widehat{\Delta}^{\alpha/2}_{d,\lambda} u(x) := \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d :\, \varepsilon < |y-x| < \lambda\}} (u(y) - u(x)) \frac{\mathcal{A}(d, \alpha)}{|x-y|^{d+\alpha}}\, dy. \tag{1.8}$$

When $\lambda = 1$, we will simply denote $\widehat{\Delta}^{\alpha/2}_{d,1}$ by $\widehat{\Delta}^{\alpha/2}_{d}$. Then w_p is $\widehat{\Delta}^{\alpha/2}_{d}$-differentiable in $\mathbb{H}^d_+$ for every $p > 0$. Observe that $\widehat{X}^a := X^0 + a\widehat{Y}^{1/a}$ is a Lévy process obtained from $X^a = X^0 + aY$ by suppressing all its jumps of size larger than 1 and that the infinitesimal generator of $\widehat{X}^a$ is $\Delta + a^\alpha \widehat{\Delta}^{\alpha/2}_{d}$. From this, we can obtain suitable exit distribution estimates for the Lévy process $\widehat{X}^a$. The desired estimates for X^a can then be obtained from that for $\widehat{X}^a$ by adding back those jumps of X^a of size larger than 1. Such an idea has already been used in [22] to study Schramm-Löwner evolutions driven by one-dimensional symmetric stable processes. We remark that the BHP in Theorem 1.4 for the case of $a = 1$ has also been mentioned in Remark 5.2 of Guan [27]. However, no precise statement (such as the range of α) nor a proof is given in that paper.

Remark 1.6 We point out here that even though the form of the BHP in Theorem 1.4 of this paper resembles the one for Δ, it is unlikely that it can be proved through a general perturbation technique by viewing $\Delta + \Delta^{\alpha/2}$ as a perturbation of Δ. Indeed, if such a general perturbation technique worked, it is reasonable to expect that it would have also worked for $\Delta + \widehat{\Delta}^{\alpha/2}_{d}$, which can be viewed as a *smaller* perturbation of Δ. But, by modifying the counter-example in Section 6 of [30], one can show that there is a rich class of non-negative harmonic functions of $\Delta + \widehat{\Delta}^{\alpha/2}_{d}$ for which the conclusion of Theorem 1.4 does not hold for general non-convex but smooth open set D.

The rest of the paper is organized as follows. In Section 2, we derive estimates on $\widehat{\Delta}_d^{\alpha/2} w_p$. These estimates are then used in Section 3 to obtain exit distribution (or harmonic measure) estimates for the finite range process $\widehat{X}^a$ and then for the desired process X^a. In Section 4, we first give the proof of Proposition 1.3, and then establish a Carleson estimate for non-negative harmonic functions of $\Delta + a^\alpha \Delta^{\alpha/2}$ in Lipschitz open sets. Then using these results, the proof of Theorem 1.4 is presented.

Throughout this paper, we use the capital letters $C_1, C_2, \cdots$ to denote constants in the statement of the results, and their labeling will be fixed. The lowercase constants $c_1, c_2, \cdots$ will denote generic constants used in the proofs, whose exact values are not important and can change from one appearance to another. The labeling of the constants $c_1, c_2, \cdots$ starts anew in every proof. The dependence of the lower case constants on the dimension $d \geq 1$ and $\alpha \in (0, 2)$ may not be mentioned explicitly. The constant $M > 0$ will be fixed throughout this paper. We will use ":=" to denote a definition, which is read as "is defined to be". For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For every function f, let $f^+ := f \vee 0$. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx or $m_d(dx)$ to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure and $\mathrm{diam}(A)$ to denote the diameter of the set A.

2 Truncated fractional Laplacian estimates for power functions

In this section, we give some estimates which will be used later. Recall that the fractional Laplacian $\Delta^{\alpha/2}$ and the truncated fractional Laplacian $\widehat{\Delta}_d^{\alpha/2} := \widehat{\Delta}_{d,1}^{\alpha/2}$ are defined in (1.3) and (1.8), respectively.

Lemma 2.1 *For $x \in \mathbb{R}^d$ and $p > 0$, set $w_p(x) := (x_1^+)^p$. Then there are constants $R_* \in (0, 1)$, $C_1 > C_2 > 0$ depending only on p, d and α such that for every $x \in \mathbb{R}^d$ with $x_1 \in (0, R_*]$*

$$|\widehat{\Delta}_d^{\alpha/2} w_p(x)| \leq C_1 \qquad \text{for } p > \alpha, \tag{2.1}$$

$$|\widehat{\Delta}_d^{\alpha/2} w_p(x)| \leq C_1 \, |\log x_1| \qquad \text{for } p = \alpha, \tag{2.2}$$

$$C_2 x_1^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} w_p(x) \leq C_1 x_1^{p-\alpha} \qquad \text{for } \alpha/2 < p < \alpha, \tag{2.3}$$

$$-C_1 \leq \widehat{\Delta}_d^{\alpha/2} w_p(x) \leq -C_2 \qquad \text{for } p = \alpha/2, \tag{2.4}$$

and

$$-C_1 x_1^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} w_p(x) \leq -C_2 x_1^{p-\alpha} \qquad \text{for } 0 < p < \alpha/2. \tag{2.5}$$

Proof. First note that using integration by parts and a change of variable, we get that for $p, x > 0$ and $\varepsilon \in (0, 1/(x+1))$,

$$\begin{aligned} \int_0^{1-\varepsilon} \frac{z^p - 1}{(1-z)^{\alpha+1}} dz =& \frac{1}{\alpha} \int_0^{1-\varepsilon} (z^p - 1) d(1-z)^{-\alpha} \\ =& \frac{1}{\alpha}(z^p - 1)(1-z)^{-\alpha}\Big|_0^{1-\varepsilon} - \frac{p}{\alpha} \int_0^{1-\varepsilon} z^{p-1}(1-z)^{-\alpha} dz \\ =& \frac{(1-\varepsilon)^p - 1}{\alpha \varepsilon^\alpha} + \frac{1}{\alpha} - \frac{p}{\alpha} \int_0^{1-\varepsilon} \frac{z^{p-1}}{(1-z)^\alpha} dz \end{aligned} \tag{2.6}$$

and

$$\int_{1+\varepsilon}^{\frac{x+1}{x}} \frac{z^p - 1}{(z-1)^{\alpha+1}} dz = -\frac{1}{\alpha} \int_{1+\varepsilon}^{\frac{x+1}{x}} (z^p - 1) d(z-1)^{-\alpha} \tag{2.7}$$

$$= -\frac{1}{\alpha}(z^p - 1)(z-1)^{-\alpha}\Big|_{1+\varepsilon}^{\frac{x+1}{x}} + \frac{p}{\alpha}\int_{1+\varepsilon}^{\frac{x+1}{x}} z^{p-1}(z-1)^{-\alpha} dz$$

$$= \frac{(1+\varepsilon)^p - 1}{\alpha\varepsilon^\alpha} + \frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}(x+1)^p x^{\alpha-p} + \frac{p}{\alpha}\int_{\frac{x}{x+1}}^{\frac{1}{\varepsilon+1}} z^{\alpha-p-1}(1-z)^{-\alpha} dz.$$

For $p > 0$ and $x \in (0,1)$, by a change of variable

$$\mathcal{A}(1,\alpha)^{-1}\widehat{\Delta}_1^{\alpha/2} w_p(x) = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \frac{w_p(y) - w_p(x)}{|x-y|^{1+\alpha}} \mathbf{1}_{\{\varepsilon<|y-x|\le 1\}}\, dy$$

$$= \lim_{\varepsilon\downarrow 0} \int_0^{x+1} \frac{y^p - x^p}{|x-y|^{1+\alpha}} \mathbf{1}_{\{|y-x|>\varepsilon\}}\, dy - x^p \int_{x-1}^0 \frac{dy}{|x-y|^{1+\alpha}}$$

$$= x^{p-\alpha} \lim_{\varepsilon\downarrow 0} \int_0^{\frac{x+1}{x}} \frac{z^p - 1}{|z-1|^{1+\alpha}} \mathbf{1}_{\{|z-1|>\varepsilon/x\}}\, dz - x^p \int_{x-1}^0 (x-y)^{-1-\alpha} dy$$

$$= x^{p-\alpha} \lim_{\varepsilon\downarrow 0} \left(\int_0^{1-\varepsilon} \frac{z^p - 1}{(1-z)^{1+\alpha}} dz + \int_{1+\varepsilon}^{\frac{x+1}{x}} \frac{z^p - 1}{(z-1)^{1+\alpha}} dz \right) - \alpha^{-1}(x^{p-\alpha} - x^p).$$

So we have by (2.6)–(2.7) that for $p > 0$ and $x \in (0,1)$,

$$\frac{\alpha}{\mathcal{A}(1,\alpha)} \widehat{\Delta}_1^{\alpha/2} w_p(x) \tag{2.8}$$

$$= x^{p-\alpha} \lim_{\varepsilon\downarrow 0} \left(1 + \frac{(1-\varepsilon)^p + (1+\varepsilon)^p - 2}{\varepsilon^\alpha} \right) - (x^{p-\alpha} - x^p) + x^p$$

$$- (x+1)^p + p x^{p-\alpha} \left(\int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dy \right)$$

$$= 2x^p - (x+1)^p + p x^{p-\alpha} \left(\int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dz \right).$$

Note that for $p > \alpha$,

$$\sup_{x\in(0,1]} x^{p-\alpha} \left| \int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dz \right| < \infty.$$

So when $p > \alpha$,

$$\sup_{x\in(0,1)} |\widehat{\Delta}_1^{\alpha/2} w_p(x)| < \infty. \tag{2.9}$$

When $p = \alpha$, there exists an $r_* > 0$ such that for $0 < x < r_*$

$$\left| \int_{\frac{x}{x+1}}^1 \frac{z^{-1} - z^{\alpha-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{\alpha-1}}{(1-z)^\alpha} dz \right| \tag{2.10}$$

$$\le \int_{\frac{x}{x+1}}^1 \frac{z^{-1}}{(1-z)^\alpha} dz \le (1+r_*)^\alpha \log((1+r_*)/x).$$

It is easy to see that

$$\sup_{x\in[r_*,1]}\left|\int_{\frac{x}{x+1}}^1 \frac{z^{-1}-z^{\alpha-1}}{(1-z)^\alpha}dz-\int_0^{\frac{x}{x+1}}\frac{z^{\alpha-1}}{(1-z)^\alpha}dz\right|<\infty. \tag{2.11}$$

On the other hand, when $p\in(0,\alpha)$,

$$\sup_{x\in(0,1]}\left|\int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1}-z^{p-1}}{(1-z)^\alpha}dz-\int_0^{\frac{x}{x+1}}\frac{z^{p-1}}{(1-z)^\alpha}dz\right|<\infty. \tag{2.12}$$

As

$$\lim_{x\to 0+}\left(\int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1}-z^{p-1}}{(1-z)^\alpha}dz-\int_0^{\frac{x}{x+1}}\frac{z^{p-1}}{(1-z)^\alpha}dz\right)\begin{cases}>0 & \text{if } p\in(\alpha/2,\alpha)\\ <0 & \text{if } p\in(0,\alpha/2),\end{cases}$$

while for $p=\alpha/2$,

$$\int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1}-z^{p-1}}{(1-z)^\alpha}dz-\int_0^{\frac{x}{x+1}}\frac{z^{p-1}}{(1-z)^\alpha}dz=-\int_0^{\frac{x}{x+1}}\frac{z^{\alpha/2-1}}{(1-z)^\alpha}dz,$$

we conclude from (2.8)–(2.12) that there are constants $r_1\in(0,1)$ and $c_1>c_2>0$ depending on p and α so that when $p=\alpha$,

$$|\widehat{\Delta}_1^{\alpha/2}w_p(x)<c_1\,|\log x| \quad\text{for } x\in(0,r_1]\quad\text{and}\quad \sup_{x\in(r_1,1)}|\widehat{\Delta}_1^{\alpha/2}w_p(x)|<\infty, \tag{2.13}$$

when $p=\alpha/2$,

$$-c_1\le \widehat{\Delta}_1^{\alpha/2}w_p(x)<-c_2 \quad\text{for } x\in(0,r_1]\quad\text{and}\quad \sup_{x\in(r_1,1)}|\widehat{\Delta}_1^{\alpha/2}w_p(x)|<\infty, \tag{2.14}$$

when $p\in(\alpha/2,\alpha)$,

$$c_2\,x^{p-\alpha}<\widehat{\Delta}_1^{\alpha/2}w_p(x)<c_1\,x^{p-\alpha} \quad\text{for } x\in(0,r_1]\quad\text{and}\quad \sup_{x\in(r_1,1)}|\widehat{\Delta}_1^{\alpha/2}w_p(x)|<\infty, \tag{2.15}$$

and for $p\in(0,\alpha/2)$,

$$-c_1\,x^{p-\alpha}<\widehat{\Delta}_1^{\alpha/2}w_p(x)<-c_2\,x^{p-\alpha} \quad\text{for } x\in(0,r_1]\quad\text{and}\quad \sup_{x\in(r_1,1)}|\widehat{\Delta}_1^{\alpha/2}w_p(x)|<\infty. \tag{2.16}$$

On the other hand, for $x\ge 1$,

$$\begin{aligned}
\widehat{\Delta}_1^{\alpha/2}w_p(x) =&\mathcal{A}(1,\alpha)\lim_{\varepsilon\downarrow 0}\int_{x-1}^{x+1}\frac{w_p(y)-w_p(x)}{|x-y|^{1+\alpha}}\,\mathbf{1}_{\{|x-y|>\varepsilon\}}\,dy\\
=&\mathcal{A}(1,\alpha)\,x^{p-\alpha}\lim_{\varepsilon\downarrow 0}\int_{\frac{x-1}{x}}^{\frac{x+1}{x}}\frac{y^p-1}{|y-1|^{1+\alpha}}\,\mathbf{1}_{\{|y-1|>\varepsilon\}}\,dy\\
=&\mathcal{A}(1,\alpha)\,x^{p-\alpha}\lim_{\varepsilon\downarrow 0}\left(\int_{\frac{x-1}{x}}^{1-\varepsilon}\frac{y^p-1}{(1-y)^{1+\alpha}}dy+\int_{1+\varepsilon}^{\frac{x+1}{x}}\frac{y^p-1}{(y-1)^{1+\alpha}}dy\right)\\
=&\mathcal{A}(1,\alpha)\,x^{p-\alpha}\int_0^{1/x}\frac{(1+u)^p+(1-u)^p-2}{u^{1+\alpha}}du.
\end{aligned}$$

Note the above integrand

$$\frac{(1+u)^p+(1-u)^p-2}{u^{1+\alpha}}$$

is of the order $u^{1-\alpha}$ near zero. So for $p>0$ and $\alpha\in(0,2)$, there is a constant $c_3=c_3(p,\alpha)>0$ so that

$$|\widehat{\Delta}_1^{\alpha/2}w_p(x)|\le c_3x^{p-2}\qquad \text{for } x\ge 1. \tag{2.17}$$

With $r_1\in(0,1)$ as in (2.13)–(2.16), the above inequality in fact holds for $x>r_1$.

The estimates (2.9)–(2.16) prove the lemma in dimension $d=1$. Now we consider the case $d\ge 2$. For each fixed $x\in\mathbb{R}^d$, we use the spherical coordinates

$$(y_1,\dots,y_d):=x+(r\cos\theta_1, r\sin\theta_1\cos\theta_2,\dots,r\sin\theta_1\dots\cos\theta_{d-1}, r\sin\theta_1\dots\sin\theta_{d-1})$$

where $r\ge 0$, $0\le\theta_1,\dots,\theta_{d-2}<\pi$ and $0\le\theta_{d-1}<2\pi$. Let

$$\phi(\widehat{\theta}):=\phi(\theta_1,\dots,\theta_{d-2}):=\sin^{d-2}\theta_1\sin^{d-3}\theta_2\dots\sin\theta_{d-2}.$$

Then for $x\in\mathbb{R}^d$ with $x_1>0$ we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d:\,1>|y-x|>\varepsilon\}}((y_1^+)^p-x_1^p)\frac{dy}{|x-y|^{d+\alpha}}\\
=&\lim_{\varepsilon\downarrow 0}\int_0^\pi d\theta_1\cdots\int_0^\pi d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}\int_\varepsilon^1\frac{((r\cos\theta_1+x_1)^+)^p-x_1^p}{r^{d+\alpha}}r^{d-1}dr\\
=&\lim_{\varepsilon\downarrow 0}\int_0^{\pi/2} d\theta_1\int_0^\pi d\theta_2\cdots\int_0^\pi d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(\cos\theta_1)^p\\
&\qquad\qquad\times\int_\varepsilon^1\frac{((r+\frac{x_1}{\cos\theta_1})^+)^p-(\frac{x_1}{\cos\theta_1})^p}{r^{1+\alpha}}dr\\
&+\lim_{\varepsilon\downarrow 0}\int_{\pi/2}^{\pi} d\theta_1\int_0^\pi d\theta_2\cdots\int_0^\pi d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(-\cos\theta_1)^p\\
&\qquad\qquad\times\int_\varepsilon^1\frac{((-r-\frac{x_1}{\cos\theta_1})^+)^p-(-\frac{x_1}{\cos\theta_1})^p}{r^{1+\alpha}}dr.
\end{aligned}$$

By the change of variable $r = t - x_1/\cos\theta_1$ for $\theta \in [0,\pi/2)$ and $r = -t - x_1/\cos\theta_1 = -t + x_1/\cos(\pi-\theta_1)$ for $\theta \in (\pi/2,\pi]$, we get

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d\colon 1>|y-x|>\varepsilon\}}((y_1^+)^p - x_1^p)\frac{dy}{|x-y|^{d+\alpha}}\\
=&\lim_{\varepsilon\downarrow 0}\int_0^{\pi/2}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(\cos\theta_1)^p\\
&\qquad\times\int_{\varepsilon+\frac{x_1}{\cos\theta_1}}^{1+\frac{x_1}{\cos\theta_1}}\frac{(t^+)^p-(\frac{x_1}{\cos\theta_1})^p}{|t-x_1/\cos\theta_1|^{1+\alpha}}dt\\
&+\lim_{\varepsilon\downarrow 0}\int_{\pi/2}^{\pi}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(\cos(\pi-\theta_1))^p\\
&\qquad\times\int_{-1+\frac{x_1}{\cos(\pi-\theta_1)}}^{-\varepsilon+\frac{x_1}{\cos(\pi-\theta_1)}}\frac{(t^+)^p-(\frac{x_1}{\cos(\pi-\theta_1)})^p}{|t-x_1/\cos(\pi-\theta_1)|^{1+\alpha}}dt\\
=&\lim_{\varepsilon\downarrow 0}\int_0^{\pi/2}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(\cos\theta_1)^p\\
&\qquad\times\int_{\varepsilon+\frac{x_1}{\cos\theta_1}}^{1+\frac{x_1}{\cos\theta_1}}\frac{(t^+)^p-(\frac{x_1}{\cos\theta_1})^p}{|t-x_1/\cos\theta_1|^{1+\alpha}}dt\\
&+\lim_{\varepsilon\downarrow 0}\int_0^{\pi/2}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(\cos\theta_1)^p\\
&\qquad\times\int_{-1+\frac{x_1}{\cos\theta_1}}^{-\varepsilon+\frac{x_1}{\cos\theta_1}}\frac{(t^+)^p-(\frac{x_1}{\cos\theta_1})^p}{|t-x_1/\cos\theta_1|^{1+\alpha}}dt\\
=&\int_0^{\pi/2}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}(\cos\theta_1)^p\\
&\qquad\times\left(\lim_{\varepsilon\downarrow 0}\int_{\{t\in\mathbb{R}\colon 1>|t-\frac{x_1}{\cos\theta_1}|>\varepsilon\}}\frac{(t^+)^p-(\frac{x_1}{\cos\theta_1})^p}{|t-x_1/\cos\theta_1|^{1+\alpha}}dt\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\widehat{\Delta}_d^{\alpha/2}w_p(x) =&\frac{\mathcal{A}(d,\alpha)}{\mathcal{A}(1,\alpha)}\int_0^{\pi/2}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}\\
&\qquad\times(\cos\theta_1)^p\widehat{\Delta}_1^{\alpha/2}w_p\left(\frac{x_1}{\cos\theta_1}\right)\\
=&\frac{\mathcal{A}(d,\alpha)}{\mathcal{A}(1,\alpha)}\int_0^{\arccos(x_1/r_1)}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}\\
&\qquad\times(\cos\theta_1)^p\widehat{\Delta}_1^{\alpha/2}w_p\left(\frac{x_1}{\cos\theta_1}\right)\\
&+\frac{\mathcal{A}(d,\alpha)}{\mathcal{A}(1,\alpha)}\int_{\arccos(x_1/r_1)}^{\pi/2}d\theta_1\int_0^{\pi}d\theta_2\cdots\int_0^{\pi}d\theta_{d-2}\int_0^{2\pi}\phi(\widehat{\theta})d\theta_{d-1}\\
&\qquad\times(\cos\theta_1)^p\widehat{\Delta}_1^{\alpha/2}w_p\left(\frac{x_1}{\cos\theta_1}\right).
\end{aligned}$$

The conclusion (2.1)–(2.5) now follow immediately from the above equality and the estimates (2.9)–(2.17), where we use (2.17) to bound the second integral above by $c_4x_1^3/r_1^3$ for some positive constant

c_4. □

Remark 2.2 A careful evaluation of (2.8) in fact shows that $\lim_{x\to 0^+} \widehat{\Delta}_1^{\alpha/2} w_p(x) = c \neq 0$ when $p > \alpha$. At first glance, this may look surprising, as in the Brownian motion case (which corresponds to $\alpha = 2$), $\Delta x_1^p = p(p-1)x_1^{p-2}$. The bound in (2.1) is due to the non-local nature of the operator $\widehat{\Delta}_d^{\alpha/2}$ for $\alpha \in (0,2)$. However a more careful analysis of (2.8) reveals that for $p > 0$,

$$\widehat{\Delta}_1^{\alpha/2} w_p(x) \asymp (2-\alpha)\big((p-\alpha)^{-1} - 1\big) + p(p-1)x^{p-\alpha} \qquad \text{for } x \in (0, r_1)$$

as $\alpha \uparrow 2$. It is not difficult to see that as $\alpha \uparrow 2$, $\widehat{\Delta}_1^{\alpha/2} w_p(x)$ converges to $\Delta w_p(x)$. □

Recall that for $\lambda > 0$, the operator $\widehat{\Delta}_{d,\lambda}^{\alpha/2}$ is defined by (1.8). Note that

$$\widehat{\Delta}_{d,\lambda}^{\alpha/2} u(x) = \lambda^{-\alpha} (\widehat{\Delta}_d^{\alpha/2} u(\lambda \cdot))(\lambda^{-1} x). \tag{2.18}$$

Thus, from Lemma 2.1 and (2.18), we get the following corollary.

Corollary 2.3 *For $x \in \mathbb{R}^d$ and $p > 0$, set $w_p(x) := (x_1^+)^p$. Then there are constants $R_* \in (0, 1/2)$, $C_1 > C_2 > 0$ depending only on p, d and α such that for every $\lambda > 0$ and $x \in \mathbb{R}^d$ with $x_1 \in (0, \lambda R_*)$,*

$$|\widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x)| \le C_1 \lambda^{p-\alpha} \qquad \textit{for } p > \alpha, \tag{2.19}$$

$$|\widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x)| \le C_1 |\log(x_1/\lambda)|, \qquad \textit{for } p = \alpha, \tag{2.20}$$

$$C_2 x_1^{p-\alpha} \le \widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x) \le C_1 x_1^{p-\alpha} \qquad \textit{for } \alpha/2 < p < \alpha, \tag{2.21}$$

$$-C_1 \lambda^{-\alpha/2} \le \widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x) \le -C_2 \lambda^{-\alpha/2} \qquad \textit{for } p = \alpha/2, \tag{2.22}$$

and

$$-C_1 x_1^{p-\alpha} \le \widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x) \le -C_2 x_1^{p-\alpha} \qquad \textit{for } 0 < p < \alpha/2. \tag{2.23}$$

3 Estimates on harmonic measures

Recall that for any open set $U \subset \mathbb{R}^d$, $\tau_U^a = \inf\{t > 0 : X_t^a \notin U\}$ is the first exit time from U by X^a.

Lemma 3.1 *For every $b \in (0, \infty)$, there exist $C_3 = C_3(M, b) > 0$ and $C_4 = C_4(M, b) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $a \in (0, M]$ and $r \in (0, b]$,*

$$C_3 r^2 \le \mathbb{E}_{x_0}\left[\tau_{B(x_0,r)}^a\right] \le C_4 r^2. \tag{3.1}$$

Proof. See Lemmas 2.2 and 2.3 in [38] or Lemmas 2.3 and 2.4 in [21] for a proof. □

In the remainder of this section, we assume D is a $C^{1,1}$ open set with characteristics (R, Λ). Recall that we are always assuming that $R \le 1$ and $\Lambda \ge 1$. For notational convenience, throughout the rest of this section, we put

$$R_0 = R_0(R,\Lambda) = \frac{R}{\sqrt{1+\Lambda^2}} \quad \text{and} \quad r_0 = r_0(R,\Lambda) = \frac{R_0}{4\sqrt{1+\Lambda^2}} = \frac{R}{4(1+\Lambda^2)}.$$

Define

$$\rho_Q(x) := x_d - \phi_Q(\widetilde{x}),$$

where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_Q. Note that for every $Q \in \partial D$ and $x \in B(Q,R) \cap D$ we have

$$(1+\Lambda^2)^{-1/2}\,\rho_Q(x) \,\leq\, \delta_D(x) \,\leq\, \rho_Q(x). \tag{3.2}$$

Recall that R_* is the constant in Lemma 2.1.

Lemma 3.2 *Fix $Q \in \partial D$ and the coordinate system CS_Q so that*

$$B(Q,R) \cap D = \{y = (\widetilde{y},\, y_d) \in B(0,R) \text{ in } CS_Q : y_d > \phi(\widetilde{y})\}\,.$$

For $p > \alpha/2$, let

$$h_p(y) := (\rho_Q(y))^p\, \mathbf{1}_{D\cap B(Q,R_0)}(y).$$

Then there exist $C_i = C_i(\alpha, p, \Lambda, R) > 0$, $i = 5, 6, 7$, independent of the choice of the point $Q \in \partial D$ such that

(i) *in the case $\frac{\alpha}{2} < p < \alpha$, for all $x \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, we have*

$$C_6\,(\rho_Q(x))^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} h_p(x) \leq C_5\,(\rho_Q(x))^{p-\alpha}\,; \tag{3.3}$$

(ii) *in the case $p > \alpha$, for all $x \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, we have*

$$|\widehat{\Delta}_d^{\alpha/2} h_p(x)| \leq C_7; \tag{3.4}$$

(iii) *in the case $p = \alpha$, for all $x \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, we have*

$$|\widehat{\Delta}_d^{\alpha/2} h_p(x)| \leq C_7 |\log(\rho_Q(x))|\,. \tag{3.5}$$

Proof. In this proof our coordinate system is always CS_Q. Fix $x = (\widetilde{x}, x_d) \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, and choose a point $x_0 \in \partial D$ satisfying $\widetilde{x} = \widetilde{x_0}$. Denote by $\overrightarrow{n}(x_0)$ the inward unit normal vector at x_0 for ∂D and set $\Phi(y) = \langle y - x_0, \overrightarrow{n}(x_0)\rangle$ for $y \in \mathbb{R}^d$. Then $\Pi := \{y : \Phi(y) = 0\}$ is the hyperplane tangent to ∂D at the point x_0. The function $\Gamma^* : \mathbb{R}^{d-1} \to \mathbb{R}$ describing the plane Π is given by $\Gamma^*(\tilde{y}) = \phi_Q(\tilde{x}_0) + \nabla\phi_Q(\tilde{x}_0)(\tilde{y} - \tilde{x}_0)$, and it holds that $\big\langle (\widetilde{y}, \Gamma^*(\widetilde{y})) - x_0,\, \overrightarrow{n}(x_0)\big\rangle = 0$. We also let

$$\begin{aligned} A &:= \big\{y : \Gamma^*(\widetilde{y}) < y_d < \phi_Q(\widetilde{y}) \text{ and } |\widetilde{y} - \widetilde{x}| < r_0\big\} \\ &\qquad \bigcup \big\{y : \Gamma^*(\widetilde{y}) > y_d > \phi_Q(\widetilde{y}) \text{ and } |\widetilde{y} - \widetilde{x}| < r_0\big\}, \\ E &:= \big\{y \in D \setminus A :\; |\widetilde{y} - \widetilde{x}| < r_0 \text{ and } \rho_Q(y) < r_0(2+\Lambda)\big\}. \end{aligned}$$

Note that, if $|x - y| < r_0$ and $y \in D$,

$$\rho_Q(y) \leq |y_d - x_d| + |x_d - \phi_Q(\widetilde{x})| + |\phi_Q(\widetilde{y}) - \phi_Q(\widetilde{x})| < r_0(2+\Lambda).$$

On the other hand if $|\widetilde{y}-\widetilde{x}|<r_0$ and $\rho_Q(y)<r_0(2+\Lambda)$, then

$$|y|^2=|\widetilde{y}|^2+|y_d|^2\le(2r_0)^2+(r_0(2+\Lambda)+|\phi_Q(\widetilde{y})|)^2\le\frac{4+(2+3\Lambda)^2}{16(1+\Lambda^2)}R_0^2<R_0^2.$$

Consequently, we have

$$D\cap B(x,r_0)\subset D\cap\big\{y:\,|\widetilde{y}-\widetilde{x}|<r_0 \text{ and } \rho_Q(y)<r_0(2+\Lambda)\big\}\subset D\cap B(0,R_0). \tag{3.6}$$

Let $\overline{h}(y):=\overline{h}_x(y):=(y_d-\Gamma^*(\widetilde{y}))^+$ for $y\in\mathbb{R}^d$. Since $\nabla\phi_Q(\widetilde{x})=\nabla\Gamma^*(\widetilde{x})$, by the mean value theorem and the $C^{1,1}$ condition on ϕ_Q,

$$\begin{aligned}|\overline{h}(y)-\rho_Q(y)|\;&=\;|\phi_Q(\widetilde{y})-\Gamma^*(\widetilde{y})|\\&=|\phi_Q(\widetilde{y})-\phi_Q(\widetilde{x})-\nabla\phi_Q(\widetilde{x})\cdot(\widetilde{y}-\widetilde{x})|\;\le\;\Lambda|\widetilde{y}-\widetilde{x}|^2,\qquad y\in E.\end{aligned} \tag{3.7}$$

For $y\in\mathbb{R}^d$, define $\delta_\Pi(y):=\mathrm{dist}(y,\Pi)$ and $D_{\Gamma^*}=\big\{y\in\mathbb{R}^d:y_d>\Gamma^*(\widetilde{y})\big\}$. Let

$$b_x:=\big(1+|\nabla\phi_Q(\widetilde{x})|^2\big)^{1/2}\qquad\text{and}\qquad h_{x,p}(y):=(\overline{h}(y))^p\quad\text{for } p>\alpha/2.$$

Note that $1\le b_x\le\sqrt{1+\Lambda^2}$ and $h_{x,p}(x)=h_p(x)$.

Recall that R_* and $C_1>C_2>0$ are the constants in Lemma 2.1. Since $\overline{h}(y)=b_x\delta_\Pi(y)$ on D_{Γ^*}, by Lemma 2.1, it holds that for $y\in D_{\Gamma^*}$ and $\delta_\Pi(y)<R_*$,

$$\begin{aligned}C_2\,b_x^p(\delta_\Pi(y))^{p-\alpha}\le\widehat{\Delta}_d^{\alpha/2}h_{x,p}(y)=\;&b_x^p\widehat{\Delta}_d^{\alpha/2}(\delta_\Pi(y))^p\\\le\;&C_1\,b_x^p(\delta_\Pi(y))^{p-\alpha}\quad\text{when } \alpha/2<p<\alpha,\end{aligned} \tag{3.8}$$

$$|\widehat{\Delta}_d^{\alpha/2}h_{x,p}(y)|\;=\;b_x^p|\widehat{\Delta}_d^{\alpha/2}(\delta_\Pi(y))^p|\;\le\;C_1\,b_x^p\;\le\;C_1(1+\Lambda^2)^{p/2}\qquad\text{when } p>\alpha, \tag{3.9}$$

$$\begin{aligned}|\widehat{\Delta}_d^{\alpha/2}h_{x,p}(y)|=&\,b_x^p|\widehat{\Delta}_d^{\alpha/2}(\delta_\Pi(y))^p|\;\le\;C_1\,b_x^p|\log(\delta_\Pi(y))|\\\le&\,C_1(1+\Lambda^2)^{p/2}|\log(\delta_\Pi(y))|\qquad\qquad\text{when } p=\alpha.\end{aligned} \tag{3.10}$$

Note that $b_x\delta_\Pi(x)=\rho_Q(x)$. Applying (3.2) and (3.8) to the point x gives that, for $\alpha/2<p<\alpha$

$$\begin{aligned}C_2\rho_Q(x)^{p-\alpha}\le C_2b_x^\alpha\rho_Q(x)^{p-\alpha}\le&\,\widehat{\Delta}_d^{\alpha/2}h_{x,p}(x)\\\le&\,C_1b_x^\alpha\rho_Q(x)^{p-\alpha}\le C_1(1+\Lambda^2)^{\alpha/2}\rho_Q(x)^{p-\alpha}.\end{aligned} \tag{3.11}$$

Note that by (3.6),

$$\begin{aligned}&|\widehat{\Delta}_d^{\alpha/2}(h_p-h_{x,p})(x)|\\=\;&\mathcal{A}(d,\alpha)\Big|\lim_{\varepsilon\downarrow0}\int_{\{1\ge|y-x|>\varepsilon\}}\frac{(h_p(y)-h_{p,x}(y))}{|x-y|^{d+\alpha}}\,dy\Big|\\\le\;&\mathcal{A}(d,\alpha)\Big|\int_{\{1\ge|y-x|>r_0\}}\frac{(h_p(y)-h_{p,x}(y))}{|x-y|^{d+\alpha}}\,dy\Big|\\&+\mathcal{A}(d,\alpha)\lim_{\varepsilon\downarrow0}\int_{\{r_0\ge|y-x|>\varepsilon\}}\frac{|h_p(y)-h_{p,x}(y)|}{|x-y|^{d+\alpha}}\,dy\\\le\;&\mathcal{A}(d,\alpha)\Big|\int_{\{1\ge|y-x|>r_0\}}\frac{(h_p(y)-h_{p,x}(y))}{|x-y|^{d+\alpha}}\,dy\Big|\\&+\mathcal{A}(d,\alpha)\int_A\frac{h_p(y)+h_{p,x}(y)}{|x-y|^{d+\alpha}}\,dy+\mathcal{A}(d,\alpha)\int_E\frac{|h_p(y)-h_{p,x}(y)|}{|x-y|^{d+\alpha}}\\=:\;&I_1+I_2+I_3.\end{aligned} \tag{3.12}$$

We claim that, if $p > \alpha/2$, then

$$I_1 + I_2 + I_3 \le c_0 \tag{3.13}$$

for some constant $c_0 = c_0(\alpha, p, \Lambda, R)$. Together with (3.9)–(3.12) this will establish the desired estimates (3.3)–(3.5) with constants depending on α, p, Λ and R.

Clearly I_1 is bounded by some positive constant.

For $y \in A$, we have

$$\begin{aligned} |h_{x,p}(y)| + |h_p(y)| \ &\le \ |y_d - \Gamma^*(\widetilde{y})|^p + |y_d - \phi_Q(\widetilde{y})|^p \ \le \ 2|\phi_Q(\widetilde{y}) - \Gamma^*(\widetilde{y})|^p \\ \le \ 2|\phi_Q(\widetilde{y}) - \phi_Q(\widetilde{x}) - \nabla\phi_Q(\widetilde{x}) \cdot (\widetilde{y} - \widetilde{x})|^p \ &\le \ 2\Lambda^p |\widetilde{y} - \widetilde{x}|^{2p}. \end{aligned} \tag{3.14}$$

Furthermore, since, on $\{|\widetilde{y} - \widetilde{x}| = r \le r_0\}$, $|\phi_Q(\widetilde{y}) - \Gamma^*(\widetilde{y})| \le \Lambda |\widetilde{y} - \widetilde{x}|^2 = \Lambda r^2$,

$$m_{d-1}\left(\{y : |\widetilde{y} - \widetilde{x}| = r, \Gamma^*(\widetilde{y}) < y_d < \phi_Q(\widetilde{y}) \text{ or } \Gamma^*(\widetilde{y}) > y_d > \phi_Q(\widetilde{y})\}\right) \le c_1 r^d$$

for some constant $c_1 > 0$ if $r \le r_0$. This together with (3.14) yields that

$$\begin{aligned} I_2 \ &\le \ \mathcal{A}(d,\alpha) \int_0^{r_0} \int_{|\widetilde{y} - \widetilde{x}| = r} \mathbf{1}_A(y) \frac{|h_{x,p}(y)| + |h_p(y)|}{|\widetilde{y} - \widetilde{x}|^{d+\alpha}} \ m_{d-1}(dy) dr \\ &\le \ c_2 \int_0^{r_0} r^{-d+2p-\alpha} m_{d-1}(\{y \in A : |\widetilde{y} - \widetilde{x}| = r\}) dr \\ &\le \ c_1 c_2 \int_0^{r_0} r^{2p-\alpha} dr \le c_3 \,. \end{aligned}$$

Note that, since $E \subset D \cap B(0, R_0)$ by (3.6), we have $h_p(y) = (\rho_Q(y))^p$ for $y \in E$. Thus, we have that for $y \in E$

$$|h_{x,p}(y) - h_p(y)| = |(\overline{h}(y))^p - (\rho_Q(y))^p| \le c_4 (\overline{h}(y))^{(p-1)_-} |\overline{h}(y) - \rho_Q(y)|, \tag{3.15}$$

where $(p-1)_- := (p-1) \wedge 0$. In the last inequality above, we have used the inequalities

$$|b^p - a^p| \le b^{p-1}|b - a| \qquad \text{for } a, b > 0, \ 0 < p \le 1$$

and

$$|b^p - a^p| \le (p+1)|b - a| \qquad \text{for } a, b \in (0,1), \ p > 1.$$

For $y = (\widetilde{y}, y_d) \in \mathbb{R}^d$, we use an affine coordinate system $z = (\widetilde{z}, z_d)$ to represent it so that $z_d = y_d - \Gamma^*(\widetilde{y})$ and $\widetilde{z}$ are the coordinates in an orthogonal coordinate system centered at x_0 for the $(d-1)$-dimensional hyperplane Π for the point $(\widetilde{y}, \Gamma^*(\widetilde{y}))$. Denote such an affine transformation $y \mapsto z$ by $z = \Psi(y)$. It is clear that there is a constant $c_5 = c_5(\Lambda, R) > 1$ so that for every $y \in \mathbb{R}^d$,

$$c_5^{-1}|\widetilde{y} - \widetilde{x}| \le |\widetilde{z}| \le c_5 |\widetilde{y} - \widetilde{x}|, \qquad c_5^{-1}|y - x| \le |\Psi(y) - \Psi(x)| \le c_5 |y - x|$$

and that

$$\Psi(E) \subset \{z = (\widetilde{z}, z_d) \in \mathbb{R}^d : \ |\widetilde{z}| < c_5 r_0 \text{ and } 0 < z_d \le c_5 r_0\}.$$

Denote $x_d - \Gamma^*(\widetilde{x})$ by w; that is, $\Psi(x) = (\widetilde{0}, w)$. Hence by (3.7) and (3.15) and applying the transform Ψ, we have by using polar coordinates for $\widetilde{z}$ on the hyperplane Π,

$$
\begin{aligned}
I_3 &\leq c_6 \int_E \frac{\overline{h}(y)^{(p-1)_-}\,|\widetilde{y}-\widetilde{x}|^2}{|y-x|^{d+\alpha}}\,dy \leq c_7 \int_{\Psi(E)} \frac{z_d^{(p-1)_-}\,|\widetilde{z}|^2}{|z-(\widetilde{0},w)|^{d+\alpha}}\,dz \\
&\leq c_8 \int_0^{c_5 r_0} z_d^{(p-1)_-} \left(\int_0^{c_5 r_0} \frac{r^d}{(r+|z_d-w|)^{d+\alpha}} dr \right) dz_d \\
&\leq c_8 \int_0^{c_5 r_0} z_d^{(p-1)_-} \left(\int_0^{c_5 r_0} \frac{1}{(r+|z_d-w|)^{\alpha}} dr \right) dz_d \\
&\leq c_9 \int_0^{c_5 r_0} z_d^{(p-1)_-} \left(\frac{1}{|z_d-w|^{\alpha-1}} - \frac{1}{(c_5 r_0 + |z_d-w|)^{\alpha-1}} \right) dz_d \\
&< c_{10} \int_0^{c_5 r_0} \frac{1}{z_d^{(1-p)^+}\,|z_d-w|^{\alpha-1}} dz_d \leq c_{11} < \infty,
\end{aligned}
$$

where all constants depend on α, p, Λ and R. The last inequality is due to the fact that since $p > 0$, $0 < \alpha < 2$ and $(1-p)^+ + \alpha - 1 = \max\{\alpha - p, \alpha - 1\} < 1$, by the dominated convergence theorem, $\phi(w) := \int_0^{c_5 r_0} z_d^{-(1-p)^+}\,|z_d - w|^{1-\alpha} dz_d$ is a strictly positive continuous function in $w \in [0, c_5 r_0]$ and hence is bounded. Thus we have proved the claim (3.13), hence completing the proof of the lemma. □

Since D is a $C^{1,1}$ open set with characteristics (R, Λ), for every $\lambda \geq 1$, λD is a $C^{1,1}$ open set with uniform characteristics (R, Λ). Thus, by the previous lemma and (2.18), we get the following as a corollary.

Corollary 3.3 *Fix $Q \in \partial D$ and the coordinate system CS_Q so that*

$$B(Q,R) \cap D = \left\{ y = (\widetilde{y},\, y_d) \in B(0,R) \text{ in } CS_Q : y_d > \phi(\widetilde{y}) \right\}.$$

Let

$$h_p(y) := (\rho_Q(y))^p\, \mathbf{1}_{D \cap B(Q,R_0)}(y).$$

Then there exist $C_i = C_i(\alpha, p, \Lambda, R) > 0$, $i = 5, 6, 7$, independent of the choice of the point $Q \in \partial D$ and $\lambda \geq 1$ such that

(i) *in the case $\frac{\alpha}{2} < p < \alpha$, for all $x \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, we have*

$$C_6\,(\rho_Q(x))^{p-\alpha} \leq \widehat{\Delta}^{\alpha/2}_{d,\lambda} h_p(x) \leq C_5\,(\rho_Q(x))^{p-\alpha}\,; \tag{3.16}$$

(ii) *in the case $p > \alpha$, for all $x \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, we have*

$$|\widehat{\Delta}^{\alpha/2}_{d,\lambda} h_p(x)| \leq C_7 \lambda^{p-\alpha}; \tag{3.17}$$

(iii) *in the case $p = \alpha$, for all $x \in D$ such that $\rho_Q(x) < r_0 \wedge R_*$ and $|\widetilde{x}| < r_0$, we have*

$$|\widehat{\Delta}^{\alpha/2}_{d,\lambda} h_p(x)| \leq C_7\,|\log(\rho_Q(x)/\lambda)|\,. \tag{3.18}$$

The following scaling property of X^a will be used below: If $(X_t^{a,D}, t \geq 0)$ is the subprocess in D of the independent sum of a Brownian motion and a symmetric α-stable process on $\mathbb{R}^d$ with weight a, then $(\lambda X^{a,D}_{\lambda^{-2}t}, t \geq 0)$ is the subprocess in λD of the independent sum of a Brownian motion and a symmetric α-stable process on $\mathbb{R}^d$ with weight $a\lambda^{(\alpha-2)/\alpha}$. So for any $\lambda > 0$, we have

$$p_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(t,x,y) = \lambda^{-d} p_D^a(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } t>0 \text{ and } x,y \in \lambda D. \tag{3.19}$$

By integrating the above equation with respect to t, we get

$$G_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(x,y) = \lambda^{2-d} G_D^a(\lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } x,y \in \lambda D \tag{3.20}$$

where

$$G_D^a(x,y) := \int_0^\infty p_D^a(t,x,y)dt$$

is the Green function of X^a in D. It is well known that the Lévy measure of X^1 has the intensity

$$J^1(x,y) = j^1(|x-y|) = \mathcal{A}(d,\alpha)|x-y|^{-(d+\alpha)}.$$

Thus by a scaling argument, we get that the Lévy intensity of X^a is

$$J^a(x,y) = j^a(|x-y|) = a^\alpha \mathcal{A}(d,\alpha)|x-y|^{-(d+\alpha)},$$

which gives the Lévy system (1.4) of X^a.

By a λ-truncated symmetric α-stable process in $\mathbb{R}^d$ we mean a pure jump symmetric Lévy process $\widehat{Y}^\lambda = \{\widehat{Y}^\lambda_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$ with Lévy density $\mathcal{A}(d,\alpha)|x|^{-d-\alpha}\, \mathbf{1}_{\{|x|<\lambda\}}$. Note that the Lévy exponent ψ^λ of $\widehat{Y}^\lambda$, defined by

$$\mathbb{E}_x\left[e^{i\xi\cdot(\widehat{Y}^\lambda_t - \widehat{Y}^\lambda_0)}\right] = e^{-t\psi^\lambda(\xi)} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

is given by

$$\psi^\lambda(\xi) = \mathcal{A}(d,\alpha) \int_{\{|y|<\lambda\}} \frac{1-\cos(\xi\cdot y)}{|y|^{d+\alpha}} dy. \tag{3.21}$$

Suppose that $\widehat{Y}^{\lambda/a}$ is a (λ/a)-truncated symmetric α-stable process in $\mathbb{R}^d$ which is independent of the Brownian motion X^0. For any $a > 0$, we define

$$\widehat{X}^{a,\lambda}_t := X^0_t + a\widehat{Y}^{\lambda/a}_t, \qquad t \geq 0.$$

Note that from (3.21) we can easily check that for any $b > 0$,

$$\psi^\lambda(b\xi) = b^\alpha \psi^{\lambda b}(\xi) \qquad \text{for every } \xi \in \mathbb{R}^d. \tag{3.22}$$

Thus for any $a > 0$ and $\xi, x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}_x\left[e^{i\xi\cdot(\widehat{X}^{a,\lambda}_t - \widehat{X}^{a,\lambda}_0)}\right] &= e^{-t|\xi|^2}\mathbb{E}_x\left[e^{i(a\xi)\cdot(\widehat{Y}^{\lambda/a}_t - \widehat{Y}^{\lambda/a}_0)}\right] \\ &= e^{-t(|\xi|^2 + \psi^{\lambda/a}(a\xi))} = e^{-t(|\xi|^2 + a^\alpha\psi^\lambda(\xi))}.\end{aligned}$$

Therefore $\widehat{X}^{a,\lambda}$ has the same distribution as the Lévy process obtained from X^a by removing jumps of size larger than λ. The above observation also gives us that the infinitesimal generator of $\widehat{X}^{a,\lambda}$ is $\Delta + a^\alpha \widehat{\Delta}^{\alpha/2}_{d,\lambda}$, and the Lévy intensity for $\widehat{X}^{a,\lambda}$ is

$$J^{a,\lambda}(x,y) := a^\alpha \mathcal{A}(d,\alpha)|x-y|^{-(d+\alpha)}\,\mathbf{1}_{\{|x-y|<\lambda\}}\,.$$

The Lévy intensity describes the jumps of the process $\widehat{X}^{a,\lambda}$ through a Lévy system: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s,y,y)=0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of $\widehat{X}^{a,\lambda}$) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x\left[\sum_{s\le T} f(s, \widehat{X}^{a,\lambda}_{s-}, \widehat{X}^{a,\lambda}_s)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, \widehat{X}^{a,\lambda}_s, y) J^{a,\lambda}(\widehat{X}^{a,\lambda}_s, y)dy\right) ds\right]. \tag{3.23}$$

For our reader's convenience, we summarize some notations below.

Process	*Generator*	*Lévy (jumping) kernel*
X^0	Δ	0
Y	$\Delta^{\alpha/2}$	$\mathcal{A}(d,\alpha)\,\lvert z\rvert^{-d-\alpha}$
aY	$a^\alpha\Delta^{\alpha/2}$	$a^\alpha\mathcal{A}(d,\alpha)\,\lvert\lvert z\rvert^{-d-\alpha}$
$\widehat{Y}^\lambda$	$\widehat{\Delta}^{\alpha/2}_{d,\lambda}$	$\mathcal{A}(d,\alpha)\,\lvert z\rvert^{-d-\alpha}\,\mathbf{1}_{\{\lvert z\rvert<\lambda\}}$
$X^a := X^0 + aY$	$\Delta + a^\alpha\Delta^{\alpha/2}$	$a^\alpha\mathcal{A}(d,\alpha)\,\lvert z\rvert^{-d-\alpha}$
$\widehat{X}^{a,\lambda} := X^0 + a\widehat{Y}^{\lambda/a}$	$\Delta + a^\alpha\widehat{\Delta}^{\alpha/2}_{d,\lambda}$	$a^\alpha\mathcal{A}(d,\alpha)\,\lvert z\rvert^{-d-\alpha}\,\mathbf{1}_{\{\lvert z\rvert<\lambda\}}$
$\widehat{X}^a := \widehat{X}^{a,1}$	$\Delta + a^\alpha\widehat{\Delta}^{\alpha/2}_{d}$	$a^\alpha\mathcal{A}(d,\alpha)\,\lvert z\rvert^{-d-\alpha}\,\mathbf{1}_{\{\lvert z\rvert<1\}}$.

For any open set $U \subset \mathbb{R}^d$, let $\widehat{\tau}^{a,\lambda}_U = \inf\{t>0 : \widehat{X}^{a,\lambda}_t \notin U\}$ be the first exit time from U by $\widehat{X}^{a,\lambda}$, and denote by $\widehat{X}^{a,\lambda,U}$ the subprocess of $\widehat{X}^{a,\lambda}$ killed upon leaving U. When $\lambda=1$, we simply write $\widehat{X}^a$ for $\widehat{X}^{a,1}$ and $\widehat{\tau}^a_U$ for $\widehat{\tau}^{a,1}_U$. The following scaling property will be used in the next lemma: by (3.22), we see that for every $\lambda, a, b > 0$ and $\xi, x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}_x\left[e^{i\xi\cdot(b(\widehat{X}^{a,\lambda}_{b^{-2}t} - \widehat{X}^{a,\lambda}_0))}\right] &= e^{-t|\xi|^2}\mathbb{E}_x\left[e^{i(ab\xi)\cdot(\widehat{Y}^{\lambda/a}_{b^{-2}t} - \widehat{Y}^{\lambda/a}_0)}\right] \\ &= e^{-t|\xi|^2}e^{-b^{-2}t\psi^{\lambda/a}(ab\xi)} = e^{-t(|\xi|^2 + a^\alpha b^{\alpha-2}\psi^{b\lambda}(\xi))}.\end{aligned}$$

Thus, if $\{\widehat{X}^{a,\lambda,D}_t, t\ge 0\}$ is the subprocess of $\{\widehat{X}^{a,\lambda}_t, t\ge 0\}$ in D, then $\{b\widehat{X}^{a,\lambda,D}_{b^{-2}t}, t\ge 0\}$ is the subprocess of $\{\widehat{X}^{ab^{(\alpha-2)/\alpha},b\lambda}_t, t\ge 0\}$ in bD. In particular, if $\{\widehat{X}^{a,D}_t, t\ge 0\}$ is the subprocess of $\{\widehat{X}^a_t, t\ge 0\}$ in D, then $\{\lambda\widehat{X}^{a,D}_{\lambda^{-2}t}, t\ge 0\}$ is the subprocess of $\{\widehat{X}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_t, t\ge 0\}$ in λD. So for any $\lambda>0$, we have

$$\widehat{p}^{\,a\lambda^{(\alpha-2)/\alpha},\lambda}_{\lambda D}(t,x,y) = \lambda^{-d}\widehat{p}^{\,a,1}_D(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } t>0 \text{ and } x,y\in\lambda D \tag{3.24}$$

where $\widehat{p}^{\,a,\lambda}_D(t,x,y)$ is the transition density of $\widehat{X}^{a,\lambda,D}$. By integrating the above equation with respect to t, we get

$$\widehat{G}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_{\lambda D}(x,y) = \lambda^{2-d}\widehat{G}^{a,1}_D(\lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } x,y\in\lambda D \tag{3.25}$$

where

$$\widehat{G}_D^{a,\lambda}(x,y) := \int_0^\infty \widehat{p}_D^{\,a,\lambda}(t,x,y)dt$$

is the Green function of $\widehat{X}^{a,\lambda}$ in D.

Recall that $\rho_Q(x) := x_d - \phi_Q(\widetilde{x})$ for every $Q \in \partial D$ and $x \in \{y = (\widetilde{y}, y_d) \in B(Q,R) : y_d > \phi_Q(\widetilde{y})\}$. We define for $r_1, r_2 > 0$

$$D_Q(r_1, r_2) := \{y \in D : r_1 > \rho_Q(y) > 0, \, |\widetilde{y}| < r_2\}.$$

Lemma 3.4 *There are positive constants $\delta_0 = \delta_0(R, M, \Lambda, \alpha) \in (0, r_0)$, $C_8 = C_8(R, M, \Lambda, \alpha)$ and $C_9 = C_9(R, M, \Lambda, \alpha)$ such that for every $a \in (0, M]$, $\lambda \geq 1$, $Q \in \partial D$ and $x \in D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)$ with $\widetilde{x} = 0$,*

$$\mathbb{P}_x \left(\widehat{X}^a_{\widehat{\tau}^a_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}} \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0) \right) \geq C_8 \lambda \delta_D(x), \tag{3.26}$$

$$\mathbb{P}_x \left(\widehat{X}^a_{\widehat{\tau}^a_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}} \in D \right) \leq C_9 \lambda \delta_D(x) \tag{3.27}$$

and

$$\mathbb{E}_x \left[\widehat{\tau}^a_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)} \right] \leq C_9 \lambda^{-1} \delta_D(x). \tag{3.28}$$

Proof. To derive the estimates in the lemma, it will be convenient to consider the scaled process $\lambda \widehat{X}^a_{\lambda^{-2}t}$, which has the same distribution as $\widehat{X}^{a\lambda^{(\alpha-2)/\alpha},\lambda}$. The latter has infinitesimal generator $\Delta + a^\alpha \lambda^{\alpha-2} \widehat{\Delta}^{\alpha/2}_{d,\lambda}$.

Without loss of generality, we assume $Q = 0$ and let $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be the $C^{1,1}$-function satisfying $\phi(\widetilde{0}) = \nabla\phi(\widetilde{0}) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda$, $|\nabla\phi(\widetilde{y}) - \nabla\phi(\widetilde{z})| \leq \Lambda|\widetilde{y} - \widetilde{z}|$ and CS_Q be the corresponding coordinate system such that

$$B(Q,R) \cap D = \left\{ (\widetilde{y}, \, y_d) \in B(0,R) \text{ in } CS_Q : y_d > \phi(\widetilde{y}) \right\}.$$

Note that, since D is a $C^{1,1}$ open set with characteristics (R, Λ), for every $\lambda \geq 1$, λD is a $C^{1,1}$ open set with the same characteristics (R, Λ). Let $\phi_\lambda(\widetilde{y}) := \phi(\lambda^{-1}\widetilde{y}) : \mathbb{R}^{d-1} \to \mathbb{R}$. Then ϕ_λ satisfies $\phi_\lambda(\widetilde{0}) = \nabla\phi_\lambda(\widetilde{0}) = 0$, $\|\nabla\phi_\lambda\|_\infty \leq \Lambda$, $|\nabla\phi_\lambda(\widetilde{y}) - \nabla\phi_\lambda(\widetilde{z})| \leq \Lambda|\widetilde{y} - \widetilde{z}|$ and

$$B(Q,R) \cap \lambda D = \left\{ y \in B(0,R) \text{ in } CS_Q : \; y_d > \phi_\lambda(\widetilde{y}) \right\} \quad \text{ for all } \lambda \geq 1.$$

We let $p > 0$ be such that $p \neq \alpha$ and $1 < p < (2 \wedge (3 - \alpha))$, and define

$$\begin{aligned} \rho_\lambda(y) &:= y_d - \phi_\lambda(\widetilde{y}), \\ h_\lambda(y) &:= \rho_\lambda(y) \mathbf{1}_{B(0,R_0) \cap \lambda D}(y), \\ h_{\lambda,p}(y) &:= h_\lambda(y)^p \; = \; (\rho_\lambda(y))^p \, \mathbf{1}_{B(0,R_0) \cap \lambda D}(y), \\ D(\lambda, r_1, r_2) &:= \{y \in \lambda D : \, 0 < \rho_\lambda(y) < r_1 \text{ and } |\widetilde{y}| < r_2\}. \end{aligned}$$

Since $\rho_\lambda(y) \leq \sqrt{1 + \Lambda^2} \, \delta_{\lambda D}(y)$ in view of (3.2), we have $0 \leq h_\lambda \leq R \leq 1$. It is easy to see that $D(\lambda, r_1, r_2)$ is contained in $D \cap B(0, R/4)$ for every $r_1, r_2 \leq r_0$. Note that the (vector-valued) Lipschitz function $\nabla\phi_\lambda$ is differentiable almost everywhere. So for a.e. $y \in B(0, R_0) \cap \lambda D$,

$$\Delta h_\lambda(y) = \Delta(y_d - \phi_\lambda(\widetilde{y})) = -\Delta\phi_\lambda(\widetilde{y}) \tag{3.29}$$

and

$$\begin{aligned}\Delta h_{\lambda,p}(y) &= \Delta(y_d - \phi_\lambda(\widetilde{y}))^p \\ &= p(p-1)(1+|\nabla\phi_\lambda(\widetilde{y})|^2)(\rho_\lambda(y))^{p-2} - p\,(\rho_\lambda(y))^{p-1}\Delta\phi_\lambda(\widetilde{y}) \\ &\geq p(p-1)(1+|\nabla\phi_\lambda(\widetilde{y})|^2)(\rho_\lambda(y))^{p-2} - p\,(\rho_\lambda(y))^{p-1}\|\Delta\phi_\lambda\|_\infty.\end{aligned}$$

Thus, since $p \in (1,2)$, we can choose a positive constant $\delta_1 = \delta_1(R, M, \Lambda, \alpha) \in (0, r_0)$, independent of λ, so that there is $c_1 > 0$ such that

$$\Delta h_{\lambda,p}(y) \geq c_1(\rho_\lambda(y))^{p-2} > 0 \qquad \text{for a.e. } y \in D(\lambda, \delta_1, r_0)\,. \tag{3.30}$$

We divide the rest of the proof into three steps.

Step 1: Constructing suitable superharmonic and subharmonic functions with respect to $\Delta + a^\alpha\lambda^{\alpha-2}\widehat{\Delta}^{\alpha/2}_{d,\lambda}$. Let ψ be a smooth positive function on $\mathbb{R}^d$ with bounded first and second order partial derivatives such that $\psi(y) = 2^{p+1}|\widetilde{y}|^2/r_0^2$ for $|y| < r_0/4$ and $2^{p+1} \leq \psi(y) \leq 2^{p+2}$ for $|y| \geq r_0/2$. Now we consider

$$u_{1,\lambda}(y) := h_\lambda(y) + h_{\lambda,p}(y)$$

and

$$u_{2,\lambda}(y) := h_\lambda(y) + \psi(y) - h_{\lambda,p}(y).$$

Observe that since $0 \leq h_\lambda \leq 1$ and $p \geq 1$, both $u_{1,\lambda}$ and $u_{2,\lambda}$ are non-negative. By Taylor's expansion with remainder of order 2,

$$\left|\left(\Delta + a^\alpha\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right)\psi(y)\right| \leq |\Delta\psi(y)| + M^\alpha\left|\widehat{\Delta}^{\alpha/2}_{d,\lambda}\psi(y)\right| \leq c_2(\alpha, M) < \infty. \tag{3.31}$$

Note that the constant c_2 above is independent of λ. Moreover, since $\lambda \geq 1$, $p > \alpha/2$ and $p \neq \alpha$, by (3.16) and (3.17) there exist $c_3 = c_3(R,\Lambda) > 0$ and $\delta_2 = \delta_2(R,\Lambda) \in (0, \delta_1]$ independent of λ such that

$$\widehat{\Delta}^{\alpha/2}_{d,\lambda} h_{\lambda,p}(y) \geq -c_3\lambda^{p-\alpha} \quad \text{for } y \in D(\lambda, \delta_2, r_0).$$

Thus by using (3.30), the fact that $p < 2$ and the inequality above, and by choosing δ_2 smaller if necessary, we get

$$\begin{aligned}\left(\Delta + a^\alpha\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) h_{\lambda,p}(y) &\geq c_1\rho_\lambda(y)^{p-2} - M^\alpha c_3\lambda^{(p-\alpha)+(\alpha-2)} \\ &\geq c_1\rho_\lambda(y)^{p-2} - M^\alpha c_3 \geq \frac{c_1}{2}\rho_\lambda(y)^{p-2}\end{aligned} \tag{3.32}$$

for a.e. $y \in D(\lambda, \delta_2, r_0)$. Furthermore by (3.16)-(3.18) and (3.29), there exist $c_4 = c_4(M) > 0$ and $\delta_3 \in (0, \delta_2)$ independent of $\lambda \geq 1$ such that for a.e. $y \in D(\lambda, \delta_3, r_0)$

$$\begin{aligned}&\left|\left(\Delta + a^\alpha\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) h_\lambda(y)\right| \\ \leq\;& c_4\left(1 + \lambda^{(1-\alpha)^+ + (\alpha-2)}\rho_\lambda(y)^{(1-\alpha)\wedge 0} + \mathbf{1}_{\{\alpha=1\}}\,\lambda^{-1}|\log(\rho_\lambda(y)/\lambda)|\right) \\ \leq\;& c_4\left(1 + \lambda^{(1-\alpha)^+ + (\alpha-2)}\rho_\lambda(y)^{(1-\alpha)\wedge 0} + e^{-1} + \mathbf{1}_{\{\alpha=1\}}\,|\log\rho_\lambda(y)|\right).\end{aligned} \tag{3.33}$$

Thus by (3.31)-(3.33) and the fact that $p < 2 \wedge (3-\alpha)$, there exists $\delta_4 \in (0, \delta_3)$ independent of $\lambda \geq 1$ such that

$$
\begin{aligned}
&\left(\Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) u_{2,\lambda}(y) \qquad (3.34)\\
\leq \quad & c_2 + c_4 \left(2 + |\log \rho_\lambda(y)| + \rho_\lambda(y)^{(1-\alpha)\wedge 0}\right) - \frac{c_1}{2} \rho_\lambda(y)^{p-2} \leq -1
\end{aligned}
$$

for a.e. $y \in D(\lambda, \delta_4, r_0)$.

On the other hand, we have from (3.17) and (3.18),

$$
\begin{aligned}
&\left(\Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) h_\lambda(y)\\
\geq \quad & -\|\Delta \phi_\lambda\|_\infty - c_5 M^\alpha (\lambda^{(1-\alpha)+(\alpha-2)} + \lambda^{-1} \log \lambda + \lambda^{-1} |\log \rho_\lambda(y)|)\\
\geq \quad & -\|\Delta \phi_\lambda\|_\infty - c_5 M^\alpha (1 + e^{-1} + |\log \rho_\lambda(y)|)
\end{aligned}
$$

for a.e. $y \in D(\lambda, \delta_4, r_0)$. Combining the inequality above with (3.32), by choosing δ_4 smaller if necessary, we have for a.e. $y \in D(\lambda, \delta_4, r_0)$,

$$
\left(\Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) u_{1,\lambda}(y) \geq -\|\Delta \phi\|_\infty - c_5 M^\alpha (2 + |\log \rho_\lambda(y)|) + \frac{c_1}{2} \rho_\lambda(y)^{p-2} \geq 0. \qquad (3.35)
$$

Step 2: Translating super-/sub-harmonic functions into super-/sub-martingale properties for $\widehat{X}^{a\lambda^{(\alpha-2)/\alpha}, \lambda}$. For notational convenience, we let

$$
\widetilde{X}^{a,\lambda} := \widehat{X}^{a\lambda^{(\alpha-2)/\alpha}, \lambda} \quad \text{and} \quad \widetilde{\tau}^{a,\lambda}_U := \widehat{\tau}^{a\lambda^{(\alpha-2)/\alpha}, \lambda}_U.
$$

We claim that the estimates (3.34) and (3.35) imply that

$$
t \mapsto u_{2,\lambda}\left(\widetilde{X}^{a,\lambda}_{t \wedge \widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\right) \text{ is a bounded supermartingale,} \qquad (3.36)
$$

$$
\mathbb{E}_x \left[\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}\right] \leq \rho_\lambda(x), \qquad (3.37)
$$

and

$$
t \mapsto u_{1,\lambda}\left(\widetilde{X}^{a,\lambda}_{t \wedge \widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\right) \text{ is a bounded submartingale.} \qquad (3.38)
$$

Observe that if v is a bounded C^2-function on $\mathbb{R}^d$ with bounded second order partial derivatives, then by Ito's formula and the Lévy system (3.23),

$$
M^v_t = v(\widetilde{X}^{a,\lambda}_t) - v(\widetilde{X}^{a,\lambda}_0) - \int_0^t \left(\Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) v(\widetilde{X}^{a,\lambda}_s) ds \qquad (3.39)
$$

is a martingale (see the proof of Proposition 4.1 in [6] for the derivation of a similar assertion). If the functions $u_{2,\lambda}$ and $u_{1,\lambda}$ were C^2 with bounded second order partial derivatives, then the claims (3.36), (3.37) and (3.38) would just follow from (3.39) and the estimates (3.34) and (3.35). However they are not C^2 since D is $C^{1,1}$ and they are truncated on the outside of $B(0, R_0) \cap \lambda D$. So we will use a mollifier. Let g be a non-negative smooth function with compact support in $\mathbb{R}^d$ whose value only depends on $|x|$ such that $g(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}^d} g(x) dx = 1$. For $k \geq 1$, define $g_k(x) = 2^{kd} g(2^k x)$. Set

$$
u^{(k)}_{i,\lambda}(z) := (g_k * u_{i,\lambda})(z) := \int_{\mathbb{R}^d} g_k(y) u_{i,\lambda}(z-y) dy, \qquad i = 1, 2.
$$

As

$$\left(\Delta + a^{\alpha}\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) u^{(k)}_{i,\lambda} = g_k * \left(\Delta + a^{\alpha}\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) u_{i,\lambda} \qquad \text{for } i = 1, 2,$$

we have by (3.34) and (3.35) that

$$\left(\Delta + a^{\alpha}\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) u^{(k)}_{1,\lambda} \geq 0 \quad \text{and} \quad \left(\Delta + a^{\alpha}\lambda^{(\alpha-2)}\widehat{\Delta}^{\alpha/2}_{d,\lambda}\right) u^{(k)}_{2,\lambda} \leq -1$$

on $D_k(\lambda, \delta_4, r_0) := \left\{y : \delta_4 - 2^{-k} > \rho_\lambda(y) > 2^{-k} \text{ and } |\widetilde{y}| < r_0 - 2^{-k}\right\}$.

Since $u^{(k)}_{i,\lambda}$, $i = 1, 2$, are bounded smooth functions on $\mathbb{R}^d$ with bounded first and second order partial derivatives, it follows from (3.39) that

$$t \mapsto u^{(k)}_{2,\lambda}\left(\widetilde{X}^{a,\lambda}_{t\wedge\widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)}}\right) + t \wedge \widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)} \text{ is a positive supermartingale}$$

and

$$t \mapsto u^{(k)}_{1,\lambda}\left(\widetilde{X}^{a,\lambda}_{t\wedge\widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)}}\right) \text{ is a bounded submartingale.}$$

Since for $i = 1, 2$, $u_{i,\lambda}$ is bounded and continuous, $u^{(k)}_{i,\lambda}$ converges uniformly to $u_{i,\lambda}$. Thus

$$t \mapsto u_{2,\lambda}\left(\widetilde{X}^{a,\lambda}_{t\wedge\widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)}}\right) + t \wedge \widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)} \text{ is a positive supermartingale} \tag{3.40}$$

and

$$t \mapsto u_{1,\lambda}\left(\widetilde{X}^{a,\lambda}_{t\wedge\widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)}}\right) \text{ is a bounded submartingale.}$$

Since $D_k(\lambda, \delta_4, r_0)$ increases to $D(\lambda, \delta_4, r_0)$, we conclude that (3.36) and (3.38) hold. Moreover, for each fixed $k \geq 1$ and $t > 0$, we have from (3.40) that

$$\mathbb{E}_x\left[u_{2,\lambda}\left(\widetilde{X}^{a,\lambda}_{t\wedge\widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)}}\right) + t \wedge \widetilde{\tau}^{a,\lambda}_{D_k(\lambda,\delta_4,r_0)}\right] \leq u_{2,\lambda}(x).$$

Since $u_{2,\lambda} \geq 0$, by first letting $k \to \infty$ and then $t \to \infty$, we get $\mathbb{E}_x\left[\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}\right] \leq u_{2,\lambda}(x)$. Since $\widetilde{x} = 0$, $\psi(x) = 0$ and so $u_{2,\lambda}(x) \leq \rho_\lambda(x)$. This proves (3.37).

Step 3: Deriving the desired exit distribution estimates by utilizing the super-/sub-martingale property. Since $\psi \geq 2^{p+1}$ on $|\widetilde{y}| \geq r_0$ and $\psi(x) = 0$, we have by (3.36),

$$\begin{aligned}
\rho_\lambda(x) &\geq u_{2,\lambda}(x) \\
&\geq \mathbb{E}_x\left[u_{2,\lambda}\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\right); \widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}} \in (\lambda D) \setminus D(\lambda, \infty, r_0)\right] \\
&\geq (2^{p+1} - 1)\,\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}} \in (\lambda D) \setminus D(\lambda, \infty, r_0)\right).
\end{aligned}$$

We also have from (3.38)

$$\begin{aligned}
\rho_\lambda(x) \leq \rho_\lambda(x) + \rho_\lambda(x)^p = u_{1,\lambda}(x) &\leq \mathbb{E}_x\left[u_{1,\lambda}\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\right)\right] \\
&\leq 2\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}} \in \lambda D\right).
\end{aligned}$$

Combining the two displays above, we get

$$\begin{aligned}
&\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in D(\lambda,\infty,r_0)\right) \\
=&\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in \lambda D\right)-\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in (\lambda D)\setminus D(\lambda,\infty,r_0)\right) \\
\geq&\frac{2^{p+1}-3}{2(2^{p+1}-1)}\rho_\lambda(x).
\end{aligned} \tag{3.41}$$

By (3.23),

$$\begin{aligned}
&\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in D(\lambda,\infty,r_0)\setminus D(\lambda,2\delta_4,r_0)\right) \\
=&\mathbb{E}_x\left[\int_0^{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\int_{D(\lambda,\infty,r_0)\setminus D(\lambda,2\delta_4,r_0)}\frac{(a\lambda^{(\alpha-2)/\alpha})^\alpha\mathcal{A}(d,\alpha)}{|\widetilde{X}^{a,\lambda}_s-y|^{d+\alpha}}\mathbf{1}_{\{|\widetilde{X}^{a,\lambda}_s-y|<\lambda\}}dyds\right] \\
\leq&\mathbb{E}_x\left[\int_0^{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\int_{D(\lambda,\infty,r_0)\setminus D(\lambda,2\delta_4,r_0)}\frac{a^\alpha\lambda^{\alpha-2}\mathcal{A}(d,\alpha)}{|\widetilde{X}^{a,\lambda}_s-y|^{d+\alpha}}dyds\right] \\
\leq&c_6\mathcal{A}(d,\alpha)a^\alpha\lambda^{\alpha-2}\left(\int_{D(\lambda,\infty,r_0)\setminus D(\lambda,2\delta_4,r_0)}|y|^{-d-\alpha}dy\right)\mathbb{E}_x\left[\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}\right] \\
\leq&c_7\mathcal{A}(d,\alpha)a^\alpha\lambda^{\alpha-2}\left(\int_{D(\lambda,2\delta_4,r_0)\setminus D(\lambda,3\delta_4/2,r_0)}|y|^{-d-\alpha}dy\right)\mathbb{E}_x\left[\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}\right] \\
\leq&c_8\,\mathbb{E}_x\left[\int_0^{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\int_{D(\lambda,2\delta_4,r_0)\setminus D(\lambda,3\delta_4/2,r_0)}\frac{a^\alpha\lambda^{\alpha-2}\mathcal{A}(d,\alpha)}{|\widetilde{X}^{a,\lambda}_s-y|^{d+\alpha}}\mathbf{1}_{\{|\widetilde{X}^{a,\lambda}_s-y|<\lambda\}}dyds\right] \\
=&c_8\,\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in D(\lambda,2\delta_4,r_0)\setminus D(\lambda,3\delta_4/2,r_0)\right).
\end{aligned} \tag{3.42}$$

Thus from (3.41)-(3.42)

$$\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in D(\lambda,2\delta_4,r_0)\right)\geq c_9\rho_\lambda(x). \tag{3.43}$$

Recall that $0\leq h_{\lambda,p}\leq 1$. If $|y|>r_0/2$, then $\psi(y)\geq 2^{p+1}$, we have

$$u_{2,\lambda}(y)=\psi(y)+h_\lambda(y)-h_{\lambda,p}(y)\geq\psi(y)-h_{\lambda,p}(y)\geq 2^p\geq 1\qquad\text{for } y\in B(0,r_0/2)^c\,.$$

Furthermore, for $y\in B(0,R_0)$ such that $\delta_4\leq\rho_\lambda(y)<R_0$,

$$u_{2,\lambda}(y)=\psi(y)+h_\lambda(y)-h_{\lambda,p}(y)\geq\rho_\lambda(y)-\rho_\lambda(y)^p\geq c_{10}\,,$$

where $c_{10}\in(0,1)$ depends on δ_4 and R. By using the last two observations, it holds that $u_{2,\lambda}\geq c_{10}>0$ on $(\lambda D)\setminus D(\lambda,\delta_4,r_0)$. Therefore, by (3.36) we get

$$\rho_\lambda(x)\geq u_{2,\lambda}(x)\geq\mathbb{E}_x\left[u_{2,\lambda}\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\right)\right]\geq c_{10}\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in\lambda D\right). \tag{3.44}$$

Since the process $\{\lambda(\widehat{X}^a_{\lambda^{-2}t}-\widehat{X}^a_0), t\geq 0\}$ under $\mathbb{P}_x$ has the same distribution as $\{\widehat{X}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_t - \widehat{X}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_0, t\geq 0\}$ under $\mathbb{P}_{\lambda x}$, we have from (3.43) that for $x\in D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)$

$$
\begin{aligned}
&\mathbb{P}_x\left(\widehat{X}^a_{\tau^a_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}}\in D\right)\\
\geq\ &\mathbb{P}_x\left(\widehat{X}^a_{\tau^a_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}}\in D_Q(2\lambda^{-1}\delta_4,\lambda^{-1}r_0)\setminus D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)\right)\\
=\ &\mathbb{P}_{\lambda x}\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in D(\lambda,2\delta_4,r_0)\setminus D(\lambda,\delta_4,r_0)\right)\\
\geq\ &c_9\rho_\lambda(\lambda x)\ \geq\ c_{11}\delta_{\lambda D}(\lambda x)\ =\ c_{11}\lambda\delta_D(x)\,,
\end{aligned}
$$

and, from (3.44)

$$
\begin{aligned}
\mathbb{P}_x\left(\widehat{X}^a_{\tau^a_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}}\in D\right) &= \mathbb{P}_{\lambda x}\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}}\in \lambda D\right)\\
&\leq c_{12}\rho_\lambda(\lambda x)\ \leq\ c_{13}\delta_{\lambda D}(\lambda x)\ =\ c_{13}\lambda\delta_D(x).
\end{aligned}
$$

Finally by (3.25) and (3.37),

$$
\begin{aligned}
\mathbb{E}_x\left[\widehat{\tau}^a_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}\right] &= \int_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}\widehat{G}^{a,1}_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}(x,y)dy\\
&= \lambda^{d-2}\int_{D_Q(\lambda^{-1}\delta_4,\lambda^{-1}r_0)}\widehat{G}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_{D(\lambda,\delta_4,r_0)}(\lambda x,\lambda y)dy\\
&= \lambda^{-2}\int_{D(\lambda,\delta_4,r_0)}\widehat{G}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_{D(\lambda,\delta_4,r_0)}(\lambda x,z)dz\\
&= \lambda^{-2}\mathbb{E}_{\lambda x}\left[\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_4,r_0)}\right]\\
&\leq \lambda^{-2}\rho_\lambda(\lambda x)\leq c_{14}\lambda^{-2}\delta_{\lambda D}(\lambda x)\ =\ c_{14}\lambda^{-1}\delta_D(x).
\end{aligned}
$$

This completes the proof by taking $\delta_0=\delta_4$, $C_8=c_{11}$, and $C_9=\max\{c_{13},c_{14}\}$. □

We now derive exit distribution estimates for the process X^a from those for $\widehat{X}^a$ in Lemma 3.4. Recall that $r_0=R_0/(4(1+\Lambda^2))$.

Lemma 3.5 *There are positive constants $\delta_0=\delta_0(R,M,\Lambda,\alpha)\in(0,r_0)$, $C_8=C_8(R,M,\Lambda,\alpha)$ and $C_{10}=C_{10}(R,M,\Lambda,\alpha)$ such that for every $a\in(0,M]$, $\lambda\geq 1$, $Q\in\partial D$ and $x\in D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)$ with $\widetilde{x}=0$,*

$$\mathbb{P}_x\left(X^a_{\tau^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}}\in D_Q(2\lambda^{-1}\delta_0,\lambda^{-1}r_0)\right)\geq C_8\lambda\delta_D(x), \tag{3.45}$$

$$\mathbb{P}_x\left(X^a_{\tau^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}}\in D\right)\leq C_{10}\lambda\delta_D(x) \tag{3.46}$$

and

$$\mathbb{E}_x\left[\tau^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}\right]\ \leq\ C_{10}\lambda^{-1}\delta_D(x). \tag{3.47}$$

Proof. Without loss of generality, we assume $Q = 0$ and let $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be the $C^{1,1}$-function satisfying $\phi(\widetilde{0}) = \nabla\phi(\widetilde{0}) = 0$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(\widetilde{y}) - \nabla\phi(\widetilde{z})| \le \Lambda|\widetilde{y} - \widetilde{z}|$ and CS_Q be the corresponding coordinate system such that

$$B(Q,R) \cap D = \{(\widetilde{y},\, y_d) \in B(0,R) \text{ in } CS_Q : \; y_d > \phi(\widetilde{y})\}.$$

Let δ_0, C_8 and C_9 be the constants from the statement of Lemma 3.4. Since $\mathrm{diam}(D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)) \le \frac{1}{2}$, we have that

$$|x-y|^{-d-\alpha}\, \mathbf{1}_{\{|x-y|<1\}} = |x-y|^{-d-\alpha} \quad \text{for all } x, y \in D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0).$$

Let

$$j(x) := a^\alpha \mathcal{A}(d,\alpha)|x|^{-(d+\alpha)}\mathbf{1}_{\{|x|\ge 1\}}.$$

Note that $\int_{\mathbb{R}^d} j(x)dx < \infty$. Thus we can write $X^a_t = \widehat{X}^a_t + Z^a_t$ where Z^a_t is a compound Poisson process with the Lévy density $j(x)$, independent of $\widehat{X}^a_t$. Since the jump size of Z^a is greater than or equal to 1 and $\mathrm{diam}(D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)) \le \frac{1}{2}$, we see from (3.28) that

$$\mathbb{E}_x\left[\tau^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}\right] \le \mathbb{E}_x\left[\widehat{\tau}^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}\right] \le C_9\lambda^{-1}\delta_D(x)\,.$$

Moreover we have from (3.26) that

$$\begin{aligned} &\mathbb{P}_x\left(X^a_{\tau^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}} \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0)\right)\\ =&\mathbb{P}_x\left(\widehat{X}^a_{\widehat{\tau}^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}} \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0)\right) \ge C_8\lambda\delta_D(x). \end{aligned}$$

We recall the notations from the proof of the previous lemma:

$$\begin{aligned} \rho_\lambda(x) &:= y_d - \phi(\lambda^{-1}\widetilde{y}),\\ D(\lambda, r_1, r_2) &:= \{y \in CS_Q : r_1 > \rho_\lambda(y) > 0,\, |\widetilde{y}| < r_2\},\\ \widetilde{X}^{a,\lambda} = \widehat{X}^{a\lambda^{(\alpha-2)/\alpha},\lambda} &\qquad \text{and} \qquad \widetilde{\tau}^{a,\lambda}_U := \widehat{\tau}^{a\lambda^{(\alpha-2)/\alpha},\lambda}_U. \end{aligned}$$

Let $a(\lambda) := a\lambda^{(\alpha-2)/\alpha}$, which is no larger than M. By (1.4),

$$\begin{aligned} &\mathbb{P}_x\left(X^{a(\lambda)}_{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}} \in (\lambda D) \setminus D(\lambda, 2\delta_0, 2r_0)\right) \\ &= \mathbb{E}_x\left[\int_0^{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}} \int_{(\lambda D)\setminus D(\lambda,2\delta_0,2r_0)} \frac{a(\lambda)^\alpha \mathcal{A}(d,\alpha)}{|X^{a(\lambda)}_s - y|^{d+\alpha}}dyds\right] \\ &\le c_1\mathcal{A}(d,\alpha)(a(\lambda))^\alpha \left(\int_{(\lambda D)\setminus D(\lambda,2\delta_0,2r_0)} |y|^{-d-\alpha}dy\right)\mathbb{E}_x\left[\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}\right] \\ &\le c_2\mathcal{A}(d,\alpha)(a(\lambda))^\alpha \left(\int_{D(\lambda,2\delta_0,r_0)\setminus D(\lambda,3\delta_0/2,r_0)} |y|^{-d-\alpha}dy\right)\mathbb{E}_x\left[\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}\right] \\ &\le c_3\mathbb{E}_x\left[\int_0^{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}} \int_{D(\lambda,2\delta_0,r_0)\setminus D(\lambda,3\delta_0/2,r_0)} \frac{a(\lambda)^\alpha \mathcal{A}(d,\alpha)}{|X^{a(\lambda)}_s - y|^{d+\alpha}}dyds\right] \\ &= c_3\mathbb{P}_x\left(X^{a(\lambda)}_{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}} \in D(\lambda, 2\delta_0, r_0) \setminus D(\lambda, 3\delta_0/2, r_0)\right). \end{aligned} \tag{3.48}$$

Thus by the above inequality and (3.44), we have

$$\begin{aligned}
&\mathbb{P}_x\left(X^{a(\lambda)}_{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}}\in\lambda D\right) \qquad (3.49)\\
&=\mathbb{P}_x\left(X^{a(\lambda)}_{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}}\in(\lambda D)\setminus D(\lambda,2\delta_0,2r_0)\right)+\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_0,r_0)}}\in D(\lambda,2\delta_0,2r_0)\right)\\
&\le c_3\mathbb{P}_x\left(X^{a(\lambda)}_{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}}\in D(\lambda,2\delta_0,r_0)\setminus D(\lambda,3\delta_0/2,r_0)\right)+\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_0,r_0)}}\in\lambda D\right)\\
&=c_3\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_0,r_0)}}\in D(\lambda,2\delta_0,r_0)\setminus D(\lambda,3\delta_0/2,r_0)\right)+\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_0,r_0)}}\in\lambda D\right)\\
&\le(c_3+1)\mathbb{P}_x\left(\widetilde{X}^{a,\lambda}_{\widetilde{\tau}^{a,\lambda}_{D(\lambda,\delta_0,r_0)}}\in\lambda D\right)\le c_4\rho_\lambda(x).
\end{aligned}$$

Since $(\lambda X^a_{\lambda^{-2}t}, t\ge 0)$ is the independent sum of a Brownian motion and a symmetric α-stable process on $\mathbb{R}^d$ with weight $a(\lambda)$, we have from (3.49) that for $x\in D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)$

$$\begin{aligned}
\mathbb{P}_x\left(X^a_{\tau^a_{D_Q(\lambda^{-1}\delta_0,\lambda^{-1}r_0)}}\in D\right)&=\mathbb{P}_{\lambda x}\left(X^{a(\lambda)}_{\tau^{a(\lambda)}_{D(\lambda,\delta_0,r_0)}}\in\lambda D\right)\\
&\le c_4\rho_\lambda(\lambda x)\;\le\; c_5\delta_{\lambda D}(\lambda x)\;=\;c_5\lambda\delta_D(x).
\end{aligned}$$

The proof is finished by taking $C_{10}=\max\{C_9,c_5\}$. □

4 Boundary Harnack principle

In this section, we give the proof of the BHP for the independent sum of a Brownian motion and a symmetric stable process. We first prove the Carleson estimate for the independent sum of a Brownian motion and a symmetric stable process on Lipschitz open sets.

We recall that an open set D in $\mathbb{R}^d$ (when $d\ge 2$) is said to be a Lipschitz open set if there exist a localization radius $R_1>0$ and a constant $\Lambda_1>0$ such that for every $Q\in\partial D$, there exist a Lipschitz function $\phi=\phi_Q:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\phi(0)=0$, $|\phi(x)-\phi(y)|\le\Lambda_1|x-y|$, and an orthonormal coordinate system CS_Q: $y=(y_1,\ldots,y_{d-1},y_d)=:(\widetilde{y},\,y_d)$ with its origin at Q such that

$$B(Q,R_1)\cap D=\{y=(\widetilde{y},y_d)\in B(0,R_1)\text{ in }CS_Q: y_d>\phi(\widetilde{y})\}.$$

The pair (R_1,Λ_1) is called the characteristics of the Lipschitz open set D. Note that a Lipschitz open set can be unbounded and disconnected. For Lipschitz open set D and every $Q\in\partial D$ and $x\in B(Q,R_1)\cap D$, we define

$$\rho_Q(x):=x_d-\phi_Q(\tilde{x})\,,$$

where $(\tilde{x},x_d)$ is the coordinates of x in CS_Q.

We recall that $X^a_t=X^0_t+aY_t$ is a Lévy process with characteristic exponent $\Phi^a(x)=|x|^2+a^\alpha|x|^\alpha$. This process may be obtained by subordinating a d-dimensional Brownian motion $W=(W_t,\,t\ge 0)$ by an independent subordinator $T^a_t:=t+a^2T_t$ where $T=(T_t,\,t\ge 0)$ is an $\alpha/2$-stable subordinator. More precisely, the processes X^a_t and $W_{T^a_t}$ have the same distribution. Note that the Laplace exponent corresponding to T^a is equal to $\phi^a(\lambda) = \lambda + a^\alpha \lambda^{\alpha/2}$. Let $\mathcal{M}_{\alpha/2}(t) := \sum_{n=0}^\infty (-1)^n t^{n\alpha/2}/\Gamma(1+n\alpha/2)$. It follows by a straightforward integration that

$$\int_0^\infty e^{-\lambda t} \mathcal{M}_{1-\alpha/2}(a^{2\alpha/(2-\alpha)}t)\, dt = \frac{1}{\phi^a(\lambda)}\,,$$

which shows that the potential density u^a of the subordinator T^a is given by

$$u^a(t) = \mathcal{M}_{1-\alpha/2}(a^{2\alpha/(2-\alpha)}t)\,.$$

Since, for any $a > 0$, ϕ^a is a complete Bernstein function, we know that $u^a(\cdot)$ is a completely monotone function. In particular, $u^a(\cdot)$ is a decreasing function. Since $u^a(t) = u^1(a^{2\alpha/(2-\alpha)}t)$, we know that $a \mapsto u^a(t)$ is a decreasing function. Therefore, if $0 < a_1 < a_2$, then $u^{a_1}(t) \ge u^{a_2}(t)$ for all $t > 0$. We will need this fact in the proof of next lemma.

Lemma 4.1 *Let $D \subset \mathbb{R}^d$ be a Lipschitz open set with the characteristics (R_1, Λ_1). There exists a constant $\delta = \delta(R_1, \Lambda_1, M) > 0$ such that for all $a \in [0, M]$ and $Q \in \partial D$, $x \in D$ with $\rho_Q(x) < R_1/2$,*

$$\mathbb{P}_x(X^a_{\tau(x)} \in D^c) \ge \delta\,,$$

where $\tau(x) := \tau^a_{D\cap B(x,2\rho_Q(x))} = \inf\{t > 0 :\ X^a_t \notin D \cap B(x, 2\rho_Q(x))\}$.

Proof. Clearly,

$$\begin{aligned}
\mathbb{P}_x\left(X^a_{\tau(x)} \in D^c\right) &\ge \mathbb{P}_x\left(X^a_{\tau(x)} \in D^c \cap B(x, 2\rho_Q(x))\right)\\
&\ge \mathbb{P}_x\left(X^a_{\tau(x)} \in \partial D \cap B(x, 2\rho_Q(x))\right).
\end{aligned}$$

Let $D_x := D \cap B(x, 2\rho_Q(x))$ and W^{D_x} be the subprocess of Brownian motion W killed upon leaving D_x. The process Z^a defined by $Z^a_t := W^{D_x}(T^a_t)$, where T^a_t is an independent subordinator described in the paragraph before the statement of the lemma, is called a subordinate killed Brownian motion in D_x. We will use ζ to denote the lifetime of Z^a. It is known from [40] that

$$\begin{aligned}
\mathbb{P}_x\left(X^a_{\tau(x)} \in \partial D \cap B(x, 2\rho_Q(x))\right) &\ge \mathbb{P}_x\left(Z^a_{\zeta-} \in \partial D \cap B(x, 2\rho_Q(x))\right)\\
&= \mathbb{E}_x\left[u^a(\widetilde{\tau}_{D_x});\, W_{\widetilde{\tau}_{D_x}} \in \partial D \cap B(x, 2\rho_Q(x))\right]\,.
\end{aligned}$$

Here and below, $\widetilde{\tau}_U := \inf\{t > 0 :\ W_t \notin U\}$ is the exit time of W from U. Denote $C_x := \partial D \cap B(x, 2\rho_Q(x))$. Then

$$\begin{aligned}
\mathbb{E}_x\left[u^a(\widetilde{\tau}_{D_x});\, W_{\widetilde{\tau}_{D_x}} \in C_x\right] &\ge \mathbb{E}_x\left[u^a(\widetilde{\tau}_{D_x});\, W_{\widetilde{\tau}_{D_x}} \in C_x, \widetilde{\tau}_{D_x} \le t\right]\\
&\ge u^a(t)\mathbb{P}_x\left[W_{\widetilde{\tau}_{D_x}} \in C_x, \widetilde{\tau}_{D_x} \le t\right]\\
&\ge u^a(t)\left(\mathbb{P}_x(W_{\widetilde{\tau}_{D_x}} \in C_x) - \mathbb{P}_x(\widetilde{\tau}_{D_x} > t)\right)\,,
\end{aligned} \tag{4.1}$$

where $t > 0$ will be chosen later.

Since D is a Lipschitz open set with characteristics (R_1, Λ_1), there exist $\eta = \eta(\Lambda, R_1) > 0$ and a cone

$$\mathcal{C} := \left\{y = (y_1, \dots, y_d) \in \mathbb{R}^d : y_d < 0,\, (y_1^2 + \cdots + y_{d-1}^2)^{1/2} < \eta|y_d|\right\} \tag{4.2}$$

such that for every $z \in \partial D$, there is a cone $\mathcal{C}_z$ with vertex z, isometric to $\mathcal{C}$, satisfying $\mathcal{C}_z \cap B(Q, R_1) \subset D^c$. Then by the scaling property of W and symmetry considerations, we have

$$\begin{aligned}\mathbb{P}_x(W_{\widetilde{\tau}_{D_x}} \in C_x) &\geq \mathbb{P}_x\big(W_{\widetilde{\tau}_{B(x,2\rho_Q(x))}} \in \partial B(x, 2\rho_Q(x)) \cap \mathcal{C}_{(\widetilde{x},\phi_Q(\widetilde{x}))}\big) \\ &\geq \mathbb{P}_0\big(W_{\widetilde{\tau}_{B(0,2)}} \in \partial B(0,2) \cap (\mathcal{C} + (\widetilde{0}, -1))\big),\end{aligned}$$

which is strictly positive. Hence we can conclude that there exists $c_1 = c_1(D) > 0$ such that

$$\mathbb{P}_x(W_{\widetilde{\tau}_{D_x}} \in C_x) \geq c_1 \,. \tag{4.3}$$

Next,

$$\mathbb{P}_x(\widetilde{\tau}_{D_x} > t) \leq \frac{\mathbb{E}_x[\widetilde{\tau}_{D_x}]}{t} \leq \frac{\mathbb{E}_x[\widetilde{\tau}_{B(x,2\rho_Q(x))}]}{t} \leq c_2 \frac{(\rho_Q(x))^2}{t} \leq c_2 \frac{R_1^2}{t} \,, \tag{4.4}$$

for some constant $c_2 > 0$. By using (4.3) and (4.4) in (4.1), we obtain that

$$\mathbb{E}_x\left[u^a(\widetilde{\tau}_{D_x});\, W_{\widetilde{\tau}_{D_x}} \in C_x\right] \geq u^a(t)\left(c_1 - c_2\frac{R_1^2}{t}\right).$$

Now choose $t = t(R_1, \Lambda_1) > 0$ large enough so that $c_1 - c_2 R_1^2/t \geq c_1/2$. Then

$$\mathbb{E}_x\left[u(\widetilde{\tau}_{D_x});\, W_{\widetilde{\tau}_{D_x}} \in C_x\right] \geq c_1 u^a(t)/2 \geq c_1 u^M(t)/2 =: \delta.$$

The lemma is thus proved. □

Suppose that D is an open set and that U and V are bounded open sets with $V \subset \overline{V} \subset U$ and $D \cap V \neq \emptyset$. If u vanishes continuously on $D^c \cap U$, then by a finite covering argument, it is easy to see that u is bounded in an open neighborhood of $\partial D \cap V$.

Lemma 4.2 *Let D be an open set and U and V be bounded open sets with $V \subset \overline{V} \subset U$ and $D \cap V \neq \emptyset$. Suppose u is a nonnegative function in $\mathbb{R}^d$ that is harmonic in $D \cap U$ with respect to X^a and vanishes continuously on $D^c \cap U$. Then u is regular harmonic in $D \cap V$ with respect to X^a, i.e.,*

$$u(x) = \mathbb{E}_x\left[u(X^a_{\tau^a_{D\cap V}})\right] \qquad \text{for all } x \in D \cap V\,. \tag{4.5}$$

Proof. For $n \geq 1$, let $B_n = \{y \in D \cap V : \delta_D(y) > 1/n\}$. Then for large n, B_n is an non-empty open subset of $D \cap V$ whose closure is contained in $D \cap U$. Since u is harmonic in $D \cap U$ with respect to X^a, for $x \in D \cap V$ and n large enough so that $x \in B_n$, we have that

$$u(x) = \mathbb{E}_x\left[u\big(X^a_{\tau^a_{B_n}}\big)\right] = \mathbb{E}_x\left[u\big(X^a_{\tau^a_{B_n}}\big);\, \tau^a_{B_n} < \tau^a_{D\cap V}\right] + \mathbb{E}_x\left[u\big(X^a_{\tau^a_{B_n}}\big);\, \tau^a_{B_n} = \tau^a_{D\cap V}\right].$$

Hence

$$\begin{aligned}&\left|u(x) - \mathbb{E}_x\left[u\big(X^a_{\tau^a_{D\cap V}}\big)\right]\right| \\ &\leq \mathbb{E}_x\left[u\big(X^a_{\tau_{B_n}}\big);\, \tau^a_{B_n} < \tau^a_{D\cap V}\right] + \mathbb{E}_x\left[u\big(X^a_{\tau^a_{D\cap V}}\big);\, \tau^a_{B_n} < \tau^a_{D\cap V}\right].\end{aligned} \tag{4.6}$$

Since $\lim_{n\to\infty} \tau^a_{B_n} = \tau^a_{D\cap V}$ almost surely under each $\mathbb{P}_x$, the second term in (4.6) converges to $\mathbb{E}_x[u(X^a_{\tau^a_{D\cap V}});\, A]$ where $A := \cap_{n=1}^\infty \{\tau^a_{B_n} < \tau^a_{D\cap V}\}$. Note that

$$X^a_{\tau^a_{D\cap V}} \in \partial D \cap V \qquad \text{on } A.$$

Hence $u\big(X^a_{\tau^a_{D\cap V}}\big) = 0$ on A, as u is assumed to vanish on $D^c \cap U$. Consequently

$$\lim_{n\to\infty} \mathbb{E}_x \left[u\big(X^a_{\tau^a_{D\cap V}}\big);\, \tau^a_{B_n} < \tau^a_{D\cap V} \right] = 0\,.$$

For the first term in (4.6), note that $\delta_D(X^a_{\tau_{B_n}}) \le 1/n$ on $\{\tau^a_{B_n} < \tau^a_{D\cap V}\}$. Therefore, by the assumption that u vanishes continuously on $D^c \cap U$, $\lim_{n\to\infty} u(X^a_{\tau_{B_n}}) = 0$. Moreover, since u vanishes continuously on $(\partial D) \cap U$, there is $n_0 \ge 1$ so that u is bounded in $D \cap V \setminus B_{n_0}$. So by the bounded convergence theorem we have

$$\lim_{n\to\infty} \mathbb{E}_x \left[u\big(X^a_{\tau_{B_n}}\big);\, \tau^a_{B_n} < \tau^a_{D\cap V} \right] = 0\,.$$

This proves the lemma. □

Proof of Proposition 1.3. We know from the parabolic Harnack inequality from Theorem 6.7 of [21] that the Harnack inequality holds for the process $X := X^1$. That is, there exists a constant $c_1 = c_1(\alpha, M) > 0$ such that for any $r \in (0, M^{\alpha/(2-\alpha)}]$, $x_0 \in \mathbb{R}^d$ and any function $v \ge 0$ harmonic in $B(x_0, r)$ with respect to X, we have

$$v(x) \le c_1 v(y) \qquad \text{for all } x, y \in B(x_0, \frac{r}{2}). \tag{4.7}$$

Now the proposition is an easy consequence of (4.7). In fact, note that for any $a \in (0, M]$, X^a has the same distribution as $\lambda X_{\lambda^{-2}t}$, where $\lambda = a^{\alpha/(\alpha-2)} \ge M^{\alpha/(\alpha-2)}$. Consequently, if u is harmonic in $B(x_0, r)$ with respect to X^a where $r \in (0, 1]$, then $v(x) := u(\lambda x)$ is harmonic in $B(\lambda^{-1}x_0, \lambda^{-1}r)$ with respect to X and $\lambda^{-1}r \le M^{\alpha/(2-\alpha)}$. So by (4.7)

$$u(\lambda x) = v(x) \le c_1 v(y) = c_1 u(\lambda y) \qquad \text{for all } x, y \in B(\lambda^{-1}x_0, \lambda^{-1}r/2).$$

That is,

$$u(x) \le c_1 u(y) \qquad \text{for all } x, y \in B(x_0, r/2).$$

□

Theorem 4.3 (Carleson estimate) *Let $D \subset \mathbb{R}^d$ be a Lipschitz open set with the characteristics (R_1, Λ_1). Then there exists a positive constant $A = A(\alpha, \Lambda_1, R_1, M)$ such that for $a \in (0, M]$, $Q \in \partial D$, $0 < r < R_1/2$, and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$u(x) \le A u(x_0) \qquad \text{for } x \in D \cap B(Q, r/2), \tag{4.8}$$

where $x_0 \in D \cap B(Q, r)$ with $\rho_Q(x_0) = r/2$.

Proof. Fix $a \in (0, M]$. Since D is Lipschitz and $r < R_1/2$, by the uniform Harnack principle in Proposition 1.3 and a standard chain argument, it suffices to prove (4.8) for $x \in D \cap B(Q, r/12)$ and $\widetilde{x}_0 = \widetilde{Q}$. Without loss of generality, we may assume that $u(x_0) = 1$. In this proof, the constants δ, β, η and c_i's are always independent of r and a.

Choose $0 < \gamma < \alpha/(d+\alpha)$ and let

$$B_0 = D \cap B(x, 2\rho_Q(x)), \qquad B_1 = B(x, r^{1-\gamma}\rho_Q(x)^\gamma).$$

Further, set

$$B_2 = B(x_0, \rho_Q(x_0)/3), \qquad B_3 = B(x_0, 2\rho_Q(x_0)/3)$$

and

$$\tau_0 = \inf\{t > 0 : X^a_t \notin B_0\}, \qquad \tau_2 = \inf\{t > 0 : X^a_t \notin B_2\}.$$

By Lemma 4.1, there exists $\delta = \delta(R_1, \Lambda_1, M) > 0$ such that

$$\mathbb{P}_x(X^a_{\tau_0} \in D^c) \geq \delta, \quad x \in B(Q, r/4). \tag{4.9}$$

By the uniform Harnack principle in Proposition 1.3 and a chain argument, there exists β such that

$$u(x) < (\rho_Q(x)/r)^{-\beta} u(x_0), \quad x \in D \cap B(Q, r/4). \tag{4.10}$$

In view of Lemma 4.2, u is regular harmonic in B_0 with respect to X^a. So

$$u(x) = \mathbb{E}_x\big[u\big(X^a_{\tau_0}\big); X^a_{\tau_0} \in B_1\big] + \mathbb{E}_x\big[u\big(X^a_{\tau_0}\big); X^a_{\tau_0} \notin B_1\big] \qquad \text{for } x \in B(Q, r/4). \tag{4.11}$$

We first show that there exists $\eta > 0$ such that

$$\mathbb{E}_x\big[u\big(X^a_{\tau_0}\big); X^a_{\tau_0} \notin B_1\big] \leq u(x_0) \quad \text{if } x \in D \cap B(Q, r/12) \text{ with } \rho_Q(x) < \eta r. \tag{4.12}$$

Let $\eta_0 := 2^{-2(d+\alpha)/d}$, then for $\rho_Q(x) < \eta_0 r$,

$$(\rho_Q(x))^{d/(\alpha+d)} < 1/4 \quad \text{and} \quad 2\rho_Q(x) \leq r^{1-\gamma}\rho_Q(x)^\gamma - 2\rho_Q(x).$$

Thus if $x \in D \cap B(Q, r/12)$ with $\rho_Q(x) < \eta_0 r$, then $|x - y| \leq 2|z - y|$ for $z \in B_0$, $y \notin B_1$. Thus we have by (1.4) and Lemma 3.1

$$\begin{aligned}
&\mathbb{E}_x\big[u\big(X^a_{\tau_0}\big); X^a_{\tau_0} \notin B_1\big] \\
=&\mathcal{A}(d,\alpha) \int_{B_0} G^a_{B_0}(x,z) \int_{|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma} a^\alpha |z-y|^{-d-\alpha} u(y)\, dy\, dz \\
\leq& 2^{d+\alpha}\mathcal{A}(d,\alpha) \int_{B_0} G^a_{B_0}(x,z) dz \int_{|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma} a^\alpha |x-y|^{-d-\alpha} u(y)\, dy \\
\leq& 2^{d+\alpha}\mathcal{A}(d,\alpha) \mathbb{E}_x[\tau_{B(x,2\rho_Q(x))}] \int_{|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma} a^\alpha |x-y|^{-d-\alpha} u(y)\, dy \\
\leq& 2^{d+\alpha}\mathcal{A}(d,\alpha) c_1 \rho_Q(x)^2 \Bigg(\int_{|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma, |y-x_0|>2\rho_Q(x_0)/3} a^\alpha |x-y|^{-d-\alpha} u(y)\, dy \\
&\qquad + \int_{|y-x_0|\leq 2\rho_Q(x_0)/3} a^\alpha |x-y|^{-d-\alpha} u(y)\, dy \Bigg) \\
=:& c_2 \rho_Q(x)^2 (I_1 + I_2).
\end{aligned} \tag{4.13}$$

On the other hand, for $z \in B_2$ and $y \notin B_3$, we have $|z-y| \le |z-x_0|+|x_0-y| \le \rho_Q(x_0)/3+|x_0-y| \le 2|x_0-y|$. we have again by (1.4) and Lemma 3.1

$$\begin{aligned} u(x_0) &\ge \mathbb{E}_{x_0}\left[u(X^a_{\tau_2}), X^a_{\tau_2} \notin B_3\right] \qquad (4.14)\\ &\ge \mathcal{A}(d,\alpha)\int_{B_2} G^a_{B_2}(x_0,z)\int_{|y-x_0|>2\rho_Q(x_0)/3} a^\alpha |z-y|^{-d-\alpha}u(y)\,dy\,dz \\ &\ge 2^{-d-\alpha}\mathcal{A}(d,\alpha)\int_{B_2} G^a_{B_2}(x_0,z)dz\int_{|y-x_0|>2\rho_Q(x_0)/3} a^\alpha |x_0-y|^{-d-\alpha}u(y)\,dy \\ &\ge 2^{-d-\alpha}\mathcal{A}(d,\alpha)c_3(\rho_Q(x_0)/3)^2\int_{|y-x_0|>2\rho_Q(x_0)/3} a^\alpha |x_0-y|^{-d-\alpha}u(y)\,dy \\ &= c_4\rho_Q(x_0)^2\int_{|y-x_0|>2\rho_Q(x_0)/3} a^\alpha |x_0-y|^{-d-\alpha}u(y)\,dy\,. \end{aligned}$$

Suppose now that $|y-x| \ge r^{1-\gamma}\rho_Q(x)^\gamma$ and $x \in B(Q,r/4)$. Then

$$|y-x_0| \le |y-x|+r \le |y-x|+r^\gamma\rho_Q(x)^{-\gamma}|y-x| \le 2r^\gamma\rho_Q(x)^{-\gamma}|y-x|.$$

Therefore

$$\begin{aligned} I_1 &= \int_{|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma,|y-x_0|>2\rho_Q(x_0)/3} a^\alpha|x-y|^{-d-\alpha}u(y)\,dy \qquad (4.15)\\ &\le \int_{|y-x_0|>2\rho_Q(x_0)/3} (2^{-1}(\rho_Q(x)/r)^\gamma)^{-d-\alpha}a^\alpha|y-x_0|^{-d-\alpha}u(y)\,dy \\ &= 2^{d+\alpha}(\rho_Q(x)/r)^{-\gamma(d+\alpha)}\int_{|y-x_0|>2\rho_Q(x_0)/3} a^\alpha|y-x_0|^{-d-\alpha}u(y)\,dy \\ &\le 2^{d+\alpha}(\rho_Q(x)/r)^{-\gamma(d+\alpha)}c_4^{-1}\rho_Q(x_0)^{-2}u(x_0) \\ &= c_5(\rho_Q(x)/r)^{-\gamma(d+\alpha)}\rho_Q(x_0)^{-2}u(x_0)\,, \end{aligned}$$

where the last inequality is due to (4.14).

If $|y-x_0| < 2\rho_Q(x_0)/3$, then $|y-x| \ge |x_0-Q|-|x-Q|-|y-x_0| > \rho_Q(x_0)/6$. This together with the uniform Harnack principle in Proposition 1.3 implies that

$$\begin{aligned} I_2 &= \int_{|y-x_0|\le 2\rho_Q(x_0)/3} a^\alpha|x-y|^{-d-\alpha}u(y)\,dy \qquad (4.16)\\ &\le c_6\int_{|y-x_0|\le 2\rho_Q(x_0)/3} a^\alpha|x-y|^{-d-\alpha}u(x_0)\,dy \\ &\le c_6u(x_0)\int_{|y-x|>\rho_Q(x_0)/6} a^\alpha|x-y|^{-d-\alpha}\,dy \;=\; c_7a^\alpha\rho_Q(x_0)^{-\alpha}u(x_0)\,. \end{aligned}$$

Combining (4.13)-(4.16) we obtain

$$\begin{aligned} &\mathbb{E}_x[u(X^a_{\tau_0});\, X^a_{\tau_0} \notin B_1] \qquad (4.17)\\ \le\;& c_2\rho_Q(x)^2\left(c_5(\rho_Q(x)/r)^{-\gamma(d+\alpha)}\rho_Q(x_0)^{-2}u(x_0)+c_7a^\alpha\rho_Q(x_0)^{-\alpha}u(x_0)\right) \\ \le\;& c_8u(x_0)\left(\rho_Q(x)^2(\rho_Q(x)/r)^{-\gamma(d+\alpha)}\rho_Q(x_0)^{-2}+a^\alpha\rho_Q(x)^2\rho_Q(x_0)^{-\alpha}\right) \\ \le\;& c_9u(x_0)\left((\rho_Q(x)/r)^{2-\gamma(d+\alpha)}+a^\alpha\rho_Q(x)^2r^{-\alpha}\right)\,, \end{aligned}$$

where in the last inequality we used the fact that $\rho_Q(x_0)=r/2$. Choose now $\eta\in(0,\eta_0)$ so that

$$c_9\left(\eta^{2-\gamma(d+\alpha)}+\eta^2M^\alpha\right)\le 1\,.$$

Then for $x\in D\cap B(Q,r/12)$ with $\rho_Q(x)<\eta r$, we have by (4.17)

$$\begin{aligned}\mathbb{E}_x\left[u(X^a_{\tau_0});\,X^a_{\tau_0}\notin B_1\right] &\le c_9\,u(x_0)\left(\eta^{2-\gamma(d+\alpha)}+\eta^2r^{2-\alpha}M^\alpha\right)\\ &\le c_9\left(\eta^{2-\gamma(d+\alpha)}+\eta^2M^\alpha\right)u(x_0)\le u(x_0)\,.\end{aligned}$$

We now prove the Carleson estimate (4.8) for $x\in D\cap B(Q,r/12)$ by a method of contradiction. Recall that $u(x_0)=1$. Suppose that there exists $x_1\in D\cap B(x,r/12)$ such that $u(x_1)\ge K>\eta^{-\beta}\vee(1+\delta^{-1})$, where K is a constant to be specified later. By (4.10) and the assumption $u(x_1)\ge K>\eta^{-\beta}$, we have $(\rho_Q(x_1)/r)^{-\beta}>u(x_1)\ge K>\eta^{-\beta}$, and hence $\rho_Q(x_1)<\eta r$. Let B_0, B_1 and τ_0 be now defined with respect to the point x_1 instead of x. Then by (4.11), (4.12) and $K>1+\delta^{-1}$,

$$K\le u(x_1)\le\mathbb{E}_{x_1}\left[u(X^a_{\tau_0});X^a_{\tau_0}\in B_1\right]+1\,,$$

and hence

$$\mathbb{E}_{x_1}\left[u(X^a_{\tau_0});X^a_{\tau_0}\in B_1\right]\ge u(x_1)-1>\frac{1}{1+\delta}\,u(x_1)\,.$$

In the last inequality of the display above we used the assumption that $u(x_1)\ge K>1+\delta^{-1}$. If $K\ge 2^{\beta/\gamma}$, then $D^c\cap B_1\subset D^c\cap B(Q,r)$. By using the assumption that $u=0$ on $D^c\cap B(Q,r)$, we get from (4.9)

$$\begin{aligned}\mathbb{E}_{x_1}[u(X^a_{\tau_0}),X^a_{\tau_0}\in B_1] &= \mathbb{E}_{x_1}[u(X^a_{\tau_0}),X^a_{\tau_0}\in B_1\cap D]\\ &\le \mathbb{P}_x(X^a_{\tau_0}\in D)\sup_{B_1}u\le(1-\delta)\sup_{B_1}u\,.\end{aligned}$$

Therefore, $\sup_{B_1}u>u(x_1)/((1+\delta)(1-\delta))$, i.e., there exists a point $x_2\in D$ such that

$$|x_1-x_2|\le r^{1-\gamma}\rho_Q(x_1)^\gamma\quad\text{ and }\quad u(x_2)>\frac{1}{1-\delta^2}\,u(x_1)\ge\frac{1}{1-\delta^2}\,K\,.$$

By induction, if $x_k\in D\cap B(Q,r/12)$ with $u(x_k)\ge K/(1-\delta^2)^{k-1}$ for $k\ge 2$, then there exists $x_{k+1}\in D$ such that

$$|x_k-x_{k+1}|\le r^{1-\gamma}\rho_Q(x_k)^\gamma\quad\text{ and }\quad u(x_{k+1})>\frac{1}{1-\delta^2}\,u(x_k)>\frac{1}{(1-\delta^2)^k}\,K\,. \tag{4.18}$$

From (4.10) and (4.18) it follows that $\rho_Q(x_k)/r\le(1-\delta^2)^{(k-1)/\beta}K^{-1/\beta}$, for every $k\ge1$. Therefore,

$$\begin{aligned}|x_k-Q| &\le |x_1-Q|+\sum_{j=1}^{k-1}|x_{j+1}-x_j|\le\frac{r}{12}+\sum_{j=1}^{\infty}r^{1-\gamma}\rho_Q(x_j)^\gamma\\ &\le \frac{r}{12}+r^{1-\gamma}\sum_{j=1}^{\infty}(1-\delta^2)^{(j-1)\gamma/\beta}K^{-\gamma/\beta}r^\gamma\\ &= \frac{r}{12}+r^{1-\gamma}r^\gamma K^{-\gamma/\beta}\sum_{j=0}^{\infty}(1-\delta^2)^{j\gamma/\beta}\\ &= \frac{r}{12}+rK^{-\gamma/\beta}\,\frac{1}{1-(1-\delta^2)^{\gamma/\beta}}.\end{aligned}$$

Choose

$$K = \eta \vee (1+\delta^{-1}) \vee 12^{\beta/\gamma}(1-(1-\delta^2)^{\gamma/\beta})^{-\beta/\gamma}.$$

Then $K^{-\gamma/\beta}(1-(1-\delta^2)^{\gamma/\beta})^{-1} \le 1/12$, and hence $x_k \in D \cap B(Q, r/6)$ for every $k \ge 1$. Since $\lim_{k\to\infty} u(x_k) = +\infty$, this contradicts the fact that u is bounded on $B(Q, r/2)$. This contradiction shows that $u(x) < K$ for every $x \in D \cap B(Q, r/12)$. This completes the proof of the theorem. □

Proof of Theorem 1.4 . Since D is a $C^{1,1}$ open set and $r < R$, by the uniform Harnack principle in Proposition 1.3 and a standard chain argument, it suffices to prove (1.6) for $x, y \in D \cap B(Q, rr_0/8)$. In this proof, the constants η and c_i's are always independent of r and a.

We recall that $r_0 = \frac{R}{4(1+\Lambda^2)}$ and $\delta_0 \in (0, r_0)$ is the constant in the statement of Lemma 3.5.

For any $r \in (0, R]$ and $x \in D \cap B(Q, rr_0/8)$, let Q_x be the point $Q_x \in \partial D$ so that $|x - Q_x| = \delta_D(x)$ and let $x_0 := Q_x + \frac{r}{8}(x - Q_x)/|x - Q_x|$. We choose a $C^{1,1}$-function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(y) - \nabla\phi(z)| \le \Lambda|y - z|$, and an orthonormal coordinate system CS with its origin at Q_x such that

$$B(Q_x, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS : y_d > \phi(\widetilde{y})\}.$$

In the coordinate system CS we have $\widetilde{x} = \widetilde{0}$ and $x_0 = (\widetilde{0}, r/8)$. For any $b_1, b_2 > 0$, we define

$$D(b_1, b_2) := \{y = (\widetilde{y}, y_d) \text{ in } CS : 0 < y_d - \phi(\widetilde{y}) < b_1 r\delta_0/8,\ |\widetilde{y}| < b_2 r r_0/8\}.$$

It is easy to see that $D(2,2) \subset D \cap B(Q, r/2)$. In fact, since $r_0 \le \frac{1}{8\Lambda}$ and $\delta_0 \le \frac{1}{8\Lambda}$, for every $z \in D(2,2)$

$$|z - Q| \le |Q - x| + |x - Q_x| + |Q_x - z| \le \frac{r}{8} + \frac{r}{8} + |z_d - \phi(\widetilde{z})| + |\phi(\widetilde{z})| < \frac{r}{4}(1+\delta_0) \le \frac{r}{2}.$$

Thus if u is a nonnegative function on $\mathbb{R}^d$ that is harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously in $D^c \cap B(Q, r)$, then, by Lemma 4.2, u is regular harmonic in $D \cap B(Q, r/2)$ with respect to X^a, hence also in $D(2,2)$. Thus by the uniform Harnack principle in Proposition 1.3, we have

$$\begin{aligned} u(x) &= \mathbb{E}_x\left[u\big(X^a_{\tau^a_{D(1,1)}}\big)\right] \ge \mathbb{E}_x\left[u\big(X^a_{\tau^a_{D(1,1)}}\big); X^a_{\tau^a_{D(1,1)}} \in D(2,1)\right] \\ &\ge c_1 u(x_0)\mathbb{P}_x\Big(X^a_{\tau^a_{D(1,1)}} \in D(2,1)\Big) \ge c_2 u(x_0)\delta_D(x)/r. \end{aligned} \tag{4.19}$$

In the last inequality above we have used (3.45).

Let $w = (\widetilde{0}, rr_0/16)$. Then it is easy to see that there exists a constant $\eta = \eta(\Lambda, r_0, \delta_0) \in (0,1)$ such that $B(w, \eta r r_0/16) \in D(1,1)$. By (1.4) and Lemma 3.1,

$$\begin{aligned} u(w) &\ge \mathbb{E}_w\left[u\big(X^a_{\tau^a_{D(1,1)}}\big); X^a_{\tau^a_{D(1,1)}} \notin D(2,2)\right] \\ &= \mathcal{A}(d,\alpha)a^\alpha \int_{D(1,1)} G^a_{D(1,1)}(w,z)\int_{\mathbb{R}^d\setminus D(2,2)} \frac{u(y)}{|z-y|^{d+\alpha}}dydz \\ &\ge c_3 a^\alpha \mathbb{E}_w\left[\tau^a_{B(w,\eta r r_0/(16))}\right]\int_{\mathbb{R}^d\setminus D(2,2)} \frac{u(y)}{|w-y|^{d+\alpha}}dy \\ &\ge c_4 a^\alpha r^2 \int_{\mathbb{R}^d\setminus D(2,2)} \frac{u(y)}{|w-y|^{d+\alpha}}dy. \end{aligned}$$

Hence by (3.47),

$$\begin{aligned}
& \mathbb{E}_x\left[u\left(X^a_{\tau^a_{D(1,1)}}\right);\, X^a_{\tau^a_{D(1,1)}} \notin D(2,2)\right] \\
= \;& \mathcal{A}(d,\alpha)a^\alpha \int_{D(1,1)} G^a_{D(1,1)}(x,z) \int_{\mathbb{R}^d\setminus D(2,2)} \frac{u(y)}{|z-y|^{d+\alpha}}dydz \\
\le \;& c_5 a^\alpha \mathbb{E}_x[\tau^a_{D(1,1)}] \int_{\mathbb{R}^d\setminus D(2,2)} \frac{u(y)}{|w-y|^{d+\alpha}}dy \\
\le \;& c_6 a^\alpha \delta_D(x) r \int_{\mathbb{R}^d\setminus D(2,2)} \frac{u(y)}{|w-y|^{d+\alpha}}dy \;\le\; \frac{c_6\,\delta_D(x)}{c_4\, r}u(w).
\end{aligned}$$

On the other hand, by the uniform Harnack principle (Proposition 1.3) and the Carleson estimate (Theorem 4.3), we have

$$\begin{aligned}
\mathbb{E}_x\left[u\left(X^a_{\tau^a_{D(1,1)}}\right);\, X^a_{\tau^a_{D(1,1)}} \in D(2,2)\right] &\le c_7\, u(x_0)\mathbb{P}_x\left(X^a_{\tau^a_{D(1,1)}} \in D(2,2)\right) \\
&\le c_8\, u(x_0)\delta_D(x)/r.
\end{aligned}$$

In the last inequality above we have used (3.46). Combining the two inequalities above, we get

$$\begin{aligned}
u(x) &= \mathbb{E}_x\left[u\left(X^a_{\tau^a_{D(1,1)}}\right);\, X^a_{\tau^a_{D(1,1)}} \in D(2,2)\right] \\
&\quad +\mathbb{E}_x\left[u\left(X^a_{\tau^a_{D(1,1)}}\right);\, X^a_{\tau^a_{D(1,1)}} \notin D(2,2)\right] \\
&\le \frac{c_8}{r}\delta_D(x)u(x_0) + \frac{c_6\,\delta_D(x)}{c_4\, r}u(w) \\
&\le \frac{c_9}{r}\,\delta_D(x)(u(x_0)+u(w)) \\
&\le \frac{c_{10}}{r}\,\delta_D(x)u(x_0).
\end{aligned} \tag{4.20}$$

In the last inequality above we have used the uniform Harnack principle (Proposition 1.3).

From (4.19)–(4.20), we have that for every $x, y \in D \cap B(Q, rr_0/8)$,

$$\frac{u(x)}{u(y)} \le \frac{c_{10}}{c_2}\,\frac{\delta_D(x)}{\delta_D(y)},$$

which proves the theorem. □

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