HEAT KERNEL ESTIMATES FOR DIRICHLET FORMS DEGENERATE AT THE BOUNDARY

SOOBIN CHO, PANKI KIM, RENMING SONG AND ZORAN VONDRAČEK

Abstract. The goal of this paper is to prove the existence of heat kernels for two types of purely discontinuous symmetric Markov processes in the upper half-space of $\mathbb{R}^d$ with jump kernels degenerate at the boundary, and to establish sharp two-sided estimates on the heat kernels. The jump kernels are of the form $J(x,y) = \mathcal{B}(x,y)|x-y|^{-\alpha-d}$, $\alpha \in (0,2)$, where the function $\mathcal{B}$ depends on four parameters and may vanish at the boundary. Our results are the first sharp two-sided estimates for the heat kernels of non-local operators with jump kernels degenerate at the boundary.

The first type of processes we consider are conservative Markov processes on the closed half-space with jump kernel $J(x,y)$. Depending on the regions where the parameters belong, the heat kernels estimates have three different forms, two of them are qualitatively different from all previously known heat kernel estimates.

The second type of processes we consider are the processes above killed either by a critical potential or upon hitting the boundary of the half-space. We establish that their heat kernel estimates have the approximate factorization property with survival probabilities decaying as a power of the distance to the boundary, where the power depends on the constant in the critical potential.

Finally, by using the heat kernel estimates, we obtain sharp two-sided Green function estimates that cover the main result of [31]. This alternative proof provides an explanation of the anomalous form of the estimates.

AMS 2020 Mathematics Subject Classification: Primary 60J35, 60J45; Secondary 60J50, 60J76, 31C25, 35K08

Keywords and phrases: Markov processes, Dirichlet forms, jump kernel, killing potential, heat kernel, heat kernel estimates, Green function

Contents

This research is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. 2021R1A4A1027378).

Research supported in part by a grant from the Simons Foundation (#960480, Renming Song).

Research supported in part by the Croatian Science Foundation under the project 4197. (Zoran Vondraček).

1. Introduction and main results

In this paper, we study both conservative and non-conservative purely discontinuous (self-similar) Markov processes in the upper half-space of $\mathbb{R}^d$ with jump kernels of the form $J(x,y) = |x-y|^{-d-\alpha}\mathcal{B}(x,y)$, $\alpha \in (0,2)$. The function $\mathcal{B}(x,y)$ may tend to 0 when x or y tends to boundary of the half-space, and so the jump kernel may be degenerate. The main focus of the paper is to show the existence and continuity of the transition densities of the processes (or the heat kernels of the corresponding non-local operators), and to establish their sharp two-sided estimates. Heat kernel estimates for non-local operators have been the subject of many papers in the last twenty years, see [2, 4, 5, 10, 12, 13, 15, 16, 17, 19, 24, 26] and the references therein. In all of the papers mentioned above, the function $\mathcal{B}$ is assumed to be bounded between two positive constants, which can be viewed as a uniform ellipticity condition for non-local operators. To the best of our knowledge, the current paper is the first one to study sharp two-sided heat kernel estimates when the jump kernel is degenerate. Boundary Harnack principle and sharp two-sided Green function estimates for purely discontinuous Markov processes with degenerate jump kernels, which can be viewed as the elliptic counterpart of results of the current paper, have been recently obtained in [30, 31, 32].

The first type of processes we look at are conservative jump processes on $\overline{\mathbb{R}}^d_+ := \{x=(\widetilde{x},x_d) : \widetilde{x}\in\mathbb{R}^{d-1}, x_d \ge 0\}$ with jump kernel $J(x,y)=|x-y|^{-d-\alpha}\mathcal{B}(x,y)$, where the function $\mathcal{B}(x,y)$ is symmetric, homogeneous, horizontally translation invariant, and is allowed to approach 0 at the boundary at arbitrary fixed polynomial rate in terms of some non-negative parameters $\beta_1,\beta_2,\beta_3,\beta_4$. The generator of such a process is the non-local operator

$$L^{\mathcal{B}}_\alpha f(x) = \text{p.v.}\int_{\mathbb{R}^d_+} (f(y)-f(x))|x-y|^{-d-\alpha}\mathcal{B}(x,y)\,dy.$$

When $\mathcal{B}(x,y)$ is bounded between two positive constants, the heat kernel estimates for such processes are of the form $\min\{t^{-d/\alpha}, tJ(x,y)\}$. This was first established in the pioneering work [15], even for the case of metric measure spaces. This form of the estimates reflects the fact that the main contribution to the heat kernel comes from one (big) jump from x to y. This feature has been observed in all subsequent studies, see [2, 16, 10, 17, 19, 24] and the references therein. In our setting of jump kernel degenerate at the boundary, there are two novel features in the heat kernel estimates. The first one appears in the case when the involved parameters satisfy $\beta_2 < \alpha+\beta_1$, in which case the heat kernel is comparable to $\min\{t^{-d/\alpha}, tJ(x+t^{1/\alpha}\mathbf{e}_d, y+t^{1/\alpha}\mathbf{e}_d)\}$, where $\mathbf{e}_d=(\widetilde{0},1)$. In words, the form of the heat kernel estimates shows that the main contribution to the heat kernel at time t comes from one jump from the point $t^{1/\alpha}$ units above x to the point $t^{1/\alpha}$ units above y. Due to the fact the jump kernel vanishes at the boundary, it is very unlikely that the process will make one (big) jump from (or to) a point very close to the boundary. The second feature is more striking and indicates a sort of a phase-transition at the level $\beta_2=\alpha+\beta_1$: When the parameters satisfy $\beta_2 \ge \alpha+\beta_1$, in addition to the already mentioned part $\min\{t^{-d/\alpha}, tJ(x+t^{1/\alpha}\mathbf{e}_d, y+t^{1/\alpha}\mathbf{e}_d)\}$, the sharp heat kernel estimates include a part which reflects a significant contribution to the heat kernel coming from two jumps connecting x and y. The precise description is given in Theorem 1.1.

The second type of processes we look at are the ones described in the previous paragraph but killed either by a critical potential $\kappa x_d^{-\alpha}$ or upon hitting the boundary of $\mathbb{R}^d_+ := \{x=(\widetilde{x},x_d) : \widetilde{x}\in\mathbb{R}^{d-1}, x_d>0\}$ (the latter happens only when $\alpha\in(1,2)$). The generator of such a process is the non-local operator $L^{\mathcal{B}}f(x) = L^{\mathcal{B}}_\alpha f(x) - \kappa x_d^{-\alpha} f(x)$. We study the effect of such killings on the heat kernel. When $\mathcal{B}(x,y)$ is bounded between two positive constants, the effect of killing on the heat kernel of the process is, after intensive research during the last fifteen years, fairly well understood. In most cases, the heat kernel of the killed process has the so called approximate factorization property: It is comparable to the product of the heat kernel of non-killed (original) process and the survival probabilities starting from points x and y. In case of smooth open sets, the survival probability can be expressed in terms of the distance between the point and the boundary, see [4, 5, 8, 12, 13, 21, 27] and the references therein. In our setting of jump kernel degenerate at the boundary, we establish the same property: The heat kernel of the killed process enjoys the approximate factorization property with survival probabilities decaying as the q-th power of the distance to the boundary, where q is in one-to-one correspondence with the constant $\kappa \ge 0$, see Theorem 1.2. When $\kappa = 0$, this theorem generalizes [14] (for half spaces) where the factorization property for censored stable process is established. Due to the quite complicated form of the heat kernel estimates in Theorem 1.1, obtaining the factorization property in Theorem 1.2 is a formidable task.

We now introduce the precise setup and state the main results of this paper. This setup was first introduced in [30] and was motivated by the results of [28, 29] on subordinate killed Lévy processes. In fact, subordinate killed Lévy processes, whose analytical counterparts are fractional powers of Dirichlet fractional Laplacians, are the main natural examples of Markov processes with jump kernels satisfying the assumptions below.

Let $d \ge 1$ and $0 < \alpha < 2$. Recall that $\mathbb{R}^d_+ = \{(\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ and $\overline{\mathbb{R}}^d_+ = \{(\widetilde{x}, x_d) \in \mathbb{R}^d : x_d \ge 0\}$. Here and below, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $a \asymp b$ means that $c \le b/a \le c^{-1}$ for some $c \in (0, 1)$. For $b_1, b_2, b_3, b_4 \ge 0$, let

$$B_{b_1,b_2,b_3,b_4}(x, y) := \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{b_1} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{b_2} \times \log^{b_3}\left(e + \frac{(x_d \vee y_d) \wedge |x-y|}{x_d \wedge y_d \wedge |x-y|}\right) \log^{b_4}\left(e + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right). \quad (1.1)$$

Let $J(x, y) = |x-y|^{-d-\alpha}\mathcal{B}(x, y)$, where the function $\mathcal{B} : \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+ \to [0, \infty)$ will be assumed to satisfy some or all of the following hypotheses:

(A1) $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in \overline{\mathbb{R}}^d_+$.

(A2) If $\alpha \ge 1$, then there exist $\theta > \alpha - 1$ and $C_1 > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \le C_1 \left(\frac{|x-y|}{x_d \wedge y_d}\right)^{\theta}, \quad x, y \in \mathbb{R}^d_+.$$

(A3)(I) There exist $C_2 \ge 1$ and $\beta_1, \beta_2, \beta_3, \beta_4 \ge 0$, with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C_2^{-1} B_{\beta_1,\beta_2,\beta_3,\beta_4}(x, y) \le \mathcal{B}(x, y) \le C_2, \quad x, y \in \overline{\mathbb{R}}^d_+.$$

(II) There exists $C_3 > 0$ such that

$$\mathcal{B}(x, y) \le C_3 B_{\beta_1,\beta_2,\beta_3,\beta_4}(x, y), \quad x, y \in \overline{\mathbb{R}}^d_+,$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are the same constants as in (I).

(A4) For all $x, y \in \overline{\mathbb{R}}^d_+$ and $a > 0$, $\mathcal{B}(ax, ay) = \mathcal{B}(x, y)$. In case $d \ge 2$, for all $x, y \in \overline{\mathbb{R}}^d_+$ and $\widetilde{z} \in \mathbb{R}^{d-1}$, $\mathcal{B}(x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)) = \mathcal{B}(x, y)$.

These four hypotheses were introduced in [30], and with the same notation as above repeated in [31, 32]. Condition **(A2)** is not needed in Theorems 1.1 and 1.3, while in Theorems 1.2 and 1.4 it is used through several results from [30, 31, 32].

Throughout this paper, we always assume that $\mathcal{B}(x, y)$ satisfies **(A1)**, **(A3)**(I) *and* **(A4)**.

Consider the following symmetric form

$$\mathcal{E}^0(u, v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} (u(x) - u(y))(v(x) - v(y))J(x, y)dydx.$$

Since $\mathcal{B}(x, y)$ is bounded, $C_c^\infty(\overline{\mathbb{R}}^d_+)$ is closable in $L^2(\overline{\mathbb{R}}^d_+, dx) = L^2(\mathbb{R}^d_+, dx)$ by Fatou's lemma. Let $\overline{\mathcal{F}}$ be the closure of $C_c^\infty(\overline{\mathbb{R}}^d_+)$ in $L^2(\mathbb{R}^d_+, dx)$ under the norm $\mathcal{E}^0_1 := \mathcal{E}^0 + (\cdot, \cdot)_{L^2(\mathbb{R}^d_+, dx)}$. Then

$(\mathcal{E}^0, \overline{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d_+, dx)$. Let $\overline{Y} = (\overline{Y}_t, t \ge 0; \mathbb{P}_x, x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}')$ be the Hunt process associated with $(\mathcal{E}^0, \overline{\mathcal{F}})$, where $\mathcal{N}'$ is an exceptional set.

Here is our first main result. The heat kernel estimates are expressed in different but equivalent forms, each providing a different viewpoint. Recall that $\mathbf{e}_d = (\widetilde{0}, 1) \in \mathbb{R}^d$.

Theorem 1.1. *Suppose that* **(A1)**, **(A3)** *and* **(A4)** *hold. Then the process $\overline{Y}$ can be refined to be a conservative Feller process with strong Feller property starting from every point in $\overline{\mathbb{R}}^d_+$ and has a jointly continuous heat kernel $\overline{p} : (0,\infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+ \to (0,\infty)$. Moreover, the heat kernel $\overline{p}$ has the following estimates: For all $(t,x,y) \in (0,\infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,*

$$\overline{p}(t,x,y) \asymp t^{-d/\alpha} \wedge \Big[tJ(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \\ + \mathbf{1}_{\{\beta_2 \ge \alpha + \beta_1\}} t^2 \int_{(x_d \vee y_d \vee t^{1/\alpha}) \wedge (|x-y|/4)}^{|x-y|/2} J(x + t^{1/\alpha}\mathbf{e}_d, x + r\mathbf{e}_d)\, J(x + r\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) r^{d-1} dr \Big]. \quad (1.2)$$

Furthermore the heat kernel estimates in (1.2) *can be rewritten in terms of B_{b_1,b_2,b_3,b_4} explicitly by considering three cases separately:*

(i) If $\beta_2 < \alpha + \beta_1$, then for all $(t,x,y) \in (0,\infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\overline{p}(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{tB_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)}{|x-y|^{d+\alpha}} \right) \\ \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d). \quad (1.3)$$

(ii) If $\beta_2 > \alpha + \beta_1$, then for all $(t,x,y) \in (0,\infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\overline{p}(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \Big[B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \\ + \left(1 \wedge \frac{t}{|x-y|^{\alpha}} \right) B_{\beta_1,\beta_1,0,\beta_3}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) + t^{1/\alpha}) \wedge |x-y|} \right) \Big]. \quad (1.4)$$

(iii) If $\beta_2 = \alpha + \beta_1$, then for all $(t,x,y) \in (0,\infty) \times \overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$,

$$\overline{p}(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \Big[B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \\ + \left(1 \wedge \frac{t}{|x-y|^{\alpha}} \right) B_{\beta_1,\beta_1,0,\beta_3+\beta_4+1}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) + t^{1/\alpha}) \wedge |x-y|} \right) \Big]. \quad (1.5)$$

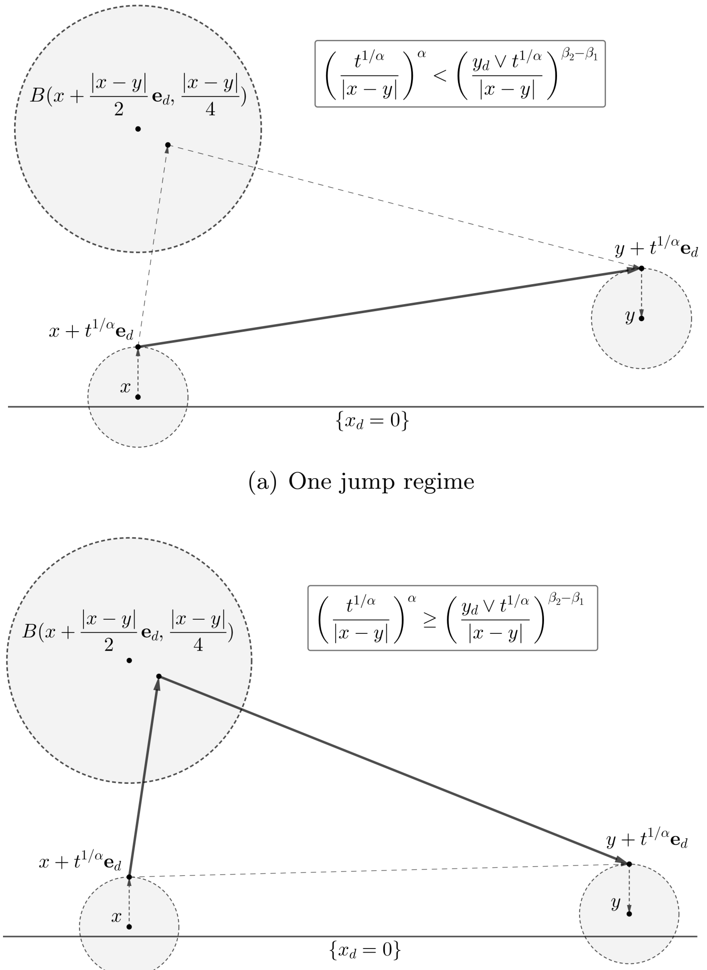
Note that if x or y is close to the boundary, then $J(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)$ is not comparable to $J(x,y)$ in our setting. Thus, even in case $\beta_2 < \alpha + \beta_1$, the form of the heat kernel is different from the usual form. The appearance of the two terms in the brackets on the right-hand sides of (1.4)–(1.5) reflects the fact that the dominant contribution to the heat kernel may come from either one jump or two jumps. Moreover, it is easy to see that neither of these two terms dominates the other one for all (t,x,y). Below we illustrate this feature when $\beta_2 > \alpha + \beta_1$.

Let $\beta_2 > \alpha + \beta_1$. When $|x-y| > 6t^{1/\alpha}$, the second term in the brackets in (1.4), i.e.,

$$\left(1 \wedge \frac{t}{|x-y|^{\alpha}} \right) B_{\beta_1,\beta_1,0,\beta_3}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) + t^{1/\alpha}) \wedge |x-y|} \right) \quad (1.6)$$

is comparable to

$$t|x-y|^{d+\alpha} \int_{B(x+2^{-1}|x-y|\mathbf{e}_d,\, 4^{-1}|x-y|)} J(x + t^{1/\alpha}\mathbf{e}_d, z)\, J(z, y + t^{1/\alpha}\mathbf{e}_d) dz, \quad (1.7)$$



(a) One jump regime

(b) Two jumps regime

FIGURE 1. Dominant path from x to y (with $x_d \leq y_d$) at time $t < (|x-y|/6)^{\alpha}$ when $\beta_2 > \alpha + \beta_1$, $\beta_3 = \beta_4 = 0$

see Remark 6.5 below. In the special case when $\beta_3 = \beta_4 = 0$ (and $\beta_2 > \alpha + \beta_1$), Figure 1, where the comparability of (1.6) and (1.7) is used, illustrates the regions where one jump and two jumps dominate.

The different forms of the heat kernel estimates in Theorem 1.1 are consequences of the relationship among β_1, β_2 and α only, the values of β_3 and β_4 do not play any role in this. To reduce technicalities, the reader, on a first reading, may assume $\beta_3 = \beta_4 = 0$, without loosing the essential features of this paper. Note that, even in this special case, a logarithmic term appears in the estimates when $\beta_2 = \alpha + \beta_1$. We need the logarithmic terms in the assumptions to cover the important examples in [21, 29].

Let $\mathcal{F}^0$ be the closure of $C_c^\infty(\mathbb{R}^d_+)$ in $L^2(\mathbb{R}^d_+, dx)$ under the norm $\mathcal{E}^0_1$. Then $(\mathcal{E}^0, \mathcal{F}^0)$ is also a regular Dirichlet form on $L^2(\mathbb{R}^d_+, dx)$ and it is the part form of $(\mathcal{E}^0, \overline{\mathcal{F}})$ on $\mathbb{R}^d_+$. Let $Y^0 = (Y^0_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d_+ \setminus \mathcal{N}_0)$ be the Hunt process associated with $(\mathcal{E}^0, \mathcal{F}^0)$ where $\mathcal{N}_0$ is an exceptional set. Y^0 can be regarded as the part process of $\overline{Y}$ killed upon exiting $\mathbb{R}^d_+$.

For $\kappa \in (0, \infty)$, define

$$\mathcal{E}^\kappa(u,v) := \mathcal{E}^0(u,v) + \int_{\mathbb{R}^d_+} u(x)v(x)\kappa x_d^{-\alpha} dx,$$
$$\mathcal{F}^\kappa := \widetilde{\mathcal{F}}^0 \cap L^2(\mathbb{R}^d_+, \kappa x_d^{-\alpha} dx),$$

where $\widetilde{\mathcal{F}}^0$ is the family of all $\mathcal{E}_1^0$-quasi-continuous functions in $\mathcal{F}^0$. Then $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ is a regular Dirichlet form on $L^2(\mathbb{R}^d_+, dx)$ with $C_c^\infty(\mathbb{R}^d_+)$ as a special standard core, see [23, Theorems 6.1.1 and 6.1.2]. Let $Y^\kappa = (Y_t^\kappa, t \ge 0; \mathbb{P}_x, x \in \mathbb{R}^d_+ \setminus \mathcal{N}_\kappa)$ be the Hunt process associated with $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ where $\mathcal{N}_\kappa$ is an exceptional set. For $\kappa \in [0,\infty)$, we denote by ζ^κ the lifetime of Y^κ. Define $Y_t^\kappa = \partial$ for $t \ge \zeta^\kappa$, where ∂ is a cemetery point added to the state space $\mathbb{R}^d_+$.

We now associate with the constant κ a positive parameter q which plays an important role in the paper. For $q \in (-1, \alpha+\beta_1)$, set

$$C(\alpha,q,\mathcal{B}) = \begin{cases} \displaystyle\int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2+1)^{(d+\alpha)/2}} \int_0^1 \frac{(s^q-1)(1-s^{\alpha-q-1})}{(1-s)^{1+\alpha}} \mathcal{B}\big(((1-s)\widetilde{u},1), s\mathbf{e}_d\big) ds\, d\widetilde{u} & \text{if } d \ge 2, \\ \displaystyle\int_0^1 \frac{(s^q-1)(1-s^{\alpha-q-1})}{(1-s)^{1+\alpha}} \mathcal{B}\big(1,s\big) ds & \text{if } d=1. \end{cases}$$

If we additionally assume that **(A3)**(II) holds, then the constant $C(\alpha,q,\mathcal{B})$ is well defined and finite for every $q \in (-1,\alpha+\beta_1)$, $C(\alpha,q,\mathcal{B}) = 0$ if and only if $q \in \{0, \alpha-1\}$, and $\lim_{q\to-1} C(\alpha,q,\mathcal{B}) = \lim_{q\to\alpha+\beta_1} C(\alpha,q,\mathcal{B}) = \infty$ (see [30, Lemma 5.4 and Remark 5.5]). Note that for every $s \in (0,1)$, $q \mapsto (s^q-1)(1-s^{\alpha-q-1})$ is strictly decreasing on $(-1,(\alpha-1)/2)$ and strictly increasing on $((\alpha-1)/2, \alpha+\beta_1)$. Thus, the shape of the map $q \mapsto C(\alpha,q,\mathcal{B})$ is given as follows.

q	-1	$\cdots$	$(\alpha-1)\wedge 0$	$\cdots$	$\frac{1}{2}(\alpha-1)$	$\cdots$	$(\alpha-1)\vee 0$	$\cdots$	$\alpha+\beta_1$
$C(\alpha,q,\mathcal{B})$	∞	$\searrow$	0	$\searrow$	minimum ≤ 0	$\nearrow$	0	$\nearrow$	∞

Consequently, for every $\kappa \ge 0$, there exists a unique $q_\kappa \in [(\alpha-1)_+, \alpha+\beta_1)$ such that

$$\kappa = C(\alpha, q_\kappa, \mathcal{B}). \tag{1.8}$$

Theorem 1.2. *Suppose that* **(A1)**–**(A4)** *and* (1.8) *hold with* $q_\kappa \in [(\alpha-1)_+, \alpha+\beta_1)$. *Then the process* Y^κ *can be refined to start from every point in* $\mathbb{R}^d_+$ *and has a jointly continuous heat kernel* $p^\kappa : (0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+ \to (0,\infty)$. *Moreover, the following approximate factorization holds for all* $(t,x,y) \in (0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$*:*

$$p^\kappa(t,x,y) \asymp \Big(1 \wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q_\kappa} \Big(1 \wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q_\kappa} \overline{p}(t,x,y) \asymp \mathbb{P}_x(\zeta^\kappa > t)\mathbb{P}_y(\zeta^\kappa > t)\, \overline{p}(t,x,y), \tag{1.9}$$

where $\overline{p}(t,x,y)$ *is the heat kernel of* $\overline{Y}$.

As a consequence of Lemma 2.3 below, we will see that, when $\alpha \le 1$, the process $\overline{Y}$ started from $\mathbb{R}^d_+$ will never hit $\partial\mathbb{R}^d_+$ and is equal to Y^0. Thus for $x, y \in \mathbb{R}^d_+$ we have that $p^0(t,x,y) = \overline{p}(t,x,y)$, implying that the non-trivial content of Theorem 1.2 is for $\kappa > 0$ when $\alpha \le 1$.

Let

$$\overline{G}(x,y) = \int_0^\infty \overline{p}(t,x,y)dt \quad \text{and} \quad G^\kappa(x,y) = \int_0^\infty p^\kappa(t,x,y)dt. \tag{1.10}$$

When $\overline{G}(\cdot,\cdot)$ is not identically infinite, it is called the Green function of $\overline{Y}$, and when $G^\kappa(\cdot,\cdot)$ is not identically infinite, it is called the Green function of Y^κ.

As a consequence of the heat kernel estimates, we get the Green function estimates. The following theorem says that the Green function of $\overline{Y}$ is comparable to that of the isotropic α-stable process in $\mathbb{R}^d$ even though the jump kernel of $\overline{Y}$ may be degenerate.

Theorem 1.3. *Suppose that* **(A1)**, **(A3)**(I) *and* **(A4)** *hold. If* $d > \alpha$, *then*

$$\overline{G}(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}}, \quad x,y \in \overline{\mathbb{R}}^d_+. \tag{1.11}$$

If $d \le \alpha$, *then* $\overline{G}(x,y) = \infty$ *for all* $x,y \in \overline{\mathbb{R}}^d_+$.

When $d > (\alpha+\beta_1+\beta_2)\wedge 2$, sharp two-sided estimates on $G^\kappa(x,y)$ were obtained in [31, 32]. In the following theorem, we extend those results by removing the restriction on d and give another proof using the heat kernel estimates. The advantage of the new proof is that it explains the reason for the phase transition in the Green function estimates for $d \geq 2$.

Let

$$H_q(x,y) = \begin{cases} 1 & \text{if } q < \alpha + \frac{1}{2}(\beta_1+\beta_2), \\ \log^{\beta_4+1}\left(e + \frac{|x-y|}{(x_d \vee y_d)\wedge |x-y|}\right) & \text{if } q = \alpha + \frac{1}{2}(\beta_1+\beta_2), \\ \left(\frac{x_d \vee y_d}{|x-y|}\wedge 1\right)^{2\alpha+\beta_1+\beta_2-2q} \log^{\beta_4}\left(e + \frac{|x-y|}{(x_d \vee y_d)\wedge |x-y|}\right) & \text{if } q > \alpha + \frac{1}{2}(\beta_1+\beta_2). \end{cases}$$

Theorem 1.4. *Suppose that* **(A1)**–**(A4)** *and* (1.8) *hold with* $q_\kappa \in [(\alpha-1)_+, \alpha+\beta_1)$. *When* $\alpha \leq 1$, *suppose also that* $q_\kappa > 0$ *(or, equivalently,* $\kappa > 0$*). Then* G^κ *has the following estimates:*
(i) If $d \geq 2$, *then for all* $x, y \in \mathbb{R}^d_+$,

$$G^\kappa(x,y) \asymp \frac{H_{q_\kappa}(x,y)}{|x-y|^{d-\alpha}}\left(\frac{x_d \wedge y_d}{|x-y|}\wedge 1\right)^{q_\kappa}\left(\frac{x_d \vee y_d}{|x-y|}\wedge 1\right)^{q_\kappa}.$$

(ii) If $d = 1$, *then for all* $x, y \in \mathbb{R}^d_+$,

$$G^\kappa(x,y) \asymp \begin{cases} \frac{1}{|x-y|^{1-\alpha}}\left(\frac{x\wedge y}{|x-y|}\wedge 1\right)^{q_\kappa} & \text{if } \alpha < 1, \\ \left(\frac{x\wedge y}{|x-y|}\wedge 1\right)^{q_\kappa}\log\left(e + \frac{(x\wedge y)\vee |x-y|}{|x-y|}\right) & \text{if } \alpha = 1, \\ (x\wedge y)^{\alpha-1}\left(\frac{x\wedge y}{|x-y|}\wedge 1\right)^{q_\kappa-\alpha+1} & \text{if } \alpha > 1. \end{cases}$$

Note that when $d \geq 2$, at the threshold $q_\kappa = \alpha + \frac{1}{2}(\beta_1+\beta_2)$, there is a transition from the usual behavior of the Green function estimates to anomalous behavior.

We describe now the strategy for proving our main results and the organization of the paper.

It is well known that an appropriate Nash-type inequality implies the existence of the heat kernel (outside an exceptional set) and its α-stable-type upper bound. So we start in Section 2 with establishing a Nash-type inequality, see Proposition 2.6. To this end, we consider a certain Feller process in $\mathbb{R}^d_+$ with continuous paths, subordinate it by an independent $\alpha/2$-stable subordinator, and show a Nash-type inequality for the Dirichlet form of the subordinate process. In case $\alpha < 1$, one can estimate the Dirichlet form of the subordinate process from above by $\mathcal{E}^0$ and thus prove a Nash-type inequality for $\mathcal{E}^0$. In case $\alpha \geq 1$, we use the scaling property of the process and a truncation of the jump kernel together with the already obtained inequality for $\alpha < 1$.

In Section 3, we prove parabolic Hölder regularity for both $\overline{Y}$ and Y^κ and use this to remove the exceptional sets and extend $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$ continuously to $(0,\infty)\times\overline{\mathbb{R}}^d_+\times\overline{\mathbb{R}}^d_+$, respectively $(0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$. To prove the parabolic Hölder regularity, we need an interior lower bound on the heat kernels. The proof of this lower bound is based on an argument that has already appeared in [30, Section 3] and is given here in Proposition 3.1. Next, we prove several lower bounds on the heat kernels, mean exit times and exit distributions of the underlying process and its killed version that allow us to apply the standard arguments for establishing the parabolic Hölder regularity. We end Section 3 with the elementary Lemma 3.15 which will be used repeatedly in deriving the heat kernel upper bounds.

Following the arguments from [9, 21], in Section 4 we establish the parabolic Harnack inequality, see Theorem 4.3, and use it to establish the important preliminary off-diagonal lower bound $p^\kappa(t,x,y) \geq ctJ(x,y)$ for t small compared to $|x-y|$, x_d and y_d, see Proposition 4.5.

In Section 5 we prove the following preliminary upper bound:

$$p^\kappa(t,x,y) \le C\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q_\kappa}\left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^{q_\kappa}\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right), \quad t>0,\ x,y\in \mathbb{R}^d_+,$$

for all $q_\kappa \in [(\alpha-1)_+, \alpha+\beta_1)$, see Proposition 5.1. Proposition 5.1 is proved in two steps: Using Lemma 3.15 and several results from [30, 31, 32] (see Lemmas 5.10-5.11) we first prove Proposition 5.1 in case $q_\kappa < \alpha$. The real challenge is to extend it to the full range of q_κ by covering the case $q_\kappa \ge \alpha$. For this we use the upper bound on Green potentials of powers of the distance to the boundary. In case $d \ge 2$, such bound was proved in [31, 32], and here we prove the corresponding result for $d=1$. The restriction $q_\kappa < \alpha$ is removed in Lemma 5.13 by using a bootstrap (induction) argument.

Section 6 is devoted to proving sharp heat kernel lower bounds. The estimates are given in terms of a function $A_{b_1,b_2,b_3,b_4}(t,x,y)$, a space-time version of the function $B_{b_1,b_2,b_3,b_4}(x,y)$. We first prove a preliminary lower bound, Proposition 6.2, by using the semigroup property and some interior lower bound on the heat kernel. This bound turns out to be sufficient in case $\beta_2 < \alpha+\beta_1$. For the case $\beta_2 \ge \alpha+\beta_1$ we need a sharper lower bound obtained in the key Lemma 6.4. There we apply the semigroup property (and the preliminary lower bound) on a carefully chosen interior set on which we have good control of the terms appearing in the preliminary lower bound. By combining Proposition 6.2 and Lemma 6.4, we obtain in Proposition 6.6 the sharp lower bound in cases $\beta_2 \ne \alpha+\beta_1$. The remaining case $\beta_2 = \alpha+\beta_1$ is more delicate and is covered in Proposition 6.9.

Sharp heat kernel upper bound is more difficult to establish and Section 7 is devoted to this task. The first step is to establish in Lemma 7.2 an upper bound which includes the function $A_{\beta_1,0,\beta_3,0}$. The proof uses Lemma 3.15 and an induction argument which consecutively increases the first parameter of the function A until reaching β_1, thus making the decay successively sharper. The sharp upper bound in case $\beta_2 < \alpha+\beta_1$, respectively $\beta_2 \ge \alpha+\beta_1$, is given in Theorem 7.5 and Corollary 7.9, respectively Theorem 7.10. The proofs are based on Lemma 3.15, Lemma 7.2, and a number of delicate technical lemmas involving multiple space-time integrals of the preliminary heat kernel estimates.

In Section 8, we combine the upper bounds obtained in Section 7 with the lower bounds from Section 6 and give the proofs of Theorems 1.1–1.2.

It is well known that the Green function is the integral over time of the heat kernel. In Section 9 we first use this observation together with the estimates of $\overline{p}(t,x,y)$ obtained in Proposition 2.7 and Lemma 3.7 (see also Remark 3.12) to prove Theorem 1.3. By using the same method of integrating the heat kernel estimates of $p^\kappa(t,x,y)$ over time we establish Theorem 1.4, thus reproving and extending the main result of [31]. This new proof sheds more light on the anomalous behavior of these Green function estimates. Lemma 9.1 clearly shows that they are consequences of the different forms of the small time heat kernel estimates.

The paper ends with an appendix which contains a number of technical results not depending on the preliminary estimates of the heat kernel.

Throughout this paper, the constants β_1, β_2, β_3, β_4 will remain the same, and κ always stands for a non-negative number. The notation $C=C(a,b,\ldots)$ indicates that the constant C depends on $a,b,\ldots$. The dependence on κ, d and α may not be mentioned explicitly. Lower case letters $c_i, i=1,2,\ldots$ are used to denote the constants in the proofs and the labeling of these constants starts anew in each proof. We denote by m_d the Lebesgue measure on $\mathbb{R}^d$. For Borel subset $D\subset \mathbb{R}^d$, $\delta_D(x)$ denotes the distance between x and the boundary ∂D.

2. Nash inequality and the existence of the heat kernel

The goal of this section is (i) to prove a Nash type inequality (Proposition 2.6); (ii) to deduce the existence of the heat kernels of $\overline{Y}$ and Y^κ; and (iii) to establish their preliminary upper bounds (Proposition 2.7).

We begin the section by introducing the notation for the relevant semigroups and establishing their scale invariance and horizontal translation invariance.

For $\kappa \ge 0$, we denote by $(\overline{P}_t)_{t\ge 0}$ and $(P_t^\kappa)_{t\ge0}$ the semigroups corresponding to $(\mathcal{E}^0, \overline{\mathcal{F}})$ and $(\mathcal{E}^\kappa, \mathcal{F}^\kappa)$ respectively. $(\overline{P}_t)_{t\ge 0}$ and $(P_t^\kappa)_{t\ge0}$ define contraction semigroups on $L^p(\overline{\mathbb{R}}^d_+, dx) = L^p(\mathbb{R}^d_+, dx)$ for every $p \in [1,\infty]$, and when $p\in[1,\infty)$, these semigroups are strongly continuous. For $t>0$ and $p,q\in[1,\infty]$, define

$$\|\overline{P}_t\|_{p\to q} = \sup\left\{ \|\overline{P}_t f\|_{L^q(\mathbb{R}^d_+,dx)} : f \in L^p(\mathbb{R}^d_+, dx),\ \|f\|_{L^p(\mathbb{R}^d_+,dx)} \le 1 \right\}.$$

For $f:\mathbb{R}^d\to\mathbb{R}$ and $r>0$, define $f^{(r)}(x) = f(rx)$. The following scaling property of $(P_t^\kappa)_{t\ge0}$ comes from [30, Lemma 5.1] and [32, Lemma 2.1]. By the same proof, $(\overline{P}_t)_{t\ge0}$ also has the same scaling property. We give the proof for the reader's convenience.

Lemma 2.1. *Let $p\in[1,\infty]$ and $\kappa\ge0$. For any $f\in L^p(\mathbb{R}^d_+,dx)$, $t>0$ and $r>0$, we have*

$$\overline{P}_t f(x) = \overline{P}_{r^{-\alpha}t} f^{(r)}(x/r) \quad \textit{and} \quad P_t^\kappa f(x) = P^\kappa_{r^{-\alpha}t} f^{(r)}(x/r) \quad \textit{in } L^p(\mathbb{R}^d_+, dx).$$

In particular, we have

$$\|\overline{P}_t\|_{1\to\infty} = t^{-d/\alpha} \|\overline{P}_1\|_{1\to\infty} \quad \textit{for all } t>0. \tag{2.1}$$

Proof. Using **(A4)**, changes of the variables and the correspondence between Dirichlet forms and semigroups, we see that for any nice function f,

$$\begin{aligned}
&\lim_{t\downarrow0}\frac{1}{t}\int_{\mathbb{R}^d_+}\left(f^{(r)}(z/r) - P^\kappa_{r^{-\alpha}t} f^{(r)}(z/r)\right) f^{(r)}(z/r)dz\\
&= r^{d-\alpha}\lim_{t\downarrow0}\frac{1}{r^{-\alpha}t}\int_{\mathbb{R}^d_+}\left(f^{(r)}(x) - P^\kappa_{r^{-\alpha}t} f^{(r)}(x)\right) f^{(r)}(x)dx\\
&= \frac{r^{d-\alpha}}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(f^{(r)}(x)-f^{(r)}(y))^2\frac{\mathcal{B}(x,y)}{|x-y|^{d+\alpha}}dxdy + \kappa r^{d-\alpha}\int_{\mathbb{R}^d_+} f^{(r)}(x)^2 x_d^{-\alpha}dx\\
&= \frac{r^{2d}}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(f(rx)-f(ry))^2\frac{\mathcal{B}(rx,ry)}{|rx-ry|^{d+\alpha}}dxdy + \kappa r^{d}\int_{\mathbb{R}^d_+} f(rx)^2 (rx_d)^{-\alpha}dx\\
&= \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(f(x)-f(y))^2\frac{\mathcal{B}(x,y)}{|x-y|^{d+\alpha}}dxdy + \int_{\mathbb{R}^d_+} f(x)^2 \kappa x_d^{-\alpha}dx = \mathcal{E}^\kappa(f,f).
\end{aligned}$$

Thus, by the uniqueness of the corresponding semigroup, we deduce the scaling property of $(P_t^\kappa)_{t\ge0}$. By a similar proof, we also get the scaling property of $(\overline{P}_t)_{t\ge0}$.

Now, we have that, for every $t>0$,

$$\begin{aligned}
\|\overline{P}_t\|_{1\to\infty} &= t^{-d/\alpha}\sup\left\{\|\overline{P}_t f\|_{L^\infty(\mathbb{R}^d_+,dx)} : \|f\|_{L^1(\mathbb{R}^d_+,dx)} \le t^{d/\alpha}\right\}\\
&= t^{-d/\alpha}\sup\left\{\|\overline{P}_1 f^{(t^{1/\alpha})}\|_{L^\infty(\mathbb{R}^d_+,dx)} : \|f\|_{L^1(\mathbb{R}^d_+,dx)} \le 1\right\} = t^{-d/\alpha}\|\overline{P}_1\|_{1\to\infty},
\end{aligned}$$

which proves (2.1). The proof is complete. □

Let $d\ge2$. For $f:\mathbb{R}^d\to\mathbb{R}$ and $\widetilde{z}\in\mathbb{R}^{d-1}$, define $f_z(x) = f(x+(\widetilde{z},0))$. From **(A4)**, we also get the following horizontal translation invariance property of the semigroups $(\overline{P}_t)_{t\ge0}$ and $(P_t^\kappa)_{t\ge0}$.

Lemma 2.2. *Let $d\ge2$, $p\in[1,\infty]$ and $\kappa\ge0$. For any $f\in L^p(\mathbb{R}^d_+,dx)$, $t>0$ and $\widetilde{z}\in\mathbb{R}^{d-1}$, we have*

$$\overline{P}_t f(x) = \overline{P}_t f_z(x-(\widetilde{z},0)) \quad \textit{and} \quad P^\kappa_t f(x) = P^\kappa_t f_z(x-(\widetilde{z},0)) \quad \textit{in } L^p(\mathbb{R}^d_+,dx).$$

A consequence of the next lemma is that, in case when $\alpha\le1$, the process $\overline{Y}$ started from $\mathbb{R}^d_+$ will never hit $\partial\mathbb{R}^d_+$ and is equal to Y^0.

Lemma 2.3. *If $\alpha \le 1$, then $\mathcal{F}^0 = \overline{\mathcal{F}}$.*

Proof. Define

$$\widetilde{\mathcal{C}}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}dydx,$$
$$\mathcal{D}(\widetilde{\mathcal{C}}) := \text{closure of } C_c^\infty(\overline{\mathbb{R}}^d_+) \text{ in } L^2(\mathbb{R}^d_+, dx) \text{ under } \widetilde{\mathcal{C}} + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+,dx)}.$$

Then $(\widetilde{\mathcal{C}}, \mathcal{D}(\widetilde{\mathcal{C}}))$ is a regular Dirichlet form associated with the *reflected* α-stable process in $\overline{\mathbb{R}}^d_+$ in the sense of [6]. By **(A3)**(I), $\mathcal{E}^0(u,u) \le C_2\widetilde{\mathcal{C}}(u,u)$ for all $u \in C_c^\infty(\overline{\mathbb{R}}^d_+)$ and hence $\mathcal{D}(\widetilde{\mathcal{C}}) \subset \overline{\mathcal{F}}$. By [6, Theorem 2.5(i) and Remark 2.2(1)], since $\alpha \le 1$, $\overline{\mathbb{R}}^d_+ \setminus \mathbb{R}^d_+$ is $(\widetilde{\mathcal{C}}, \mathcal{D}(\widetilde{\mathcal{C}}))$-polar and hence is also $(\mathcal{E}^0, \overline{\mathcal{F}})$-polar. Therefore, when starting from $\mathbb{R}^d_+$, $\overline{Y}$ will never exit $\mathbb{R}^d_+$. Hence $\overline{Y}$ and Y^0 are the same when they start from $x \in \mathbb{R}^d_+$ and thus $\mathcal{F}^0 = \overline{\mathcal{F}}$. □

In order to prove the Nash type inequality, we first consider a Brownian motion on $\mathbb{R}^d_+$ killed with a critical potential and a subordinate process obtained by time changing this killed Brownian motion with an independent $\alpha/2$-stable subordinator. Then using results from [21], the Hardy inequality in [32, Proposition 3.2] and comparing the Dirichlet form corresponding to the subordinate process with $(\mathcal{E}^0, \overline{\mathcal{F}})$, we deduce the desired result.

For any $\gamma \ge 0$, denote by I_γ the modified Bessel function of the first kind which is defined by

$$I_\gamma(r) = \sum_{m=0}^\infty \frac{1}{m!\,\Gamma(\gamma+1+m)}\left(\frac{r}{2}\right)^{2m+\gamma},$$

where $\Gamma(r) := \int_0^\infty u^{r-1}e^{-u}du$ is the Gamma function. It is known that (see, e.g. [1, (9.6.7) and (9.7.1)])

$$I_\gamma(r) \asymp (1\wedge r)^{\gamma+1/2} r^{-1/2} e^r, \quad r>0. \tag{2.2}$$

Define for $t>0$ and $x=(x_1,\dots,x_d), y=(y_1,\dots,y_d) \in \mathbb{R}^d_+$,

$$q^\gamma(t,x,y) = \frac{\sqrt{x_d y_d}}{2t} I_\gamma\left(\frac{x_d y_d}{2t}\right)\exp\left(-\frac{x_d^2+y_d^2}{4t}\right)\prod_{i=1}^{d-1}\left[\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{|x_i-y_i|^2}{4t}\right)\right].$$

Note that by (2.2),

$$q^\gamma(t,x,y) \asymp \left(1\wedge \frac{x_d y_d}{t}\right)^{\gamma+1/2} t^{-d/2}\exp\left(-\frac{|x-y|^2}{4t}\right). \tag{2.3}$$

By [33, Lemma 4.1 and Theorem 4.9], $q^\gamma(t,x,y)$ is the transition density of the Feller process (killed Brownian motion with critical potential) $W^\gamma = (W^\gamma_t)_{t\ge 0}$ on $\mathbb{R}^d_+$ associated with the following regular Dirichlet form $(\mathcal{Q}^\gamma, \mathcal{D}(\mathcal{Q}^\gamma))$:

$$\mathcal{Q}^\gamma(u,v) := \int_{\mathbb{R}^d_+} \nabla u(x)\cdot\nabla v(x)dx + \left(\gamma^2 - \frac{1}{4}\right)\int_{\mathbb{R}^d_+} u(x)v(x)x_d^{-2}dx,$$
$$\mathcal{D}(\mathcal{Q}^\gamma) := \text{closure of } C_c^\infty(\mathbb{R}^d_+) \text{ in } L^2(\mathbb{R}^d_+, dx) \text{ under } \mathcal{Q}^\gamma_1 = \mathcal{Q}^\gamma + (\cdot,\cdot)_{L^2(\mathbb{R}^d_+,dx)}.$$

Let $S=(S_t)_{t\ge0}$ be an $\alpha/2$-stable subordinator independent of W^γ, and let $X^\gamma = (X^\gamma_t)_{t\ge0}$ be the subordinate process $X^\gamma_t := W^\gamma_{S_t}$. Then X^γ is a Hunt process with no diffusion part. The transition density $p^\gamma(t,x,y)$ of X^γ_t exists and is given by

$$p^\gamma(t,x,y) = \int_0^\infty q^\gamma(s,x,y)\frac{d}{ds}\mathbb{P}(S_t\le s).$$

Also, the jump kernel $J^\gamma(dx,dy)$ and the killing measure $\kappa^\gamma(dx)$ of X^γ have densities $J^\gamma(x,y)$ and $\kappa^\gamma(x)$ that are given by the following formulas (see, for instance, [34, (2.8)–(2.9)]):

$$J^\gamma(x,y)=\int_0^\infty q^\gamma(t,x,y)\,\nu_{\alpha/2}(t)dt,\qquad \kappa^\gamma(x)=\int_0^\infty\left(1-\int_{\mathbb{R}^d_+}q^\gamma(t,x,y)dy\right)\nu_{\alpha/2}(t)dt,$$

where $\nu_{\alpha/2}(t)=\frac{\alpha/2}{\Gamma(1-\alpha/2)}t^{-1-\alpha/2}$ is the Lévy density of the $\alpha/2$-stable subordinator S.

Lemma 2.4. *(i) There exists a constant $c_{\gamma,\alpha}>0$ such that $\kappa^\gamma(x)=c_{\gamma,\alpha}x_d^{-\alpha}$ for every $x\in\mathbb{R}^d_+$.*
(ii) It holds that for any $x,y\in\mathbb{R}^d_+$,

$$J^\gamma(x,y)\asymp\left(1\wedge\frac{x_d}{|x-y|}\right)^{\gamma+1/2}\left(1\wedge\frac{y_d}{|x-y|}\right)^{\gamma+1/2}\frac{1}{|x-y|^{d+\alpha}}=\frac{B_{\gamma+1/2,\gamma+1/2,0,0}(x,y)}{|x-y|^{d+\alpha}}.$$

(iii) There exists a constant $C>0$ such that

$$p^\gamma(t,x,y)\le Ct^{-d/\alpha},\quad t>0,\ x,y\in\mathbb{R}^d_+.$$

Proof. (i) Observe that the process W^γ satisfies the scaling property and horizontal translation invariance, namely, for any $t>0$, $x,y\in\mathbb{R}^d_+$, $\lambda>0$ and $\widetilde{z}\in\mathbb{R}^{d-1}$,

$$q^\gamma(\lambda^2t,\lambda x,\lambda y)=\lambda^{-d}q^\gamma(t,x,y)\quad\text{and}\quad q^\gamma(t,x+(\widetilde{z},0),y+(\widetilde{z},0))=q^\gamma(t,x,y).$$

Therefore, for every $x\in\mathbb{R}^d_+$, using the change of variables $z=(y-(\widetilde{x},0))/x_d$ in the second equality below and $s=x_d^{-2}t$ in the third, we get that

$$\begin{aligned}\kappa^\gamma(x)&=\frac{\alpha/2}{\Gamma(1-\alpha/2)}\int_0^\infty\left(1-\int_{\mathbb{R}^d_+}q^\gamma(t,(\widetilde{0},x_d),(\widetilde{y}-\widetilde{x},y_d))dy\right)t^{-\alpha/2-1}dt\\&=\frac{\alpha/2}{\Gamma(1-\alpha/2)}\int_0^\infty\left(1-x_d^d\int_{\mathbb{R}^d_+}q^\gamma(t,(\widetilde{0},x_d),x_dz)dz\right)t^{-\alpha/2-1}dt\\&=\frac{\alpha/2}{\Gamma(1-\alpha/2)}\int_0^\infty\left(1-\int_{\mathbb{R}^d_+}q^\gamma(x_d^{-2}t,(\widetilde{0},1),z)dz\right)t^{-\alpha/2-1}dt\\&=\frac{\alpha/2}{\Gamma(1-\alpha/2)}x_d^{-\alpha}\int_0^\infty\left(1-\int_{\mathbb{R}^d_+}q^\gamma(s,(\widetilde{0},1),z)dz\right)s^{-\alpha/2-1}ds=\kappa^\gamma((\widetilde{0},1))x_d^{-\alpha}.\end{aligned}$$

(ii) In view of (2.3), the condition $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ in [21] holds with $C_0=0$, $\Phi(r)=r^2$ and $h(t,x,y)=(1\wedge(x_dy_d/t))^{\gamma+1/2}$. Note that the tail of the Lévy measure of S is given by $\int_r^\infty\nu_{\alpha/2}(u)du=r^{-\alpha/2}/\Gamma(1-\alpha/2)$. Thus, the condition **(Poly-∞)** in [21] also holds and we obtain from [21, Theorem 4.1] that for $x,y\in\mathbb{R}^d_+$,

$$J^\gamma(x,y)\asymp\left(1\wedge\frac{x_dy_d}{|x-y|^2}\right)^{\gamma+1/2}\frac{1}{|x-y|^{d+\alpha}}\asymp\left(1\wedge\frac{x_d}{|x-y|}\right)^{\gamma+1/2}\left(1\wedge\frac{y_d}{|x-y|}\right)^{\gamma+1/2}\frac{1}{|x-y|^{d+\alpha}}.$$

In the second comparison, we used the fact that $(1\wedge\frac{x_d}{|x-y|})(1\wedge\frac{y_d}{|x-y|})\le(1\wedge\frac{x_dy_d}{|x-y|^2})\le2(1\wedge\frac{x_d}{|x-y|})(1\wedge\frac{y_d}{|x-y|})$ for all $x,y\in\mathbb{R}^d_+$, which can be proved elementarily by using $y_d\le x_d+|x-y|$.

(iii) Since the conditions **(Poly-∞)** and $\mathbf{HK}^{\mathbf{h}}_{\mathbf{U}}$ in [21] hold, the result follows from [21, Proposition 4.5(ii)]. □

Denote by $(\mathcal{C}^\gamma,\mathcal{D}(\mathcal{C}^\gamma))$ the regular Dirichlet form associated with the subordinate process X^γ. Then since X^γ has no diffusion part, we get from Lemma 2.4(i) that

$$\mathcal{C}^\gamma(u,v)=\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J^\gamma(x,y)dydx+c_{\gamma,\alpha}\int_{\mathbb{R}^d_+}u(x)v(x)x_d^{-\alpha}dx.$$

Also, we have $C_c^\infty(\mathbb{R}^d_+)\subset\mathcal{D}(\mathcal{C}^\gamma)$ since $\mathcal{D}(\mathcal{Q}^\gamma)\subset\mathcal{D}(\mathcal{C}^\gamma)$, see [34].

Lemma 2.5. *There exists a constant $C>0$ such that*

$$\|u\|^{2(1+\alpha/d)}_{L^2(\mathbb{R}^d_+,dx)} \le C\mathcal{C}^\gamma(u,u) \quad \text{for every } u\in C_c^\infty(\mathbb{R}^d_+) \text{ with } \|u\|_{L^1(\mathbb{R}^d_+,dx)}\le 1.$$

Proof. By [7, Theorem 2.1] (see also [11, Theorem 3.4] and [22, Theorem II.5]), the result follows from Lemma 2.4(iii). □

Proposition 2.6. *There exists a constant $C>0$ such that*

$$\|u\|^{2(1+\alpha/d)}_{L^2(\mathbb{R}^d_+,dx)} \le C\mathcal{E}^0(u,u) \quad \text{for every } u\in \overline{\mathcal{F}} \text{ with } \|u\|_{L^1(\mathbb{R}^d_+,dx)}\le 1. \tag{2.4}$$

Proof. We first assume that $\alpha<1$. Let $\gamma=\beta_1\vee\beta_2$. Using Lemmas 2.5 and 2.4(i)–(ii), the Hardy inequality in [32, Proposition 3.2] and **(A3)**(I), we get that for any $u\in C_c^\infty(\mathbb{R}^d_+)$ with $\|u\|_{L^1(\mathbb{R}^d_+,dx)}\le 1$,

$$\begin{aligned}\|u\|^{2(1+\alpha/d)}_{L^2(\mathbb{R}^d_+,dx)} &\le c_1\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))^2\frac{B_{\gamma+1/2,\gamma+1/2,0,0}(x,y)}{|x-y|^{d+\alpha}}dydx + c_{\gamma,\alpha}\int_{\mathbb{R}^d_+}u(x)^2x_d^{-\alpha}dx\\ &\le c_2\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))^2\frac{B_{\gamma+1/2,\gamma+1/2,0,0}(x,y)}{|x-y|^{d+\alpha}}dydx \le c_3\mathcal{E}^0(u,u),\end{aligned}$$

where $B_{\gamma+1/2,\gamma+1/2,0,0}$ is defined in (1.1). By Lemma 2.3, $\overline{\mathcal{F}}$ is the closure of $C_c^\infty(\mathbb{R}^d_+)$ under $\mathcal{E}^0_1$. Therefore, we conclude that (2.4) is true when $\alpha<1$.

Now, we assume that $\alpha\ge 1$. Since (2.4) is valid when $\alpha<1$ and $C_c^\infty(\overline{\mathbb{R}}^d_+)\subset\overline{\mathcal{F}}$, we get that for every $u\in C_c^\infty(\overline{\mathbb{R}}^d_+)$ with $\|u\|_{L^1(\mathbb{R}^d_+,dx)}\le 1$,

$$\begin{aligned}\mathcal{E}^0(u,u) &\ge \frac12\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))^2\frac{\mathcal{B}(x,y)}{|x-y|^{d+1/2}}\mathbf{1}_{\{|x-y|\le 1\}}dydx\\ &= \frac12\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))^2\frac{\mathcal{B}(x,y)}{|x-y|^{d+1/2}}(1-\mathbf{1}_{\{|x-y|>1\}})dydx\\ &\ge c_4\|u\|^{2(1+1/(2d))}_{L^2(\mathbb{R}^d_+,dx)} - 2C_2\|u\|^2_{L^2(\mathbb{R}^d_+,dx)}\sup_{x\in\mathbb{R}^d_+}\int_{\mathbb{R}^d_+,|x-y|>1}\frac{dy}{|x-y|^{d+1/2}}\\ &\ge c_4\|u\|^{2(1+1/(2d))}_{L^2(\mathbb{R}^d_+,dx)} - c_5\|u\|^2_{L^2(\mathbb{R}^d_+,dx)}.\end{aligned}$$

Since $\overline{\mathcal{F}}$ is the closure of $C_c^\infty(\overline{\mathbb{R}}^d_+)$ under $\mathcal{E}^0_1$, [7, Theorem 2.1] yields that $\|\overline{P}_t\|_{1\to\infty}\le c_6t^{-2d}e^{c_5t}$ for all $t>0$. By (2.1), it follows that $\|\overline{P}_t\|_{1\to\infty}=t^{-d/\alpha}\|\overline{P}_1\|_{1\to\infty}\le c_6e^{c_5}t^{-d/\alpha}$ for all $t>0$. Using [7, Theorem 2.1] again, we conclude that (2.4) holds for $\alpha\ge1$ and finish the proof. □

As a consequence of the Nash-type inequality (2.4), we get the existence and a priori upper bounds of the heat kernels $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$ of $\overline{Y}$ and Y^κ respectively.

Proposition 2.7. *Let $\kappa\ge0$. The processes $\overline{Y}$ and Y^κ have heat kernels $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$ defined on $(0,\infty)\times(\overline{\mathbb{R}}^d_+\setminus\mathcal{N})\times(\overline{\mathbb{R}}^d_+\setminus\mathcal{N})$ and $(0,\infty)\times(\mathbb{R}^d_+\setminus\mathcal{N})\times(\mathbb{R}^d_+\setminus\mathcal{N})$ respectively, where $\mathcal{N}\subset\overline{\mathbb{R}}^d_+$ is a properly exceptional set for $\overline{Y}$. Moreover, there exists a constant $C>0$ such that*

$$p^\kappa(t,x,y)\le\overline{p}(t,x,y),\quad x,y\in\mathbb{R}^d_+\setminus\mathcal{N} \tag{2.5}$$

and

$$\overline{p}(t,x,y)\le C\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right),\quad t>0,\ x,y\in\overline{\mathbb{R}}^d_+\setminus\mathcal{N}. \tag{2.6}$$

Proof. By our Nash-type inequality (2.4) and [3], $\overline{Y}$ has a heat kernel $\overline{p}(t,x,y)$ on $(0,\infty)\times(\overline{\mathbb{R}}^d_+\setminus\mathcal{N})\times(\overline{\mathbb{R}}^d_+\setminus\mathcal{N})$ for a properly exceptional set $\mathcal{N}$ and

$$\overline{p}(t,x,y)\le ct^{-d/\alpha},\quad t>0,\ x,y\in\overline{\mathbb{R}}^d_+\setminus\mathcal{N}. \tag{2.7}$$

Since $\mathcal{B}(x,y)$ is bounded above, using (2.7), we can follow the arguments given in [11, Example 5.5] line by line and conclude that (2.6) holds. According to [32] (see the discussion before Lemma 2.1 there), Y^κ can be realized as a subprocess of $\overline{Y}$. Thus, Y^κ has a heat kernel $p^\kappa(t,x,y)$ defined on $(0,\infty)\times(\mathbb{R}^d_+\setminus\mathcal{N})\times(\mathbb{R}^d_+\setminus\mathcal{N})$ and we also obtain (2.5). □

For notational convenience, we extend the domain of $p^\kappa(t,x,y)$ to $(0,\infty)\times(\overline{\mathbb{R}}^d_+\setminus\mathcal{N})\times(\overline{\mathbb{R}}^d_+\setminus\mathcal{N})$ by letting $p^\kappa(t,x,y)=0$ if $x\in\partial\mathbb{R}^d_+\setminus\mathcal{N}$ or $y\in\partial\mathbb{R}^d_+\setminus\mathcal{N}$.

3. Parabolic Hölder regularity and consequences

For $\kappa\ge 0$ and an open set $D\subset\overline{\mathbb{R}}^d_+$ relative to the topology on $\overline{\mathbb{R}}^d_+$, we denote by $\overline{Y}^D$ and $\overline{P}^D_t$ the part of the process $\overline{Y}$ killed upon exiting D and its semigroup, and by $Y^{\kappa,D}$ and $P^{\kappa,D}_t$ the part of the process Y^κ killed upon exiting $D\cap\mathbb{R}^d_+$ and its semigroup, respectively. The Dirichlet forms of $\overline{Y}^D$ and $Y^{\kappa,D}$ are $(\mathcal{E}^0,\overline{\mathcal{F}}^D)$ and $(\mathcal{E}^\kappa,\mathcal{F}^{\kappa,D})$, where $\overline{\mathcal{F}}^D=\{u\in\overline{\mathcal{F}}:u=0\text{ q.e. on }\overline{\mathbb{R}}^d_+\setminus D\}$ and $\mathcal{F}^{\kappa,D}=\{u\in\mathcal{F}^\kappa:u=0\text{ q.e. on }\mathbb{R}^d_+\setminus D\}$ respectively. For $u,v\in\mathcal{F}^{\kappa,D}$,

$$\mathcal{E}^\kappa(u,v)=\frac12\int_D\int_D((u(x)-u(y))(v(x)-v(y))J(x,y)dydx+\int_D u(x)v(x)\kappa_D(x)dx, \tag{3.1}$$

where

$$\kappa_D(x)=\int_{\mathbb{R}^d_+\setminus D}J(x,y)dy+\kappa x_d^{-\alpha}. \tag{3.2}$$

Let $\overline{\tau}_D=\inf\{t>0:\overline{Y}_t\notin D\}$ and $\tau^\kappa_D=\inf\{t>0:Y^\kappa_t\notin D\cap\mathbb{R}^d_+\}$. For $x,y\notin\mathcal{N}$, let

$$\begin{aligned}\overline{p}^D(t,x,y)&=\overline{p}(t,x,y)-\mathbb{E}_x\big[\overline{p}(t-\overline{\tau}_D,\overline{Y}_{\overline{\tau}_D},y);\ \overline{\tau}_D<t\big],\\ p^{\kappa,D}(t,x,y)&=p^\kappa(t,x,y)-\mathbb{E}_x\big[p^\kappa(t-\tau^\kappa_D,Y^\kappa_{\tau^\kappa_D},y);\ \tau^\kappa_D<t\big].\end{aligned} \tag{3.3}$$

By the strong Markov property, $\overline{p}^D(t,x,y)$ and $p^{\kappa,D}(t,x,y)$ are the transition densities of $\overline{Y}^D$ and $Y^{\kappa,D}$ respectively.

In case when D is a relatively compact open subset of $\mathbb{R}^d_+$, one can show that the process $Y^{\kappa,D}$ can start from every point in D and provide an interior lower bound for its transition density. We accomplish this by identifying the semigroup $P^{\kappa,D}_t$ with the Feynman-Kac semigroup of the part process on D of an auxiliary process. This idea has already been used in [30, Subsection 3.1]. For the benefit of the reader, we repeat some of the details here. Note that, unlike [30, Subsection 3.1], **(A3)**(II) is not assumed.

Recall that we denote by m_d the Lebesgue measure on $\mathbb{R}^d$.

Proposition 3.1. *If D is a relatively compact open subset of $\mathbb{R}^d_+$, then $Y^{\kappa,D}$ has a transition density $p^{\kappa,D}(t,x,y)$ defined for any $(t,x,y)\in(0,\infty)\times D\times D$. Furthermore, for any $T>0$ and $b\in(0,1]$, there exists a constant $C=C(b,T,D)>0$ such that*

$$p^{\kappa,D}(t,x,y)\ge C\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\quad t\in(0,T),\ x,y\in D\ \text{such that}\ \delta_D(x)\wedge\delta_D(y)>bt^{1/\alpha}.$$

Proof. For $\gamma>0$ let $J_\gamma:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ be defined by $J_\gamma(x,y)=J(x,y)$ if $x,y\in D$, and $J_\gamma(x,y)=\gamma|x-y|^{-\alpha-d}$ otherwise. It follows from **(A3)**(I) and the relative compactness of D that $J_\gamma(x,y)\asymp|x-y|^{-\alpha-d}$. Hence by [16, Theorem 1.2], there exists a Feller and strongly Feller process Z (that can start from every point in $\mathbb{R}^d$) with a continuous transition density $\widetilde{q}(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ satisfying

$$c_1^{-1}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\le\widetilde{q}(t,x,y)\le c_1\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right),\quad t>0,\ x,y\in\mathbb{R}^d,\tag{3.4}$$

for some constant $c_1\ge 1$.

Denote the part of the process Z killed upon exiting D by Z^D. The Dirichlet form of Z^D is $(\mathcal{C},\mathcal{D}_D(\mathcal{C}))$, where for $u,v\in\mathcal{D}_D(\mathcal{C})$,

$$\begin{aligned}\mathcal{C}(u,v) &= \frac12\int_D\int_D(u(x)-u(y))(v(x)-v(y))J_\gamma(x,y)\,dy\,dx+\int_D u(x)v(x)\kappa_D^Z(x)\,dx\\ &= \frac12\int_D\int_D(u(x)-u(y))(v(x)-v(y))J(x,y)\,dy\,dx+\int_D u(x)v(x)\kappa_D^Z(x)\,dx\end{aligned}$$

with

$$\kappa_D^Z(x)=\int_{\mathbb{R}^d\setminus D}J_\gamma(x,y)\,dy=\gamma\int_{\mathbb{R}^d\setminus D}|x-y|^{-d-\alpha}\,dy\,,\quad x\in D\,,\tag{3.5}$$

and $\mathcal{D}_D(\mathcal{C})=\{u\in\mathcal{D}(\mathcal{C}):\ u=0 \text{ q.e. on } \mathbb{R}^d\setminus D\}$.

Let $\delta_D=\operatorname{dist}(D,\partial\mathbb{R}^d_+)$ and let V be the $\delta_D/2$-neighborhood of D, that is $V:=\{x\in\mathbb{R}^d_+:\operatorname{dist}(x,D)<\delta_D/2\}$. Then

$$\kappa_D(x)=\int_{\mathbb{R}^d_+\setminus V}J(x,y)\,dy+\int_{V\setminus D}J(x,y)\,dy+\kappa x_d^{-\alpha}\,.$$

It follows from **(A3)**(I) and the relative compactness of D that $c_2|x-y|^{-d-\alpha}\le J(x,y)\le C_2|x-y|^{-d-\alpha}$ for all $x,y\in V$ with $c_2:=c_2(D)>0$. It is easy to see that $\sup_{x\in D}\int_{\mathbb{R}^d_+\setminus V}J(x,y)\,dy=:c_3\ <\infty$. Therefore

$$c_2\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\le\kappa_D(x)\le c_3+C_2\int_{V\setminus D}|x-y|^{-d-\alpha}dy+c_4\,,\quad x\in U\,,$$

where $c_4:=\kappa\sup_{x\in D}x_d^{-\alpha}$. Since

$$\inf_{x\in D}\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\ge m_d(V\setminus D)\operatorname{diam}(V)^{-d-\alpha}=:c_5>0\,,$$

we conclude that

$$c_2\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\le\kappa_D(x)\le c_6\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\,,\quad x\in D\,.$$

Further, since

$$\kappa_D^Z(x)=\gamma\bigg(\int_{\mathbb{R}^d\setminus V}|x-y|^{-d-\alpha}\,dy+\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\bigg)\,,\quad x\in D$$

and $\sup_{x\in D}\int_{\mathbb{R}^d\setminus V}|x-y|^{-d-\alpha}\,dy=:c_7<\infty$, we see that there is a constant $c_8>0$ such

$$\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\le\gamma^{-1}\kappa_D^Z(x)\le c_8\int_{V\setminus D}|x-y|^{-d-\alpha}\,dy\,,\quad x\in D,.$$

It follows that $c_6^{-1}\kappa_D(x)\le\gamma^{-1}\kappa_D^Z(x)\le c_8c_2^{-1}\kappa_D(x)$ for all $x\in D$ with positive constants c_2,c_6,c_8 not depending on γ. Now we choose $\gamma>0$ so small that $\gamma c_8c_2^{-1}\le 1$. With this choice we get that $\kappa_D^Z(x)\le\kappa_D(x)$ for all $x\in D$. In particular, with $c_9:=\gamma c_6^{-1}$ we see that

$$c_9\kappa_D(x)\le\kappa_D^Z(x)\le\kappa_D(x)\,,\qquad x\in D\,.\tag{3.6}$$

It follows that for $u\in C_c^\infty(D)$,

$$\mathcal{E}_1^{\kappa,D}(u,u)\ =\ \mathcal{E}^{\kappa,D}(u,u)+\int_U u(x)^2\,dx$$

$$\begin{aligned}
&= \frac{1}{2}\int_D\int_D (u(x)-u(y))^2 J(x,y)\,dy\,dx + \int_D u(x)^2\kappa_D(x)\,dx + \int_U u(x)^2 dx \\
&\asymp \frac{1}{2}\int_D\int_D (u(x)-u(y))^2 J_\gamma(x,y)\,dy\,dx + \int_D u(x)^2\kappa_D^Z(x)\,dx + \int_D u(x)^2 dx \\
&= \mathcal{C}(u,u) + \int_D u(x)^2 dx = \mathcal{C}_1(u,u)\,.
\end{aligned}$$

Since $C_c^\infty(D)$ is a core of both $(\mathcal{E}^\kappa, \mathcal{F}^{\kappa,D})$ and $(\mathcal{C}, \mathcal{D}_D(\mathcal{C}))$, we conclude that $\mathcal{F}^{\kappa,D} = \mathcal{D}_D(\mathcal{C})$.

Define $\widetilde{\kappa} : D \to \mathbb{R}$ by $\widetilde{\kappa}(x) := \kappa_D(x) - \kappa_D^Z(x)$, $x \in D$. By the choice of γ we have that $\widetilde{\kappa} \geq 0$. On the other hand, it follows from (3.5) and (3.6) that there is a constant $c_{10} > 0$ such that

$$\widetilde{\kappa}(x) \leq \kappa_D(x) \leq c_{10}\delta_D(x)^{-\alpha}\,, \quad x \in D. \tag{3.7}$$

Let $\mu(dx) = \widetilde{\kappa}(x)\,dx$ be a measure on D. Using (3.4) and (3.7) one can check that $\mu \in \mathbf{K}_1(D)$, where the class $\mathbf{K}_1(D)$ is defined in [20, Definition 2.12].

For any Borel function $f : D \to [0,\infty)$ let

$$T_t^{D,\widetilde{\kappa}} f(x) = \mathbb{E}_x\left[\exp\left(-\int_0^t \widetilde{\kappa}(Z_s^D)ds\right) f(Z_t^D)\right], \quad t > 0,\ x \in D. \tag{3.8}$$

be the Feynman-Kac transform of the semigroup corresponding to the killed process Z^D. By [20, Theorem 2.15], the semigroup $(T_t^{D,\widetilde{\kappa}})_{t\geq 0}$ has a transition density $p^{Z,D}(t,x,y)$ (with respect to the Lebesgue measure) such that for every $T > 0$ and $b \in (0,1]$ there exists a constant $c_2 = c_2(b,T,D) > 0$ such that

$$p^{Z,D}(t,x,y) \geq c_2\widetilde{q}(t,x,y), \quad t \in (0,T),\ x,y \in D \text{ such that } \delta_D(x)\wedge\delta_D(y) > bt^{1/\alpha}. \tag{3.9}$$

Finally, by computing the Dirichlet forms of $Y^{\kappa,D}$ and Z^D (for the latter use [23, Theorem 6.1.2]), we conclude that they coincide. This implies that $P_t^{\kappa.D} = T_t^{D,\widetilde{\kappa}}$. Combining the lower bound in (3.4) with (3.9), the proof is complete. □

Recall that $\mathbf{e}_d = (\widetilde{0}, 1)$. Define for $a \in (0,1]$,

$$S(a) = \{(\widetilde{z}, z_d) \in \mathbb{R}_+^d : |\widetilde{z}| < 2/a,\ a/(2a+2) < z_d < (2a+2)/a\}.$$

Note that $B(\mathbf{e}_d, 1/8) \subset S(a)$ for all $a \in (0,1]$. Moreover, for any $a \in (0,1]$, $x \in B(\mathbf{e}_d, 1/8)$ and $y = (\widetilde{y}, y_d) \in \mathbb{R}_+^d$ with $x_d \wedge y_d > a|x-y|$, we have $|\widetilde{y}| \leq |\widetilde{x}| + |y-x| < 1/8 + x_d/a < 2/a$, $y_d + y_d/a \geq x_d - |x-y| + |x-y| \geq 7/8$ and $y_d \leq x_d + |x-y| \leq x_d + x_d/a < (2a+2)/a$. Thus,

$$\begin{aligned}
&\left\{(t, (\widetilde{x}, x_d), (\widetilde{y}, y_d)) \in \mathbb{R}_+^1 \times B(\mathbf{e}_d, 1/8) \times \mathbb{R}_+^d : x_d \wedge y_d > a(t^{1/\alpha} \vee |x-y|)\right\} \\
&\subset (0, 2/a^\alpha) \times S(a) \times S(a) \qquad \text{for every } a \in (0,1].
\end{aligned} \tag{3.10}$$

Proposition 3.2. *For any $a \in (0,1]$, there exists a constant $C = C(a) > 0$ such that the following estimates hold: For any $t > 0$ and $x \in \mathbb{R}_+^d \setminus \mathcal{N}$, there is a measurable set $N_{t,x} \subset \mathbb{R}_+^d$ of zero Lebesgue measure such that for all $y \in \mathbb{R}_+^d \setminus N_{t,x}$ with $x_d \wedge y_d > a(t^{1/\alpha} \vee |x-y|)$,*

$$\overline{p}(t,x,y) \geq p^\kappa(t,x,y) \geq C\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{3.11}$$

Proof. By (2.5), it suffices to prove the second inequality in (3.11). Since $\mathcal{N}$ has zero Lebesgue measure, there exist $r > 0$ and $\widetilde{z} \in \mathbb{R}^{d-1}$ such that $r^{-1}(x - (\widetilde{z}, 0)) \in B(\mathbf{e}_d, 1/8) \setminus \mathcal{N}$. By Lemmas 2.1 and 2.2, we have

$$p^\kappa(t,x,y) = r^{-d}p^\kappa(r^{-\alpha}t, r^{-1}(x - (\widetilde{z},0)), r^{-1}(y - (\widetilde{z},0))) \qquad \text{for a.e. } y \in \mathbb{R}_+^d. \tag{3.12}$$

Therefore, by (3.10), it is enough to prove that there exists a constant $c = c(a) > 0$ such that for all $0 < t < 2/a^\alpha$, $x \in B(\mathbf{e}_d, 1/8) \setminus \mathcal{N}$ and a.e. $y \in S(a) \setminus \mathcal{N}$,

$$p^{\kappa,S(a/2)}(t,x,y) \geq c\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \tag{3.13}$$

which implies the second inequality in (3.11) by (3.3) and (3.12).

Note that $\delta_{S(a/2)}(y) \geq a/(a^2+3a+2)$ for every $y \in S(a)$. Hence for all $0 < t < 2/a^\alpha$ and $z, y \in S(a/2)$ it holds that $\delta_{S(a/2)}(z) \wedge \delta_{S(a/2)}(y) \geq (a^2/(2a^2+6a+4))t^{1/\alpha}$. Now it follows from Proposition 3.1 (with $b = (a^2/(2a^2+6a+4))$) that (3.13) holds. □

As a direct consequence of Proposition 3.2, for any $a \in (0,1]$, there exists a constant $C > 0$ such that

$$\overline{p}(t,x,y) \geq p^\kappa(t,x,y) \geq Ct^{-d/\alpha} \tag{3.14}$$

for all $t > 0$, $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$ and a.e. $y \in \mathbb{R}^d_+ \setminus \mathcal{N}$ with $x_d \wedge y_d > at^{1/\alpha}$ and $|x-y| \leq t^{1/\alpha}$.

By repeating the proofs of [21, Lemmas 6.1 and 6.3], we obtain the following two results from (3.3), Proposition 2.7 and (3.14).

Lemma 3.3. *There exist constants $C > 0$ and $\eta \in (0,1/4)$ such that for all $x \in \mathbb{R}^d_+$, $r \in (0, x_d)$, $t \in (0,(\eta r)^\alpha]$ and $z \in B(x, \eta t^{1/\alpha}) \setminus \mathcal{N}$,*

$$\overline{p}^{B(x,r)}(t,z,y) \geq p^{\kappa, B(x,r)}(t,z,y) \geq Ct^{-d/\alpha} \quad \text{for a.e. } y \in B(x, \eta t^{1/\alpha}).$$

Lemma 3.4. *There exists a constant $C > 1$ such that for all $x \in \mathbb{R}^d_+ \setminus \mathcal{N}$ and $r \in (0, x_d)$,*

$$C^{-1} r^\alpha \leq \mathbb{E}_x[\tau^\kappa_{B(x,r)}] \leq \sup_{z \in B(x,r)\setminus\mathcal{N}} \mathbb{E}_z[\overline{\tau}_{B(x,r)}] \leq Cr^\alpha. \tag{3.15}$$

The Lévy system formula (see [23, Theorem 5.3.1] and the arguments in [15, p.40]) state that for any non-negative Borel function F on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ vanishing on the diagonal and any stopping time T for Y^κ, it holds that

$$\mathbb{E}_x \sum_{s \leq T} F(Y^\kappa_{s-}, Y^\kappa_s) = \mathbb{E}_x \int_0^T \int_{\mathbb{R}^d_+} F(Y^\kappa_s, y) J(Y^\kappa_s, y) dy ds, \quad x \in \mathbb{R}^d_+ \setminus \mathcal{N}. \tag{3.16}$$

Here $Y^\kappa_{s-} = \lim_{t \uparrow s} Y^\kappa_t$ denotes the left limit of the process Y at time $s > 0$. Similarly, for any non-negative Borel function F on $\overline{\mathbb{R}}^d_+ \times \overline{\mathbb{R}}^d_+$ vanishing on the diagonal and any stopping time T for $\overline{Y}$, it holds that

$$\mathbb{E}_x \sum_{s \leq T} F(\overline{Y}_{s-}, \overline{Y}_s) = \mathbb{E}_x \int_0^T \int_{\mathbb{R}^d_+} F(\overline{Y}_s, y) J(\overline{Y}_s, y) dy ds, \quad x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}. \tag{3.17}$$

See [31, (3.3) and (3.4)] for a simper form following from (3.16), which will be used in this paper too.

For $x = (\widetilde{x}, x_d) \in \overline{\mathbb{R}}^d_+$ and $t > 0$, we define

$$\begin{aligned} V_x(t) &= \{z = (\widetilde{z}, z_d) \in \mathbb{R}^d : |\widetilde{z} - \widetilde{x}| < 2t^{1/\alpha},\ z_d \in [0, 2t^{1/\alpha})\}, \\ W_x(t) &= \{z = (\widetilde{z}, z_d) \in \mathbb{R}^d : |\widetilde{z} - \widetilde{x}| < 2t^{1/\alpha},\ z_d \in [x_d + 5t^{1/\alpha}, x_d + 8t^{1/\alpha})\}. \end{aligned} \tag{3.18}$$

In dimension 1, we abuse notation and use $V_x(t) = [0, 2t^{1/\alpha})$ and $W_x(t) = [x + 5t^{1/\alpha}, x + 8t^{1/\alpha})$.

Lemma 3.5. *(i) There exists $C > 0$ such that for all $t > 0$ and $x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$,*

$$\mathbb{P}_x\big(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\big) \geq C.$$

(ii) There exists a constant $C > 0$ such that for all $n \geq 1$, $t > 0$ and $x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$,

$$\mathbb{P}_x(\overline{\tau}_{V_x(t)} > nt) \leq 2e^{-Cn}.$$

Proof. (i) Define $V_x(t,r) = \{z \in V_x(t) : \delta_{V_x(t)}(z) > rt^{1/\alpha}\}$ for $r > 0$. For any $r \in (0,1)$, $z \in V_x(t,r)$, $u \in B(z, rt^{1/\alpha}/2)$ and $w \in W_x(t)$, we have

$$rt^{1/\alpha}/2 \leq u_d \leq 5t^{1/\alpha} \leq w_d \quad \text{and} \quad |u-w| \leq |u-z| + |z-x| + |x-w| < 16t^{1/\alpha}. \tag{3.19}$$

Using the Lévy system formula in (3.17), **(A3)**(I), (3.19) and Lemma 3.4, we get that for any $r \in (0,1)$ and $z \in V_x(t,r) \setminus \mathcal{N}$,

$$\begin{aligned}\mathbb{P}_z(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)) &\geq \mathbb{P}_z(\overline{Y}_{\overline{\tau}_{B(z,rt^{1/\alpha}/2)}} \in W_x(t)) = \mathbb{E}_z\Big[\int_0^{\overline{\tau}_{B(z,rt^{1/\alpha}/2)}} \int_{W_x(t)} J(\overline{Y}_s, w)dwds\Big] \\ &\geq c_1\mathbb{E}_z\Big[\int_0^{\overline{\tau}_{B(z,rt^{1/\alpha}/2)}} \int_{W_x(t)} \Big(\frac{r/2}{16}\Big)^{\beta_1}\Big(\frac{5}{16}\Big)^{\beta_2}(16t^{1/\alpha})^{-d-\alpha}dwds\Big] \\ &\geq c_2 r^{\beta_1} t^{-1-d/\alpha}\mathbb{E}_z\big[\overline{\tau}_{B(z,rt^{1/\alpha}/2)}\big]\int_{W_x(t)} dw \geq c_3 r^{\beta_1+\alpha}. \end{aligned} \tag{3.20}$$

By Proposition 2.7 and [25, Remark 3.3], the condition (i) in [25, Theorem 3.1] holds true with $\rho(s) = s^{1/\alpha}$. Then, since $\overline{Y}$ is conservative, by the implication (i) $\Rightarrow$ (ii) of [25, Theorem 3.1], there exists a constant $\varepsilon_0 > 0$ independent of t and x such that

$$\mathbb{P}_x(\overline{\tau}_{V_x}(t) > \varepsilon_0 t) \geq 1/2. \tag{3.21}$$

On the other hand, for all $r > 0$, we get from Proposition 2.7 that

$$\begin{aligned}\mathbb{P}_x(\overline{\tau}_{V_x(t)\setminus V_x(t,r)} > \varepsilon_0 t) &\leq \mathbb{P}_x(\overline{Y}_{\varepsilon_0 t} \in V_x(t)\setminus V_x(t,r)) \\ &\leq \int_{V_x(t)\setminus V_x(t,r)} p(\varepsilon_0 t, x, y)dy \leq c_4 t^{-d/\alpha}\int_{V_x(t)\setminus V_x(t,r)} dy \leq c_5 r. \end{aligned} \tag{3.22}$$

Set $r_0 := 1/(4c_5+1)$. Using the strong Markov property and (3.20)–(3.22), we obtain

$$\begin{aligned}&\mathbb{P}_x\big(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\big) \geq \mathbb{P}_x\big(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t),\, \overline{\tau}_{V_x(t)\setminus V_x(t,r_0)} \leq \varepsilon_0 t < \overline{\tau}_{V_x(t)}\big) \\ &\geq \mathbb{P}_x\Big(\mathbb{P}_{\overline{Y}_{\overline{\tau}_{V_x(t)\setminus V_x(t,r_0)}}}\big(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\big) \,:\, \overline{\tau}_{V_x(t)\setminus V_x(t,r_0)} \leq \varepsilon_0 t < \overline{\tau}_{V_x(t)}\Big) \\ &\geq c_3 r_0^{\beta_1+\alpha}\mathbb{P}_x\big(\overline{\tau}_{V_x(t)\setminus V_x(t,r_0)} \leq \varepsilon_0 t < \overline{\tau}_{V_x(t)}\big) \\ &\geq c_3 r_0^{\beta_1+\alpha}\big(\mathbb{P}_x\big(\overline{\tau}_{V_x(t)} > \varepsilon_0 t\big) - \mathbb{P}_x\big(\overline{\tau}_{V_x(t)\setminus V_x(t,r_0)} > \varepsilon_0 t\big)\big) \geq c_3 r_0^{\beta_1+\alpha}/4. \end{aligned}$$

(ii) By Proposition 2.7, there exists a constant $k_0 > 0$ independent of t and x such that for any $z \in V_x(t) \setminus \mathcal{N}$,

$$\mathbb{P}_z\big(\overline{\tau}_{V_x(t)} > k_0 t\big) \leq \mathbb{P}_z\big(\overline{Y}_{k_0 t} \in V_x(t)\big) \leq \int_{V_x(t)} \overline{p}(k_0 t, z, y)dy \leq c_1 t^{d/\alpha}(k_0 t)^{-d/\alpha} \leq \frac{1}{2}. \tag{3.23}$$

For $r > 0$, let $\lfloor r \rfloor := \sup\{m \in \mathbb{N} : m \leq r\}$. Now for any $n \geq 1$, using the Markov property and (3.23), we get that

$$\begin{aligned}&\mathbb{P}_x(\overline{\tau}_{V_x(t)} > nt) \leq \mathbb{P}_x(\overline{\tau}_{V_x(t)} > \lfloor n/k_0 \rfloor k_0 t) \\ &\leq \mathbb{P}_x\big(\mathbb{P}_{X_{(\lfloor n/k_0\rfloor-1)k_0 t}}(\overline{\tau}_{V_x(t)} > k_0 t) : \overline{\tau}_{V_x(t)} > (\lfloor n/k_0 \rfloor - 1)k_0 t\big) \\ &\leq 2^{-1}\mathbb{P}_x\big(\overline{\tau}_{V_x(t)} > (\lfloor n/k_0 \rfloor - 1)k_0 t\big) \leq \cdots \leq 2^{-\lfloor n/k_0 \rfloor} \leq 2e^{-(\log 2)n/k_0}. \end{aligned}$$

□

Lemma 3.6. *There exist constants $M > 1$ and $C > 0$ such that for all $t > 0$ and $x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$,*

$$\operatorname*{ess\,inf}_{z\in W_x(t)} \overline{p}(Mt, x, z) \geq Ct^{-d/\alpha}.$$

Proof. Suppose $t > 0$ and $x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$. For all $w, z \in W_x(t)$, we have $w_d \wedge z_d \geq 5t^{1/\alpha}$ and $|w - z| < 7t^{1/\alpha}$. Therefore, by Proposition 3.2, for any $M > 1$, there is $c_1 = c_1(M) > 0$ independent of t and x such that for all $w \in W_x(t) \setminus \mathcal{N}$, $z \in W_x(t)$ and $\varepsilon \in (0, t^{1/\alpha})$,

$$\inf_{t\leq s\leq Mt} \mathbb{P}_w\big(\overline{Y}_s \in B(z,\varepsilon)\big) = \inf_{t\leq s\leq Mt}\int_{B(z,\varepsilon)} \overline{p}(s, w, y)dy \geq c_1 t^{-d/\alpha} m_d(B(z,\varepsilon)). \tag{3.24}$$

By the strong Markov property and (3.24), for all $M > 1$, $z \in W_x(t)$ and $\varepsilon \in (0, t^{1/\alpha})$,

$$\begin{aligned}
\mathbb{P}_x\left(\overline{Y}_{Mt} \in B(z,\varepsilon)\right) &\ge \mathbb{E}_x\left[\mathbb{P}_{\overline{Y}_{\overline{\tau}_{V_x(t)}}}\left(\overline{Y}_{Mt-\overline{\tau}_{V_x(t)}} \in B(z,\varepsilon)\right) : \overline{\tau}_{V_x(t)} \le (M-1)t,\, \overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\right] \\
&\ge \Big(\inf_{t\le s\le Mt}\ \inf_{w\in W_x(t)\setminus\mathcal{N}} \mathbb{P}_w\big(\overline{Y}_s \in B(z,\varepsilon)\big)\Big)\, \mathbb{P}_x\left(\overline{\tau}_{V_x(t)} \le (M-1)t,\, \overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\right) \\
&\ge c_1 t^{-d/\alpha} m_d(B(z,\varepsilon))\left(\mathbb{P}_x\big(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\big) - \mathbb{P}_x\big(\overline{\tau}_{V_x(t)} > (M-1)t\big)\right). \qquad (3.25)
\end{aligned}$$

By Lemma 3.5(i)-(ii), there are constants $c_2, c_3, c_4 > 0$ such that for all $M > 1$,

$$\mathbb{P}_x\big(\overline{Y}_{\overline{\tau}_{V_x(t)}} \in W_x(t)\big) - \mathbb{P}_x\big(\overline{\tau}_{V_x(t)} > (M-1)t\big) \ge c_2 - c_3 e^{-c_4 M}.$$

Choosing $M = c_4^{-1}\log(2c_3/c_2) + 1$, we arrive at the result by (3.25) and the Lebesgue differentiation theorem. □

Lemma 3.7. *There exists $C > 0$ such that for all $t > 0$ and $x, y \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$ with $|x-y| \le t^{1/\alpha}$,*

$$\overline{p}(t,x,y) \ge C t^{-d/\alpha}.$$

Proof. Let $M > 1$ be the constant in Lemma 3.6. Note that for all $(z,w) \in W_x(t/M) \times W_y(t/M)$, we have $z_d \wedge w_d \ge 5(t/M)^{1/\alpha}$ and $|z-w| \le |z-x| + |x-y| + |y-w| \le 21(t/M)^{1/\alpha}$. Therefore, by Proposition 3.2, there is $c_1 > 0$ independent of t, x, y such that

$$\operatorname*{ess\,inf}_{z\in W_x(t/(3M))}\ \operatorname*{ess\,inf}_{w\in W_y(t/(3M))} \overline{p}(t/3,z,w) \ge c_1 t^{-d/\alpha}. \qquad (3.26)$$

By the semigroup property, (3.26) and Lemma 3.6,

$$\begin{aligned}
\overline{p}(t,x,y) &\ge \int_{W_x(t/(3M))}\int_{W_y(t/(3M))} \overline{p}(t/3,x,z)\overline{p}(t/3,z,w)\overline{p}(t/3,w,y)\,dz\,dw \\
&\ge \Big(\operatorname*{ess\,inf}_{z\in W_x(t/(3M))} \overline{p}(t/3,x,z)\Big)\Big(\operatorname*{ess\,inf}_{w\in W_y(t/(3M))} \overline{p}(t/3,y,w)\Big) \\
&\quad \times \Big(\operatorname*{ess\,inf}_{z\in W_x(t/(3M))}\ \operatorname*{ess\,inf}_{w\in W_y(t/(3M))} \overline{p}(t/3,z,w)\Big)\int_{W_x(t/(3M))}\int_{W_y(t/(3M))} dz\,dw \\
&\ge c_2 t^{-3d/\alpha+2d/\alpha} = c_2 t^{-d/\alpha}.
\end{aligned}$$

The proof is complete. □

For $x \in \overline{\mathbb{R}}^d_+$ and $r > 0$, we denote $B_+(x,r) := B(x,r) \cap \overline{\mathbb{R}}^d_+$. We observe that

$$m_d(B_+(x,r)) \asymp r^d \qquad \text{for all } x \in \overline{\mathbb{R}}^d_+,\ r > 0. \qquad (3.27)$$

Now, using (2.6) and Lemma 3.7 (instead of (3.14)), we extend the results in Lemmas 3.3 and 3.4 to $\overline{Y}$ removing the restrictions on x and r.

Lemma 3.8. *There exist constants $C > 0$ and $\eta \in (0, 1/4)$ such that for all $x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$, $r > 0$, $t \in (0, (\eta r)^\alpha]$ and $z \in B_+(x, \eta t^{1/\alpha}) \setminus \mathcal{N}$,*

$$\overline{p}^{B_+(x,r)}(t,z,y) \ge C t^{-d/\alpha} \quad \textit{for a.e. } y \in B_+(x,\eta t^{1/\alpha}) \setminus \mathcal{N}.$$

Lemma 3.9. *There exists a constant $C > 1$ such that for all $x \in \overline{\mathbb{R}}^d_+ \setminus \mathcal{N}$ and $r > 0$,*

$$C^{-1} r^\alpha \le \mathbb{E}_x\left[\overline{\tau}_{B_+(x,r)}\right] \le \sup_{z\in B_+(x,r)\setminus\mathcal{N}} \mathbb{E}_z\left[\overline{\tau}_{B_+(x,r)}\right] \le C r^\alpha. \qquad (3.28)$$

Let $\overline{X} := (T_s, \overline{Y}_s)_{s\geq 0}$ and $X^\kappa := (T_s, Y^\kappa_s)_{s\geq 0}$ be time-space processes where $T_s = T_0 - s$. The law of the time-space process $s \mapsto \overline{X}_s$ or $s \mapsto X^\kappa_s$ starting from (t,x) will be denoted by $\mathbb{P}_{(t,x)}$. For every open subset U of $[0,\infty)\times\mathbb{R}^d$, define $\overline{\tau}_U = \inf\{s>0 : \overline{X}_s \notin U\}$ and $\tau^\kappa_U = \inf\{s>0 : X^\kappa_s \notin U\}$. We also define $\overline{\sigma}_U = \inf\{s>0 : \overline{X}_s \in U\}$ and $\sigma^\kappa_U = \inf\{s>0 : X^\kappa_s \in U\}$.

A Borel function $u : [0,\infty)\times\overline{\mathbb{R}}^d_+ \to \mathbb{R}$ is said to be parabolic in $(a,b]\times B_+(x,r) \subset (0,\infty)\times\overline{\mathbb{R}}^d_+$ with respect to $\overline{Y}$ if for every relatively compact open set $U\subset (a,b]\times B_+(x,r)$ with respect to the topology on $[0,\infty)\times\overline{\mathbb{R}}^d_+$, it holds that $u(t,z) = \mathbb{E}_{(t,z)}u(\overline{X}_{\overline{\tau}_U})$ for all $(t,z)\in U$ with $z\notin\mathcal{N}$. Similarly, a Borel function $u : [0,\infty)\times\mathbb{R}^d_+ \to \mathbb{R}$ is said to be parabolic in $(a,b]\times B(x,r) \subset (0,\infty)\times\mathbb{R}^d_+$ with respect to Y^κ if for every relatively compact open set $U\subset(a,b]\times B(x,r)$, it holds that $u(t,z) = \mathbb{E}_{(t,z)}u(X^\kappa_{\tau^\kappa_U})$ for all $(t,z)\in U$ with $z\notin\mathcal{N}$.

Lemma 3.10. *(i) Let $\eta\in(0,1/4)$ be the constant from Lemma 3.3. For every $\delta\in(0,\eta]$, there exists a constant $C>0$ such that for all $x\in\mathbb{R}^d_+\setminus\mathcal{N}$, $r\in(0,x_d)$, $t\geq\delta r^\alpha$, and any compact set $A\subset[t-\delta r^\alpha, t-\delta r^\alpha/2]\times B(x,(\eta\delta/2)^{1/\alpha}r)$,*

$$\mathbb{P}_{(t,x)}\big(\sigma^\kappa_A < \tau^\kappa_{[t-\delta r^\alpha,t]\times B(x,r)}\big) \geq C\frac{m_{d+1}(A)}{r^{d+\alpha}}.$$

(ii) Let $\eta\in(0,1/4)$ be the constant from Lemma 3.8. For every $\delta\in(0,\eta]$, there exists a constant $C>0$ such that for all $x\in\overline{\mathbb{R}}^d_+\setminus\mathcal{N}$, $r>0$, $t\geq\delta r^\alpha$, and any compact set $A\subset[t-\delta r^\alpha, t-\delta r^\alpha/2]\times B_+(x,(\eta\delta/2)^{1/\alpha}r)$,

$$\mathbb{P}_{(t,x)}\left(\overline{\sigma}_A < \overline{\tau}_{[t-\delta r^\alpha,t]\times B_+(x,r)}\right) \geq C\frac{m_{d+1}(A)}{r^{d+\alpha}}.$$

Proof. By repeating the proofs of [21, Lemma 6.5] (using the Lévy system formulas in (3.16) and (3.17)), we deduce the results from Lemmas 3.3 and 3.8 respectively. □

Theorem 3.11. *(i) For any $\delta\in(0,1)$, there exist $\lambda\in(0,1]$ and $C>0$ such that for all $x\in\mathbb{R}^d_+$, $r\in(0,x_d)$, $t_0\geq 0$, and any function u on $(0,\infty)\times\mathbb{R}^d_+$ which is parabolic in $(t_0,t_0+r^\alpha]\times B(x,r)$ with respect to Y^κ and bounded in $(t_0,t_0+r^\alpha]\times\mathbb{R}^d_+$, we have*

$$|u(s,y)-u(t,z)| \leq C\left(\frac{|s-t|^{1/\alpha}+|y-z|}{r}\right)^\lambda \operatorname*{ess\,sup}_{[t_0,t_0+r^\alpha]\times\mathbb{R}^d_+}|u|, \tag{3.29}$$

for every $s,t\in(t_0+(1-\delta^\alpha)r^\alpha, t_0+r^\alpha]$ and $y,z\in B(x,\delta r)\setminus\mathcal{N}$.
(ii) For any $\delta\in(0,1)$, there exist $\lambda\in(0,1]$ and $C>0$ such that for all $x\in\overline{\mathbb{R}}^d_+$, $r>0$, $t_0\geq 0$, and any function u on $(0,\infty)\times\overline{\mathbb{R}}^d_+$ which is parabolic in $(t_0,t_0+r^\alpha]\times B_+(x,r)$ with respect to $\overline{Y}$ and bounded in $(t_0,t_0+r^\alpha]\times\overline{\mathbb{R}}^d_+$, (3.29) holds true for every $s,t\in(t_0+(1-\delta^\alpha)r^\alpha, t_0+r^\alpha]$ and $y,z\in B_+(x,\delta r)\setminus\mathcal{N}$.

Proof. Using Lemmas 3.3, 3.4 and 3.10(i) for (i), and (3.27) and Lemmas 3.8, 3.9 and 3.10(ii) for (ii), we get the desired results using the same argument as in the proof of [15, Theorem 4.14] (see also the proof of [18, Proposition 3.8]). We omit details here. □

Remark 3.12. By Theorem 3.11, since the heat kernels $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$ are parabolic with respect to $\overline{Y}$ and Y^κ respectively, they can be extended continuously to $(0,\infty)\times\overline{\mathbb{R}}^d_+\times\overline{\mathbb{R}}^d_+$ and $(0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$ respectively. As consequences, by Proposition 2.7, $\overline{Y}$ and Y^κ can be refined to be a strongly Feller processes starting from every point in $\overline{\mathbb{R}}^d_+$ and $\mathbb{R}^d_+$ respectively, and the exceptional set $\mathcal{N}$ in Proposition 2.7 can be taken to be the empty set.

In the remainder of this paper, we take the jointly continuous version of $\overline{p} : (0,\infty)\times\overline{\mathbb{R}}^d_+\times\overline{\mathbb{R}}^d_+ \to [0,\infty)$ and $p^\kappa : (0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+ \to [0,\infty)$, take the exceptional set $\mathcal{N}$ in Proposition 2.7 to be empty set, and replace the essinf in Lemma 3.6 by inf. Again, for notational convenience, we extend the domain of $p^\kappa(t,x,y)$ to $(0,\infty)\times(0,\infty)\times\overline{\mathbb{R}}^d_+\times\overline{\mathbb{R}}^d_+$ by letting $p^\kappa(t,x,y)=0$ if $x\in\partial\mathbb{R}^d_+$ or $y\in\partial\mathbb{R}^d_+$.

The following scaling and horizontal translation invariance properties of the heat kernels (the latter for $d\ge 2$), which come from Lemmas 2.1 and 2.2, will be used throughout the paper: For any $(t,x,y)\in(0,\infty)\times\overline{\mathbb{R}}^d_+\times\overline{\mathbb{R}}^d_+$, $r>0$ and $\widetilde{z}\in\mathbb{R}^{d-1}$,

$$\begin{aligned}\overline{p}(t,x,y)&=r^{-d}\overline{p}(t/r^\alpha,x/r,y/r)=\overline{p}(t,x+(\widetilde{z},0),y+(\widetilde{z},0)),\\ p^\kappa(t,x,y)&=r^{-d}p^\kappa(t/r^\alpha,x/r,y/r)=p^\kappa(t,x+(\widetilde{z},0),y+(\widetilde{z},0)).\end{aligned}\tag{3.30}$$

From Proposition 2.7, since the exceptional set $\mathcal{N}$ is removed, we obtain

Corollary 3.13. *If $d>\alpha$,*

$$G^\kappa(x,y)\le\overline{G}(x,y)\le\frac{c}{|x-y|^{d-\alpha}},\quad x,y\in\overline{\mathbb{R}}^d_+.$$

Remark 3.14. The assumption $d>(\alpha+\beta_1+\beta_2)\wedge 2$ in [31, 32] is only used to show $G^\kappa(x,y)\le c|x-y|^{-d+\alpha}$. Thus, by Corollary 3.13, all results in [31, 32] with the assumption $d>(\alpha+\beta_1+\beta_2)\wedge 2$ hold under the weaker assumption $d>\alpha$.

Since the heat kernels $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$ are jointly continuous, we get the next lemma from the strong Markov properties of $\overline{Y}$ and Y^κ. This lemma is a refined version of [13, Lemma 4.2] which was inspired by [35]. In this paper, this lemma will play an important role in the bootstrap method to prove sharp upper estimates on the heat kernels. Although the proof of next lemma is standard, we give it for the reader's convenience.

Lemma 3.15. *(i) Let V_1 and V_3 be open subsets of $\overline{\mathbb{R}}^d_+$ with $\mathrm{dist}(V_1,V_3)>0$. Set $V_2:=\overline{\mathbb{R}}^d_+\setminus(V_1\cup V_3)$. For any $x\in V_1$, $y\in V_3$ and $t>0$, it holds that*

$$\begin{aligned}\overline{p}(t,x,y)&\le\mathbb{P}_x(\overline{\tau}_{V_1}<t)\sup_{s\le t,\,z\in V_2}\overline{p}(s,z,y)\\&\quad+\mathrm{dist}(V_1,V_3)^{-d-\alpha}\int_0^t\int_{V_3}\int_{V_1}\overline{p}^{V_1}(t-s,x,u)\mathcal{B}(u,w)\overline{p}(s,y,w)\,du\,dw\,ds.\end{aligned}$$

(ii) Let V_1 and V_3 be open subsets of $\mathbb{R}^d_+$ with $\mathrm{dist}(V_1,V_3)>0$. Set $V_2:=\mathbb{R}^d_+\setminus(V_1\cup V_3)$. For any $x\in V_1$, $y\in V_3$ and $t>0$, it holds that

$$\begin{aligned}p^\kappa(t,x,y)&\le\mathbb{P}_x(\tau^\kappa_{V_1}<t<\zeta^\kappa)\sup_{s\le t,\,z\in V_2}p^\kappa(s,z,y)\\&\quad+\mathrm{dist}(V_1,V_3)^{-d-\alpha}\int_0^t\int_{V_3}\int_{V_1}p^{\kappa,V_1}(t-s,x,u)\mathcal{B}(u,w)p^\kappa(s,y,w)\,du\,dw\,ds.\end{aligned}$$

Proof. Since the proofs are the same, we only give the proof of (ii).

By the strong Markov property, the Lévy system formula in (3.16) and symmetry, we get that for any $x\in V_1$ and $y\in V_3$,

$$\begin{aligned}p^\kappa(t,x,y)&=\mathbb{E}_x\left[p^\kappa(t-\tau^\kappa_{V_1},Y^\kappa_{\tau^\kappa_{V_1}},y):\tau^\kappa_{V_1}<t<\zeta^\kappa\right]\\&=\mathbb{E}_x\left[p^\kappa(t-\tau^\kappa_{V_1},Y^\kappa_{\tau^\kappa_{V_1}},y):\tau^\kappa_{V_1}<t<\zeta^\kappa,Y^\kappa_{\tau^\kappa_{V_1}}\in V_2\right]\\&\quad+\mathbb{E}_x\left[p^\kappa(t-\tau^\kappa_{V_1},Y^\kappa_{\tau^\kappa_{V_1}},y):\tau^\kappa_{V_1}<t<\zeta^\kappa,Y^\kappa_{\tau^\kappa_{V_1}}\in V_3\right]\\&\le\mathbb{P}_x(\tau^\kappa_{V_1}<t<\zeta^\kappa)\sup_{s\le t,\,z\in V_2}p^\kappa(s,z,y)\\&\quad+\int_0^t\int_{V_3}\int_{V_1}p^{\kappa,V_1}(t-s,x,u)\frac{\mathcal{B}(u,w)}{|u-w|^{d+\alpha}}p^\kappa(s,w,y)\,du\,dw\,ds\end{aligned}$$

$$\leq \mathbb{P}_x(\tau^\kappa_{V_1} < t < \zeta^\kappa) \sup_{s\leq t,\, z\in V_2} p^\kappa(s,z,y)$$
$$+ \operatorname{dist}(V_1,V_3)^{-d-\alpha} \int_0^t \int_{V_3}\int_{V_1} p^{\kappa,V_1}(t-s,x,u)\mathcal{B}(u,w)p^\kappa(s,y,w)\,du\,dw\,ds.$$

□

4. Parabolic Harnack inequality and preliminary lower bound of heat kernels

In this section, we prove that the parabolic Harnack inequality holds for $\overline{Y}$ and Y^κ, and get some preliminary lower bounds for the heat kernels $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$.

Recall that $B_+(x,r) = B(x,r)\cap \overline{\mathbb{R}}^d_+$.

Lemma 4.1. *(i) Let $\eta\in(0,1/4)$ be the constant from Lemma 3.3 and let $\delta\in(0,2^{-\alpha-2}\eta)$. There exists a constant $C>0$ such that for all $y\in\mathbb{R}^d_+$, $R\in(0,y_d)$, $r\in(0,(\eta\delta/2)^{1/\alpha}R/2]$, $\delta R^\alpha/2\leq t-s\leq 4\delta(2R)^\alpha$, $x\in B(y,(\eta\delta/2)^{1/\alpha}R/2)$ and $z\in B(x_0,(\eta\delta/2)^{1/\alpha}R)$,*

$$\mathbb{P}_{(t,z)}\big(\sigma^\kappa_{\{s\}\times B(x,r)} \leq \tau^\kappa_{[s,t]\times B(y,R)}\big) \geq C(r/R)^d.$$

(ii) Let $\eta\in(0,1/4)$ be the constant from Lemma 3.8 and let $\delta\in(0,2^{-\alpha-2}\eta)$. There exists a constant $C>0$ such that for all $y\in\overline{\mathbb{R}}^d_+$, $R>0$, $r\in(0,(\eta\delta/2)^{1/\alpha}R/2]$, $\delta R^\alpha/2\leq t-s\leq 4\delta(2R)^\alpha$, $x\in B_+(y,(\eta\delta/2)^{1/\alpha}R/2)$ and $z\in B_+(x_0,(\eta\delta/2)^{1/\alpha}R)$,

$$\mathbb{P}_{(t,z)}\big(\overline{\sigma}_{\{s\}\times B_+(x,r)} \leq \overline{\tau}_{[s,t]\times B_+(y,R)}\big) \geq C(r/R)^d.$$

Proof. Using Lemmas 3.3 and 3.8, and (3.27), the result can be proved by the same argument as that of [21, Lemma 6.7]. We omit details here. □

In order to obtain the parabolic Harnack inequality for $\overline{Y}$ and Y^κ, we introduce two conditions:

(UBS) There exists a constant $C>0$ such that for all $x,y\in\overline{\mathbb{R}}^d_+$ and $0<r\leq|x-y|/2$,

$$\mathcal{B}(x,y)\leq \frac{C}{r^d}\int_{B_+(x,r)}\mathcal{B}(z,y)dz. \tag{4.1}$$

(IUBS) There exists a constant $C>0$ such that (4.1) holds for all $x,y\in\mathbb{R}^d_+$ and $0<r\leq(|x-y|\wedge x_d)/2$. (Note that, $B_+(x,r)=B(x,r)$ for this range of r.)

Lemma 4.2. *If* **(A3)**(II) *also holds, then* **(UBS)** *is satisfied. In particular,* **(IUBS)** *is satisfied.*

Proof. Let $x,y\in\overline{\mathbb{R}}^d_+$ and $0<r\leq|x-y|/2$. Note that for all $z\in B_+(x,r)$, $|x-y|/2\leq|z-y|\leq 2|x-y|$ by the triangle inequality. Thus, by **(A3)**, there is $c_1>0$ independent of x,y and r such that for all $z\in B_+(x,r)$ with $z_d\geq x_d$, $\mathcal{B}(x,y)\leq c_1\mathcal{B}(z,y)$. Using this, we get

$$\frac{1}{r^d}\int_{B_+(x,r)}\mathcal{B}(z,y)dz \geq \frac{\mathcal{B}(x,y)}{c_1 r^d}\int_{B(x,r):z_d\geq x_d}dz \geq c_2\mathcal{B}(x,y).$$

□

We now show that the following parabolic Harnack inequalities hold.

Theorem 4.3. *(i) Suppose that $\mathcal{B}(x,y)$ satisfies* **(IUBS)**. *Then there exist constants $\delta>0$ and $C,M\geq1$ such that for all $t_0\geq0$, $x\in\mathbb{R}^d_+$ and $R\in(0,x_d)$, and any non-negative function u on $(0,\infty)\times\mathbb{R}^d_+$ which is parabolic on $Q:=(t_0,t_0+4\delta R^\alpha]\times B(x,R)$ with respect to $\overline{Y}$ or Y^κ, we have*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1)\leq C\inf_{(t_2,y_2)\in Q_+} u(t_2,y_2), \tag{4.2}$$

where $Q_-=[t_0+\delta R^\alpha,t_0+2\delta R^\alpha]\times B(x,R/M)$ and $Q_+=[t_0+3\delta R^\alpha,t_0+4\delta R^\alpha]\times B(x,R/M)$.

(ii) Suppose that $\mathcal{B}(x,y)$ satisfies **(UBS)**. *Then there exist constants $\delta>0$ and $C,M\ge 1$ such that for all $t_0\ge 0$, $x\in\overline{\mathbb{R}}^d_+$ and $R>0$, and any non-negative function u on $(0,\infty)\times\overline{\mathbb{R}}^d_+$ which is parabolic on $Q^0:=(t_0,t_0+4\delta R^\alpha]\times B_+(x,R)$ with respect to $\overline{Y}$, we have*

$$\sup_{(t_1,y_1)\in Q^0_-} u(t_1,y_1)\le C\inf_{(t_2,y_2)\in Q^0_+} u(t_2,y_2), \tag{4.3}$$

where $Q^0_-=[t_0+\delta R^\alpha,t_0+2\delta R^\alpha]\times B_+(x,R/M)$ and $Q^0_+=[t_0+3\delta R^\alpha,t_0+4\delta R^\alpha]\times B_+(x,R/M)$.

Proof. (i) By **(IUBS)**, there is a constant $C>0$ such that for all $x,y\in\mathbb{R}^d_+$ and $0<r\le(|x-y|\wedge x_d)/2$,

$$J(x,y)=\frac{\mathcal{B}(x,y)}{|x-y|^{d+\alpha}}\le\frac{C}{r^d}\int_{B_+(x,r)}\frac{\mathcal{B}(z,y)}{|z-y|^{d+\alpha}}dz=\frac{C}{r^d}\int_{B_+(x,r)}J(z,y)dz. \tag{4.4}$$

Using Proposition 2.7 and (4.4), one can follow the proof of [21, Lemma 6.10] and see that [21, Lemma 6.10] is also valid for our case. (Note that, in the proof of [21, Lemma 6.10], a pointwise comparison for the jump kernel from [21, Proposition 6.8] was used to bound the term I_2 therein which can be replaced by (4.4).) Using this and Lemmas 3.3, 3.4, 3.10, 4.1, the result can be proved using the same argument as in the proof of [9, Lemma 5.3] (see also the proof of [18, Lemma 4.1]). We omit details here.

(ii) Since **(UBS)** implies that (4.4) is satisfied for all $x,y\in\mathbb{R}^d_+$ and $0<r\le|x-y|/2$, using Proposition 2.7 and (3.27), one can also deduce that [21, Lemma 6.10] is valid for this case with $B_+(x_0,\cdot)$ instead of $B(x_0,\cdot)$ in the definitions of Q_i, $1\le i\le 4$ therein. Then using Lemmas 3.8, 3.9, 3.10(ii), 4.1(ii), one can follow the arguments in the proof of [9, Lemma 5.3] and conclude the result. □

Using Lemma 3.3, and Theorem 4.3(i), we obtain the following lemma.

Lemma 4.4. *Suppose that $\mathcal{B}(x,y)$ satisfies* **(IUBS)**. *For any positive constants a,b, there exists a constant $C=C(a,b,\kappa)>0$ such that for all $z\in\mathbb{R}^d_+$ and $r>0$ with $B(z,2br)\subset\mathbb{R}^d_+$,*

$$\inf_{y\in B(z,br/2)}\mathbb{P}_y\big(\tau^\kappa_{B(z,br)}>ar^\alpha\big)\ \ge\ C.$$

Proof. By Lemma 3.3, there exist constants $c_1,c_2,\varepsilon_1>0$ such that for all $z\in\mathbb{R}^d_+$ and $r>0$ with $B(z,2br)\subset\mathbb{R}^d_+$,

$$\begin{aligned}\mathbb{P}_z\big(\tau^\kappa_{B(z,br/2)}>\varepsilon_1r^\alpha\big)&=\int_{B(z,br/2)}p^{\kappa,B(z,br/2)}(\varepsilon_1r^\alpha,z,w)dw\\ &\ge\int_{B(z,c_1r)}p^{\kappa,B(z,br/2)}(\varepsilon_1r^\alpha,z,w)dw\ge c_2.\end{aligned} \tag{4.5}$$

Thus it suffices to prove the lemma for $a>\varepsilon_1$. Applying the parabolic Harnack inequality (Theorem 4.3) repeatedly, we conclude that there exists $c_3>0$ such that for any $w,y\in B(z,br/2)$,

$$p^{\kappa,B(z,br)}(ar^\alpha,y,w)\ge c_3\,p^{\kappa,B(z,br)}(\varepsilon_1r^\alpha,z,w).$$

Thus, using (4.5), we deduce that for any $y\in B(z,br/2)$,

$$\begin{aligned}&\mathbb{P}_y\big(\tau^\kappa_{B(z,br)}>ar^\alpha\big)=\int_{B(z,br)}p^{\kappa,B(z,br)}(ar^\alpha,y,w)dw\\ &\ge c_3\int_{B(z,br/2)}p^{\kappa,B(z,br)}(\varepsilon_1r^\alpha,z,w)dw\ge c_3\mathbb{P}_z\big(\tau^\kappa_{B(z,br/2)}>\varepsilon_1r^\alpha\big)\ge c_2c_3.\end{aligned}$$

This proves the lemma. □

Now, we follow the proof of [8, Proposition 3.5] to get the following preliminary lower bound.

Proposition 4.5. *Suppose that $\mathcal{B}(x,y)$ satisfies* **(IUBS)**. *For any $a>0$, there exists a constant $C=C(a,\kappa)>0$ such that for any $(t,x,y)\in(0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$ with $at^{1/\alpha}\le(4|x-y|)\wedge x_d\wedge y_d$,*

$$\overline{p}(t,x,y)\ge p^\kappa(t,x,y)\ge CtJ(x,y).$$

Proof. The first inequality holds true by (2.5) and Remark 3.12.

By Lemma 4.4, starting at $z\in B(y,\,(12)^{-1}at^{1/\alpha})$, with probability at least $c_1=c_1(a)>0$, the process Y^κ does not move more than $(18)^{-1}at^{1/\alpha}$ by time t. Thus, using the strong Markov property and the Lévy system formula in (3.16), we obtain that for $a(t/2)^{1/\alpha}\le(4|x-y|)\wedge x_d\wedge y_d$,

$$\begin{aligned}&\mathbb{P}_x\left(Y^\kappa_t\in B\big(y,\,6^{-1}at^{1/\alpha}\big)\right)\\&\ge c_1\mathbb{P}_x\big(Y^\kappa_{t\wedge\tau^\kappa_{B(x,(18)^{-1}at^{1/\alpha})}}\in B(y,\,(12)^{-1}at^{1/\alpha})\text{ and }t\wedge\tau^\kappa_{B(x,(18)^{-1}at^{1/\alpha})}\text{ is a jumping time}\big)\\&=c_1\mathbb{E}_x\bigg[\int_0^{t\wedge\tau^\kappa_{B(x,(18)^{-1}at^{1/\alpha})}}\int_{B(y,\,(12)^{-1}at^{1/\alpha})}J(Y^\kappa_s,u)duds\bigg].\end{aligned}\tag{4.6}$$

By (4.4), we obtain that for $a(t/2)^{1/\alpha}\le(4|x-y|)\wedge x_d\wedge y_d$,

$$\begin{aligned}&\mathbb{E}_x\bigg[\int_0^{t\wedge\tau^\kappa_{B(x,(18)^{-1}at^{1/\alpha})}}\int_{B(y,\,(12)^{-1}at^{1/\alpha})}J(Y^\kappa_s,u)duds\bigg]\\&=\mathbb{E}_x\bigg[\int_0^t\int_{B(y,\,(12)^{-1}at^{1/\alpha})}J(Y^{\kappa,B(x,(18)^{-1}at^{1/\alpha})}_s,u)duds\bigg]\\&\ge c_2t^{d/\alpha}\int_0^t\mathbb{E}_x\left[J(Y^{\kappa,B(x,(18)^{-1}at^{1/\alpha})}_s,y)\right]ds\\&\ge c_2t^{d/\alpha}\int_{t/2}^t\int_{B(x,(72)^{-1}a(t/2)^{1/\alpha})}J(w,y)p^{\kappa,B(x,(18)^{-1}at^{1/\alpha})}(s,x,w)dwds.\end{aligned}\tag{4.7}$$

Note that, for $t/2<s<t$ and $w\in B(x,(72)^{-1}a(t/2)^{1/\alpha})$, we have

$$\delta_{B(x,(18)^{-1}at^{1/\alpha})}(w)\ge(18)^{-1}at^{1/\alpha}-(72)^{-1}a(t/2)^{1/\alpha}\ge(36)^{-1}as^{-1/\alpha}$$

and

$$|x-w|<(72)^{-1}a(t/2)^{1/\alpha}\le4^{-1}(18)^{-1}as^{1/\alpha}.$$

Thus by Lemma 3.3 and the parabolic Harnack inequality (Theorem 4.3) we see that for $t/2<s<t$ and $w\in B(x,(72)^{-1}a(t/2)^{1/\alpha})$,

$$p^{\kappa,B(x,(18)^{-1}at^{1/\alpha})}(s,x,w)\ge c_3\,t^{-d/\alpha}.\tag{4.8}$$

Combining (4.6), (4.7) with (4.8) and applying (4.4) again, we get that for $a(t/2)^{1/\alpha}\le(4|x-y|)\wedge x_d\wedge y_d$,

$$\mathbb{P}_x\left(Y^\kappa_t\in B\big(y,\,6^{-1}at^{1/\alpha}\big)\right)\ge c_4t\int_{B(x,(72)^{-1}a(t/2)^{1/\alpha})}J(w,y)dw\ge c_5t^{1+d/\alpha}J(x,y).\tag{4.9}$$

Note that for $y_d\ge at^{1/\alpha}$ and $z\in B(y,a(t/2)^{1/\alpha}/6)$,

$$z_d\ge y_d-|z-y|\ge a(1-(1/2)^{1/\alpha}/6)t^{1/\alpha}\ge(5a/6)t^{1/\alpha}\tag{4.10}$$

The proposition now follows from the Chapman-Kolmogorov equation along with (4.9) and Proposition 3.2 (using (4.10)): for $at^{1/\alpha}\le(4|x-y|)\wedge x_d\wedge y_d$,

$$\begin{aligned}p^\kappa(t,x,y)&=\int_D p^\kappa(t/2,x,z)p^\kappa(t/2,z,y)dz\ge\int_{B(y,\,a(t/2)^{1/\alpha}/6)}p^\kappa(t/2,x,z)p^\kappa(t/2,z,y)dz\\&\ge c_6t^{-d/\alpha}\,\mathbb{P}_x\left(Y^\kappa_{t/2}\in B(y,a(t/2)^{1/\alpha}/6)\right)\ge c_7\,tJ(x,y).\end{aligned}$$

□

5. Preliminary upper bounds of heat kernels

The goal of this section is to prove the following proposition.

Proposition 5.1. *Suppose that* **(A1)**–**(A4)** *and* (1.8) *hold true. Then there exists a constant* $C>0$ *such that*

$$p^\kappa(t,x,y) \le C\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q_\kappa}\left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^{q_\kappa}\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right), \quad t>0,\ x,y\in \mathbb{R}^d_+, \tag{5.1}$$

where $q_\kappa \in [(\alpha-1)_+, \alpha+\beta_1)$ *is the constant from* (1.8).

Note that, since $t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}$ is comparable to the transition density of the isotropic α-stable process in $\mathbb{R}^d$, there exists a constant $C>0$ such that for all $t,s>0$ and $x,y\in\mathbb{R}^d_+$,

$$\int_{\mathbb{R}^d_+}\left(t^{-d/\alpha}\wedge \frac{t}{|x-z|^{d+\alpha}}\right)dz \le C, \tag{5.2}$$

and

$$\int_{\mathbb{R}^d_+}\left(t^{-d/\alpha}\wedge \frac{t}{|x-z|^{d+\alpha}}\right)\left(s^{-d/\alpha}\wedge \frac{s}{|y-z|^{d+\alpha}}\right)dz \le C\left((t+s)^{-d/\alpha}\wedge \frac{t+s}{|x-y|^{d+\alpha}}\right), \tag{5.3}$$

where in (5.3) we used the semigroup property.

Before giving the proof of Proposition 5.1, we record its simple consequence.

Corollary 5.2. *There exists a constant* $C>0$ *such that*

$$\mathbb{P}_x(\zeta^\kappa>t) \le C\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q_\kappa}, \quad t>0,\ x\in\mathbb{R}^d_+.$$

Proof. By Proposition 5.1 and (5.2),

$$\begin{aligned}\mathbb{P}_x(\zeta^\kappa>t) &= \int_{\mathbb{R}^d_+} p^\kappa(t,x,y)dy\\ &\le c_1\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q_\kappa}\int_{\mathbb{R}^d_+}\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right)dy \le c_2\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q_\kappa}.\end{aligned}$$

□

When $q_\kappa=0$, Proposition 5.1 follows from Proposition 2.7. Hence, in the remainder of this section, we assume that **(A1)**–**(A4)** hold, fix $\kappa\in[0,\infty)$ such that $q_\kappa>0$ and denote by q the constant q_κ in (1.8). For the proof we will need several results from [30, 31, 32] that we now recall for the convenience of the reader.

For $r>0$, we define a subset $U(r)$ of $\overline{\mathbb{R}}^d_+$, $d\ge 2$, by

$$U(r) := \{x=(\widetilde{x},x_d)\in\mathbb{R}^d : |\widetilde{x}|<r/2,\ 0\le x_d<r/2\}.$$

When $d=1$, we abuse notation and use $U(r)=[0,r/2)$. For $t>0$ and an open set $V\subset\mathbb{R}^d_+$, denote by $Y^{\kappa,d}_t$ and $Y^{\kappa,V,d}_t$ the last coordinates of Y^κ_t and $Y^{\kappa,V}_t$ respectively.

Lemma 5.3. *There exists a constant* $C>0$ *such that for all* $x,y\in\mathbb{R}^d_+$ *satisfying* $|x-y|\ge x_d$, *it holds that*

$$\mathcal{B}(x,y) \le C x_d^{\beta_1}(|\log x_d|^{\beta_3}\vee 1)\big(1+\mathbf{1}_{|y|\ge 1}(\log|y|)^{\beta_3}\big)|x-y|^{-\beta_1}.$$

Proof. See [30, Lemma 5.2(a)]. □

Lemma 5.4. *For any $R > 0$, there exists a constant $C > 0$ such that for all $r \in (0, R]$ and $x \in U(2^{-4}r) \cap \mathbb{R}^d_+$,*

$$\mathbb{E}_x \int_0^{\tau^\kappa_{U(r)}} (Y^{\kappa,d}_t)^{\beta_1} |\log Y^{\kappa,d}_t|^{\beta_3} dt \le C x_d^q.$$

Proof. The result follows from scaling (Lemma 2.1), and [30, Lemma 5.7(a)] if $\kappa > 0$ and [32, Lemma 5.3] if $\kappa = 0$. □

Lemma 5.5. *There exists a constant $C > 0$ such that for all $r > 0$ and $x \in U(2^{-4}r) \cap \mathbb{R}^d_+$,*

$$\mathbb{P}_x\big(\tau^\kappa_{U(r)} < \zeta^\kappa\big) = \mathbb{P}_x\big(Y^\kappa_{\tau^\kappa_{U(r)}} \in \mathbb{R}^d_+\big) \le C \left(\frac{x_d}{r}\right)^q.$$

Proof. When $\kappa > 0$, the result follows from [31, Lemma 3.4]. When $\kappa = 0$, using [32, Lemma 5.5] and Lemma 5.4, one can follow the proof of [31, Lemma 3.4] and deduce the result. □

Lemma 5.6. *There exists a constant $C > 0$ such that for all $r > 0$ and $x \in U(2^{-4}r) \cap \mathbb{R}^d_+$,*

$$\mathbb{E}_x\big[\tau^\kappa_{U(r)}\big] \le C \left(\frac{x_d}{r}\right)^q.$$

Proof. The result follows from scaling (Lemma 2.1), and [30, Lemma 5.13] if $\kappa > 0$ and [32, Lemma 4.5] if $\kappa = 0$. □

Proposition 5.7. *There exists a constant $C > 0$ such that for any $w \in \partial\mathbb{R}^d_+$, $r > 0$ and any non-negative function f in $\mathbb{R}^d_+$ that is harmonic in $\mathbb{R}^d_+ \cap B(w, r)$ with respect to Y^κ and vanishes continuously on $\partial\mathbb{R}^d_+ \cap B(w, r)$, we have*

$$f(x) \le C f(\widehat{x}) \quad \text{for all } x \in \mathbb{R}^d_+ \cap B(w, r/2),$$

where $\widehat{x} \in \mathbb{R}^d_+ \cap B(w, r)$ with $\widehat{x}_d \ge r/4$.

Proof. The result follows from [30, Theorem 1.2] if $\kappa > 0$ and [32, Theorem 5.6] if $\kappa = 0$. □

After recalling the known results above, we now continue with several auxiliary lemmas leading to the proof of Proposition 5.1.

Lemma 5.8. *For all $\gamma \ge 0$, $t > 0$ and $x \in U(1) \cap \mathbb{R}^d_+$, it holds that*

$$\int_{\mathbb{R}^d_+} p^\kappa(t, x, z)(1 \wedge z_d)^\gamma dz \le \mathbb{E}_x\left[(1 \wedge Y^{\kappa,U(1),d}_t)^\gamma : \tau^\kappa_{U(1)} > t\right] + \mathbb{P}_x\big(Y^\kappa_{\tau^\kappa_{U(1)}} \in \mathbb{R}^d_+\big). \tag{5.4}$$

In particular, it holds that

$$\mathbb{P}_x(\zeta^\kappa > t) \le t^{-1}\mathbb{E}_x\big[\tau^\kappa_{U(1)}\big] + \mathbb{P}_x\big(Y^\kappa_{\tau^\kappa_{U(1)}} \in \mathbb{R}^d_+\big). \tag{5.5}$$

Proof. Since $Y^{\kappa,U(1)}_t = Y^\kappa_t$ for $t < \tau^\kappa_{U(1)}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d_+} p^\kappa(t, x, z)(1 \wedge z_d)^\gamma dz = \mathbb{E}_x\left[(1 \wedge Y^{\kappa,d}_t)^\gamma : t < \zeta^\kappa\right] \\
&= \mathbb{E}_x\left[(1 \wedge Y^{\kappa,d}_t)^\gamma : \tau^\kappa_{U(1)} > t\right] + \mathbb{E}_x\left[(1 \wedge Y^{\kappa,d}_t)^\gamma : \tau^\kappa_{U(1)} \le t < \zeta^\kappa\right] \\
&\le \mathbb{E}_x\left[(1 \wedge Y^{\kappa,U(1),d}_t)^\gamma : \tau^\kappa_{U(1)} > t\right] + \mathbb{E}_x\big[1 : \tau^\kappa_{U(1)} < \zeta^\kappa\big] \\
&= \mathbb{E}_x\left[(1 \wedge Y^{\kappa,U(1),d}_t)^\gamma : \tau^\kappa_{U(1)} > t\right] + \mathbb{P}_x\big(Y^\kappa_{\tau^\kappa_{U(1)}} \in \mathbb{R}^d_+\big).
\end{aligned}$$

By taking $\gamma = 0$ in (5.4) and using Markov's inequality, we get

$$\mathbb{P}_x(\zeta^\kappa > t) \le \mathbb{P}_x(\tau^\kappa_{U(1)} > t) + \mathbb{P}_x\big(Y^\kappa_{\tau^\kappa_{U(1)}} \in \mathbb{R}^d_+\big) \le t^{-1}\mathbb{E}_x\big[\tau^\kappa_{U(1)}\big] + \mathbb{P}_x\big(Y^\kappa_{\tau^\kappa_{U(1)}} \in \mathbb{R}^d_+\big).$$

□

Lemma 5.9. *There exists $C > 0$ such that*

$$p^\kappa(t,x,y) \le Ct^{-d/\alpha}\mathbb{P}_x(\zeta^\kappa > t/3)\mathbb{P}_y(\zeta^\kappa > t/3), \quad t > 0,\ x,y \in \mathbb{R}^d_+.$$

Proof. By the semigroup property, the symmetry of $p^\kappa(t,\cdot,\cdot)$ and Proposition 2.7, we obtain

$$\begin{aligned} p^\kappa(t,x,y) &= \int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} p^\kappa(t/3,x,z)p^\kappa(t/3,z,w)p^\kappa(t/3,y,w)dzdw \\ &\le c_1 t^{-d/\alpha}\int_{\mathbb{R}^d_+} p^\kappa(t/3,x,z)dz\int_{\mathbb{R}^d_+} p^\kappa(t/3,y,w)dw = c_1t^{-d/\alpha}\mathbb{P}_x(\zeta^\kappa > t/3)\mathbb{P}_y(\zeta^\kappa > t/3). \end{aligned}$$

□

The next lemma shows that (5.1) (hence Proposition 5.1) is a consequence of the following, seemingly weaker, inequality: There exists $C > 0$ such that

$$p^\kappa(t,x,y) \le C\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q\left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q t^{-d/\alpha}, \quad t > 0,\ x,y \in \mathbb{R}^d_+. \tag{5.6}$$

Lemma 5.10. *If* (5.6) *holds true, then* (5.1) *also holds.*

Proof. We claim that there exists a constant $c_1 > 0$ such that

$$p^\kappa(t,x,y) \le c_1\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{5.7}$$

By (3.30), we can assume $\widetilde{x} = \widetilde{0}$ and $t = t_0 = (1/2)^\alpha$. If $x_d \ge 2^{-4}t_0^{1/\alpha}$ or $|x-y| \le 4t_0^{1/\alpha}$, then (5.7) follows from Proposition 2.7 or the assumption (5.6) respectively. Hence, we assume $x_d < 2^{-4}t_0^{1/\alpha}$ and $|x-y| > 4t_0^{1/\alpha}$, and will show that

$$p^\kappa(t_0,x,y) \le c_1(t_0)\frac{x_d^q}{|x-y|^{d+\alpha}}. \tag{5.8}$$

Let $V_1 = U(t_0^{1/\alpha})$, $V_3 = \{w \in \mathbb{R}^d_+ : |w-y| < |x-y|/2\}$ and $V_2 = \mathbb{R}^d_+ \setminus (V_1 \cup V_3)$. By Lemma 5.5, we have

$$\mathbb{P}_x(\tau^\kappa_{V_1} < t_0 < \zeta^\kappa) \le \mathbb{P}_x(Y^\kappa_{\tau^\kappa_{V_1}} \in \mathbb{R}^d_+) \le c_2(t_0^{-1/\alpha}x_d)^q. \tag{5.9}$$

Also, we get from Proposition 2.7 that

$$\sup_{s\le t_0,\, z\in V_2} p^\kappa(s,z,y) \le c_3 \sup_{s\le t_0,\, z\in\mathbb{R}^d_+, |z-y|>|x-y|/2}\frac{s}{|z-y|^{d+\alpha}} = 2^{d+\alpha}c_3\frac{t_0}{|x-y|^{d+\alpha}}. \tag{5.10}$$

Next, we note that by the triangle inequality, for any $u \in V_1$ and $w \in V_3$,

$$|u-w| \ge |x-y| - |x-u| - |y-w| \ge |x-y| - t_0^{1/\alpha} - \frac{|x-y|}{2} \ge \frac{|x-y|}{4} \ge t_0^{1/\alpha} \ge u_d. \tag{5.11}$$

In particular, recalling that $\beta_1 > 0$ if $\beta_3 > 0$, we see that $\big(1 + \mathbf{1}_{|w|\ge1}(\log|w|)^{\beta_3}\big)|u-w|^{-\beta_1} \le c_4$ for $u \in V_1$ and $w \in V_3$, so by Lemma 5.3, we have that for any $u \in V_1$ and $w \in V_3$,

$$\mathcal{B}(u,w) \le c_5u_d^{\beta_1}(|\log u_d|^{\beta_3} \vee 1)\big(1 + \mathbf{1}_{|w|\ge1}(\log|w|)^{\beta_3}\big)|u-w|^{-\beta_1} \le c_6u_d^{\beta_1}|\log u_d|^{\beta_3}. \tag{5.12}$$

Thus, by **(A3)**(II), Lemma 5.4, (5.11) and (5.12) we get that

$$\int_0^{t_0}\int_{V_3}\int_{V_1} p^{\kappa,V_1}(t-s,x,u)\mathcal{B}(u,w)p^\kappa(s,y,w)dudwds$$

$$\begin{aligned}
&\le c_7 \int_0^{t_0} \left(\int_{V_1} p^{\kappa,V_1}(t-s,x,u) u_d^{\beta_1} |\log u_d|^{\beta_3} du \right) \left(\int_{V_3} p^{\kappa}(s,y,w) dw \right) ds \\
&\le c_7 \int_0^{\infty} \left(\int_{V_1} p^{\kappa,V_1}(s,x,u) u_d^{\beta_1} |\log u_d|^{\beta_3} du \right) ds \\
&= c_7 \mathbb{E}_x \int_0^{\tau^\kappa_{V_1}} (Y^{\kappa,d}_s)^{\beta_1} |\log Y^{\kappa,d}_s|^{\beta_3} ds \le c_8 x_d^q. \qquad (5.13)
\end{aligned}$$

Now (5.8) (and so (5.7)) follows from (5.9)–(5.11), (5.13) and Lemma 3.15.

Finally, by the semigroup property, symmetry, (5.3) and (5.7),

$$\begin{aligned}
p^\kappa(t,x,y) &= \int_{\mathbb{R}^d_+} p^\kappa(t/2,x,z) p^\kappa(t/2,y,z) dz \\
&\le c_1^2 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \int_{\mathbb{R}^d_+} \left((t/2)^{-d/\alpha} \wedge \frac{t/2}{|x-z|^{d+\alpha}} \right) \left((t/2)^{-d/\alpha} \wedge \frac{t/2}{|y-z|^{d+\alpha}} \right) dz \\
&\le c_9 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).
\end{aligned}$$

□

Now we prove (5.6) holds. We first consider the case $q < \alpha$.

Lemma 5.11. *If $q < \alpha$, then* (5.6) *holds true.*

Proof. By (3.30), it suffices to prove (5.6) when $t = 1$. By Lemma 5.9, it suffices to prove that there exists a constant $c_1 > 0$ such that $\mathbb{P}_x(\zeta^\kappa > 1/3) \le c_1(1 \wedge x_d)^q$ for all $x \in \mathbb{R}^d_+$. By Lemma 2.2 and the fact that $\mathbb{P}_x(\zeta > 1/3) \le 1$, without loss of generality, we can assume $\widetilde{x} = 0$ and $x_d < 2^{-5}$. Then, since $q < \alpha$, by (5.5) and Lemmas 5.5–5.6 (with $r = 1$), we get $\mathbb{P}_x(\zeta^\kappa > 1/3) \le c_2 x_d^q$. □

To remove the assumption $q < \alpha$ in Lemma 5.11, we will make use of a result from [31, 32]. Recall from Remark 3.14 that all results in [31, 32] are valid when $d > \alpha$. Now we state, and will prove later, that the desired result still holds true for $d = 1 \le \alpha$ (thus for all d).

Lemma 5.12. *Let $\gamma > q - \alpha$. There exists $C > 0$ such that for any $R > 0$, $U(R) \subset D \subset U(2R)$ and any $x = (\widetilde{0}, x_d) \in \mathbb{R}^d_+$ with $x_d \le R/10$, it holds that*

$$\mathbb{E}_x \int_0^{\tau^\kappa_D} (Y^{\kappa,D,d}_s)^\gamma ds = \int_0^\infty \int_D p^{\kappa,D}(t,x,z) z_d^\gamma dz dt \le C R^{\gamma+\alpha-q} x_d^q.$$

Proof. When $d > \alpha$, by Remark 3.14, the result follows from [31, Proposition 6.10] if $\kappa > 0$, and from [32, Proposition 6.8] if $\kappa = 0$. The proof of the case $d = 1 \le \alpha$ is postponed to the end of this section. □

Lemma 5.13. *Inequality* (5.6) *holds true.*

Proof. Again using (3.30), it suffices to prove (5.6) when $t = 1$.

Inequality (5.6) holds when $q < \alpha$ by Lemma 5.11. Now, assume that (5.6) holds for all $q < k\alpha$ for some $k \in \mathbb{N}$. We now show (5.6) also holds for $q \in [k\alpha, (k+1)\alpha)$ and hence (5.6) always holds by induction.

Fix $\varepsilon \in (0, \alpha)$ such that $q - \alpha + \varepsilon < k\alpha$. Then $q - \alpha + \varepsilon < q$ and $C(\alpha, q - \alpha + \varepsilon, \mathcal{B}) < C(\alpha, q, \mathcal{B}) = \kappa$. By (3.30) and the induction hypothesis, it holds that for any $s, u \in [0, 1/4]$ and $z, w \in \mathbb{R}^d_+$,

$$p^\kappa(1-s-u,z,w) = (1-s-u)^{-d/\alpha} p^\kappa(1, (1-s-u)^{-1/\alpha} z, (1-s-u)^{-1/\alpha} w)$$

$$\le 2^{d/\alpha} p^{C(\alpha,q-\alpha+\varepsilon,\mathcal{B})}(1,(1-s-u)^{-1/\alpha}z,(1-s-u)^{-1/\alpha}w)$$
$$\le c_3(1\wedge z_d)^{q-\alpha+\varepsilon}(1\wedge w_d)^{q-\alpha+\varepsilon}.$$

Here in the first inequality above we used the fact that $p^\kappa(1,x,y)\le p^{C(\alpha,q-\alpha+\varepsilon,\mathcal{B})}(1,x,y)$, which is a consequence of $C(\alpha,q,\mathcal{B})>C(\alpha,q-\alpha+\varepsilon,\mathcal{B})$. Therefore, by the semigroup property and symmetry, we get

$$p^\kappa(1,x,y) = 16\int_0^{1/4}\int_0^{1/4} p^\kappa(1,x,y)dsdu$$
$$= 16\int_0^{1/4}\int_0^{1/4}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+} p^\kappa(s,x,z)p^\kappa(1-s-u,z,w)p^\kappa(u,y,w)dzdwdsdu$$
$$\le 16c_3\Big(\int_0^{1/4}\int_{\mathbb{R}^d_+} p^\kappa(s,x,z)(1\wedge z_d)^{q-\alpha+\varepsilon}dzds\Big)\Big(\int_0^{1/4}\int_{\mathbb{R}^d_+} p^\kappa(u,y,w)(1\wedge w_d)^{q-\alpha+\varepsilon}dwdu\Big).$$

Thus, to conclude (5.6) by induction, it suffices to show that there exists a constant $c_4>0$ such that

$$\int_0^{1/4}\int_{\mathbb{R}^d_+} p^\kappa(s,v,z)(1\wedge z_d)^{q-\alpha+\varepsilon}dzds \le c_4(1\wedge v_d)^q, \quad v\in\mathbb{R}^d_+. \tag{5.14}$$

By Lemma 2.2, we can assume $\widetilde{v}=0$. If $v\notin U(2^{-4})$, then we get

$$(1\wedge v_d)^q \ge 2^{-5q} \ge 2^{-5q}\int_0^{1/4}\int_{\mathbb{R}^d_+} p^\kappa(s,v,z)dzds \ge 2^{-5q}\int_0^{1/4}\int_{\mathbb{R}^d_+} p^\kappa(s,v,z)(1\wedge z_d)^{q-\alpha+\varepsilon}dzds.$$

Otherwise, if $v\in U(2^{-4})$, then by (5.4), Fubini's theorem, Lemmas 5.5 and 5.12,

$$\int_0^{1/4}\int_{\mathbb{R}^d_+} p^\kappa(s,v,z)(1\wedge z_d)^{q-\alpha+\varepsilon}dzds$$
$$\le \int_0^{1/4}\mathbb{E}_v\left[(1\wedge Y_s^{\kappa,U(1),d})^{q-\alpha+\varepsilon} : \tau^\kappa_{U(1)}>s\right]ds + \int_0^{1/4}\mathbb{P}_v(Y^\kappa_{\tau^\kappa_{U(1)}}\in\mathbb{R}^d_+)ds$$
$$\le \mathbb{E}_v\int_0^{\tau^\kappa_{U(1)}}(1\wedge Y_s^{\kappa,U(1),d})^{q-\alpha+\varepsilon}ds + \frac14\mathbb{P}_v(Y^\kappa_{\tau_{U(1)}}\in\mathbb{R}^d_+) \le c_5 v_d^q.$$

This completes the proof. □

Proof of Proposition 5.1: The assertion is a direct consequence of Lemmas 5.13 and 5.10. □

In the remainder of this section we complete the proof of Lemma 5.12.

Lemma 5.14. *If $d=1\le\alpha$, then there exists $C>0$ such that for all $x,y\in(0,\infty)$,*

$$G^\kappa(x,y)\le C\left(1\wedge\frac{x\wedge y}{|x-y|}\right)^{q\wedge(\alpha-\frac12)}(x\vee y)^{\alpha-1}\log\left(e+\frac{x\vee y}{|x-y|}\right). \tag{5.15}$$

Proof. Let $\widetilde{q}:=q\wedge(\alpha-\frac12)\in(0,q]\cap[\alpha-1,\alpha)$. By symmetry and scaling (3.30), we can assume $|x-y|=1$ and $x\le y$ without loss of generality. Then $y=x+1>x\vee 1$. It suffices to show that

$$G^\kappa(x,y)\le Cx^{\widetilde{q}}y^{\alpha-1}\log(e+y).$$

Note that $1+2\widetilde{q}>1+\widetilde{q}\ge\alpha$. Since $C(\alpha,\widetilde{q},\mathcal{B})\le C(\alpha,q,\mathcal{B})=\kappa$ implies that $p^\kappa(t,x,y)\le p^{C(\alpha,\widetilde{q},\mathcal{B})}(t,x,y)$, using the fact $y>x\vee 1$ and Lemmas 5.11 and 5.10, we get

$$G^\kappa(x,y)\le\int_0^\infty p^{C(\alpha,\widetilde{q},\mathcal{B})}(t,x,y)dt$$

$$\le c_1\left(x^{\widetilde{q}}\int_0^1 t^{(\alpha-\widetilde{q})/\alpha}dt + x^{\widetilde{q}}\int_1^{y^\alpha} t^{-(1+\widetilde{q})/\alpha}dt + x^{\widetilde{q}}y^{\widetilde{q}}\int_{y^\alpha}^\infty t^{-(1+2\widetilde{q})/\alpha}dt\right)$$
$$\le c_1x^{\widetilde{q}}\left(1+\log(y^\alpha)+\frac{\alpha}{1+2\widetilde{q}-\alpha}y^{\alpha-\widetilde{q}-1}\right) \le c_2x^{\widetilde{q}}y^{\alpha-1}\log(e+y).$$

The proof is complete. □

We now improve (5.15) by removing the term $\alpha-\frac{1}{2}$ from the power of the first factor.

Lemma 5.15. *If $d=1\le\alpha$, then there exists $C>0$ such that for all $x,y\in(0,\infty)$,*

$$G^\kappa(x,y)\le C\left(1\wedge\frac{x\wedge y}{|x-y|}\right)^q (x\vee y)^{\alpha-1}\log\left(e+\frac{x\vee y}{|x-y|}\right).$$

Proof. Let $r=2^{-6}$. By symmetry, (3.30) and Lemma 5.14, without loss of generality, we can assume $|x-y|=1$ and $x<y\wedge(r/2)$. Note that $y=x+1\in(1,1+r/2)$. It suffices to show that

$$G^\kappa(x,y)\le Cx^q.$$

For $z\in(0,r/2)$ and $w\in(r,\infty)$, we have $z<w/2\le w-z<w$ and $w\vee y\asymp w$. Thus, for any $z\in(0,r/2)$, using Lemmas 5.3 and 5.14, we obtain

$$\int_r^\infty G^\kappa(w,y)\mathcal{B}(z,w)(w-z)^{-1-\alpha}dw$$
$$\le c_1z^{\beta_1}(|\log z|^{\beta_3}\vee 1)\int_r^\infty \frac{G^\kappa(w,y)}{w^{1+\alpha+\beta_1}}(1+\mathbf{1}_{\{w\ge1\}}(\log w)^{\beta_3})dw$$
$$\le c_2z^{\beta_1}|\log z|^{\beta_3}\int_r^\infty \log\left(e+\frac{w}{|w-y|}\right)\frac{(1+\mathbf{1}_{\{w\ge1\}}(\log w)^{\beta_3})}{w^{2+\beta_1}}dw.$$

Hence, by the Lévy system formula and Lemma 5.4, we get

$$\mathbb{E}_x\left[G^\kappa(Y^\kappa_{\tau^\kappa_{(0,r/2)}},y);Y^\kappa_{\tau^\kappa_{(0,r/2)}}\notin(0,r)\right]$$
$$\le c_2\mathbb{E}_x\int_0^{\tau^\kappa_{(0,r/2)}}(Y^{\kappa,d}_t)^{\beta_1}|\log(Y^{\kappa,d}_t)|^{\beta_3}dt\int_r^\infty \log\left(e+\frac{w}{|w-y|}\right)\frac{(1+\mathbf{1}_{\{w\ge1\}}(\log w)^{\beta_3})}{w^{2+\beta_1}}dw$$
$$\le c_3x^q\int_r^\infty \log\left(e+\frac{w}{|w-y|}\right)\frac{(1+\mathbf{1}_{\{w\ge1\}}(\log w)^{\beta_3})}{w^{2+\beta_1}}dw. \tag{5.16}$$

Since $w-y\ge w/2$ for $w\in(2y,\infty)$ and $y\in(1,1+r/2)$, using a change of the variables, we obtain

$$\int_r^\infty \log\left(e+\frac{w}{|w-y|}\right)\frac{(1+\mathbf{1}_{\{w\ge1\}}(\log w)^{\beta_3})}{w^{2+\beta_1}}dw$$
$$\le\frac{1+(\log(2+r))^{\beta_3}}{r^{2+\beta_1}}\int_r^{2y}\log\left(e+\frac{2+r}{|w-y|}\right)dw+\log(e+2)\int_{2y}^\infty\frac{1+(\log w)^{\beta_3}}{w^{2+\beta_1}}dw$$
$$\le\frac{2(1+(\log(2+r))^{\beta_3})}{r^{2+\beta_1}}\int_0^{1+r/2}\log\left(e+\frac{2+r}{v}\right)dv+c_4\le c_5.$$

Thus, by (5.16), it holds that

$$\mathbb{E}_x\left[G^\kappa(Y^\kappa_{\tau^\kappa_{(0,r/2)}},y);Y^\kappa_{\tau^\kappa_{(0,r/2)}}\notin(0,r)\right]\le c_3c_5x^q. \tag{5.17}$$

Since $z\mapsto G^\kappa(z,y)$ is harmonic in $(0,2r)$ with respect to Y^κ and vanishes continuously as $z\to0$ by Lemma 5.14, we get from Proposition 5.7 that $G^\kappa(z,y)\le c_6G^\kappa(r,y)$ for all $z\in(0,r)$. Therefore, using Lemmas 5.5 and 5.14, since $y\in(1,1+r/2)$, we obtain

$$\mathbb{E}_x\left[G^\kappa(Y^\kappa_{\tau^\kappa_{(0,r/2)}},y);Y^\kappa_{\tau^\kappa_{(0,r/2)}}\in(0,r)\right]\le c_6G^\kappa(r,y)\mathbb{P}_x(Y^\kappa_{\tau^\kappa_{(0,r/2)}}\in(0,r))$$

$$\le c_7\left(\frac{x}{r}\right)^q\left(\frac{r}{y-r}\right)^{q\wedge(\alpha-\frac{1}{2})}y^{\alpha-1}\log\left(e+\frac{y}{y-r}\right)\le c_8x^q. \tag{5.18}$$

Combining (5.17) and (5.18) and using the harmonicity of $z\mapsto G^\kappa(z,y)$ in $(0,2r)$, we arrive at

$$G^\kappa(x,y)=\mathbb{E}_x\left[G^\kappa(Y^\kappa_{\tau^\kappa_{(0,r/2)}},y);Y^\kappa_{\tau^\kappa_{(0,r/2)}}\notin(0,r)\right]+\mathbb{E}_x\left[G^\kappa(Y^\kappa_{\tau^\kappa_{(0,r/2)}},y);Y^\kappa_{\tau^\kappa_{(0,r/2)}}\in(0,r)\right]\le c_9x^q.$$

The proof is complete. □

Proof of Lemma 5.12 for $d=1\le\alpha$: Assume that $d=1\le\alpha$. Since $D\subset U(2R)=[0,R)$, using Fubini's theorem and Lemma 5.15, we have

$$\begin{aligned}
&\int_0^\infty\int_D p^{\kappa,D}(t,x,z)z^\gamma dzdt\le\int_0^R\int_0^\infty p^{\kappa,D}(t,x,z)dt\,z^\gamma dz\le\int_0^R G^\kappa(x,z)z^\gamma dz\\
&\le c_1\int_0^{x/2}\frac{z^{q+\gamma}}{|x-z|^q}x^{\alpha-1}\log\left(e+\frac{x}{|x-z|}\right)dz+c_1\int_{x/2}^{2x}z^\gamma(2x)^{\alpha-1}\log\left(e+\frac{2x}{|x-z|}\right)dz\\
&\quad+c_1\int_{2x}^R\frac{x^q}{|x-z|^q}z^{\gamma+\alpha-1}\log\left(e+\frac{z}{|x-z|}\right)dz\\
&=:c_1(I+II+III).
\end{aligned}$$

Since $\gamma+\alpha-q>0$ and $x\le R/10$, we see that

$$I\le x^{q+\gamma+\alpha-1}\int_0^{x/2}\frac{1}{(x/2)^q}\log\left(e+\frac{x}{(x/2)}\right)dz\le 2^{q-1}\log(e+2)x^{\gamma+\alpha}\le c_2R^{\gamma+\alpha-q}x^q.$$

Next, using the change of the variables $z=xy$, we also get

$$\begin{aligned}
II&\le(2x)^{\gamma+\alpha-1}\int_{x/2}^{2x}\log\left(e+\frac{2x}{|x-z|}\right)dz=2^{\gamma+\alpha-1}x^{\gamma+\alpha}\int_{1/2}^2\log\left(e+\frac{2}{|1-y|}\right)dy\\
&\le c_3x^{\gamma+\alpha}\le c_3R^{\gamma+\alpha-q}x^q.
\end{aligned}$$

Lastly, since $|x-z|\ge z/2$ for all $z\ge 2x$, we obtain

$$III\le 2^q\log(e+2)x^q\int_{2x}^R z^{\gamma+\alpha-q-1}dz\le c_4R^{\gamma+\alpha-q}x^q.$$

The proof is complete. □

6. Sharp lower bounds of heat kernels

In this and the next section, whenever we consider $\overline{Y}$, we assume that **(A1)**, **(A3)** *and* **(A4)** *hold, and whenever we consider Y^κ, we assume that all* **(A1)**–**(A4)** *hold.* Note that **(IUBS)** holds under this setting by Lemma 4.2.

We fix $\kappa\in[0,\infty)$. The following notational convenience will be used throughout this and the next section. When we consider Y^κ, we assume that $\kappa>0$ if $\alpha\in(0,1]$ (see Lemma 2.3), and denote by q the strictly positive constant $q_k\in((\alpha-1)_+,\alpha+\beta_1)$ from (1.8). Additionally, we write Y, $p(t,x,y)$, $p^D(t,x,y)$, τ_D and ζ instead of Y^κ, $p^\kappa(t,x,y)$, $p^{\kappa,D}(t,x,y)$, τ^κ_D and ζ^κ. When we consider $\overline{Y}$, the letter q denotes 0, and we write Y, $p(t,x,y)$, $p^D(t,x,y)$, τ_D and ζ instead of $\overline{Y}$, $\overline{p}(t,x.y)$, $\overline{p}^D(t,x,y)$, $\overline{\tau}_D$ and ∞.

Recall the definitions of $V_x(t)$ and $W_x(t)$ from (3.18). We let $V_x:=V_x(1)$ and $W_x:=W_x(1)$.

Lemma 6.1. *There exist constants $M>1$ and $C>0$ such that for all $t>0$ and $x\in\mathbb{R}^d_+$,*

$$\inf_{z\in W_x(t)}p(Mt,x,z)\ge C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q t^{-d/\alpha}.$$

Proof. When $q=0$, the result is given in Lemma 3.6.

Suppose $q>0$. By the scaling property (3.30), it suffices to prove the lemma for $t=1$. If $x_d \ge 1$, then the result follows from Proposition 3.2. Assume $x_d<1$. By Proposition 3.2, for any $M>1$, there is $c_1=c_1(M)>0$ such that

$$\inf_{w,z\in W_x,\,1\le s\le M} p(s,w,z) \ge c_1 \quad \text{for all } x\in\mathbb{R}^d_+. \tag{6.1}$$

Using the strong Markov property and (6.1), we see that for all $M>1$ and $z\in W_x$,

$$\begin{aligned} p(M,x,z) &\ge \mathbb{E}_x\left[p(M-\tau_{V_x},Y_{\tau_{V_x}},z):\tau_{V_x}\le M-1,\,Y_{\tau_{V_x}}\in W_x\right] \\ &\ge \left(\inf_{w\in W_x,\,1\le s\le M} p(s,w,z)\right)\mathbb{P}_x\left(\tau_{V_x}\le M-1,\,Y_{\tau_{V_x}}\in W_x\right) \\ &\ge c_1\left(\mathbb{P}_x(Y_{\tau_{V_x}}\in W_x)-\mathbb{P}_x(\tau_{V_x(t)}>M-1)\right). \end{aligned}$$

Note that, by [30, Lemma 5.10] for $\kappa>0$ and [32, Theorem 1.1] for $\kappa=0$, we have

$$\mathbb{P}_x(Y_{\tau_{V_x}}\in W_x)\ge 2c_2 x_d^q$$

and, by Corollary 5.2, we also have

$$\mathbb{P}_x(\tau_{V_x}>M-1)\le \mathbb{P}_x(\zeta>M-1)\le c_3(x_d/(M-1)^{1/\alpha})^q.$$

Thus, we can choose $M=1+(c_3/c_2)^{\alpha/q}$ so that $2c_2-c_3(M-1)^{-q/\alpha}=c_2$, which implies $p(M,x,z)\ge c_2x_d^q$. □

For any $b_1,b_2,b_3,b_4\ge 0$, we define for $t\ge 0$ and $x,y\in\overline{\mathbb{R}}^d_+$,

$$\begin{aligned} A_{b_1,b_2,b_3,b_4}(t,x,y) &:= \left(\frac{(x_d\wedge y_d)\vee t^{1/\alpha}}{|x-y|}\wedge 1\right)^{b_1}\left(\frac{(x_d\vee y_d)\vee t^{1/\alpha}}{|x-y|}\wedge 1\right)^{b_2} \\ &\quad\times \log^{b_3}\left(e+\frac{((x_d\vee y_d)\vee t^{1/\alpha})\wedge|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right)\log^{b_4}\left(e+\frac{|x-y|}{((x_d\vee y_d)\vee t^{1/\alpha})\wedge|x-y|}\right). \end{aligned} \tag{6.2}$$

We first note that $A_{b_1,b_2,b_3,b_4}(0,x,y)=B_{b_1,b_2,b_3,b_4}(x,y)$ for $x,y\in\mathbb{R}^d_+$ and

$$A_{b_1,b_2,b_3,b_4}(t,x,y)\asymp B_{b_1,b_2,b_3,b_4}(x+t^{1/\alpha}\mathbf{e}_d,y+t^{1/\alpha}\mathbf{e}_d) \quad \text{for } t\ge 0,x,y\in\mathbb{R}^d_+, \tag{6.3}$$

since $a\vee b\asymp a+b$ for all $a,b\ge 0$. Note also that for any $a>0$, there exists $c>0$ such that

$$A_{b_1,b_2,b_3,b_4}(t,x,y)\ge c(a\wedge 1)^{b_1+b_2}, \tag{6.4}$$

for all $t>0$ and $x,y\in\mathbb{R}^d_+$ with $(x_d\wedge y_d)+t^{1/\alpha}\ge a|x-y|$.

Proposition 6.2. *There exists a constant $C>0$ such that for all $t>0$ and $x,y\in\mathbb{R}^d_+$,*

$$p(t,x,y)\ge C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).$$

Proof. Without loss of generality, we assume $x_d\le y_d$. Let $M>1$ be the constant in Lemma 6.1. By the semigroup property and Lemma 6.1,

$$\begin{aligned} p(t,x,y) &\ge \int_{W_x(t/(3M))}\int_{W_y(t/(3M))} p(t/3,x,z)\,p(t/3,z,w)\,p(t/3,w,y)dzdw \\ &\ge c_1t^{2d/\alpha}\left(\inf_{z\in W_x(t/(3M))}p(t/3,x,z)\right)\left(\inf_{w\in W_y(t/(3M))}p(t/3,y,w)\right) \\ &\quad\times\left(\inf_{(z,w)\in W_x(t/(3M))\times W_y(t/(3M))}p(t/3,z,w)\right) \\ &\ge c_2\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\inf_{(z,w)\in W_x(t/(3M))\times W_y(t/(3M))}p(t/3,z,w). \end{aligned} \tag{6.5}$$

For all $(z,w) \in W_x(t/(3M)) \times W_y(t/(3M))$, we have $z_d \geq x_d + 5(t/(3M))^{1/\alpha}$, $w_d \geq y_d + 5(t/(3M))^{1/\alpha}$, $|z-w| \leq |z-x| + |x-y| + |y-w| \leq |x-y| + 20(t/(3M))^{1/\alpha}$ and $|z-w| \geq |x-y| - |z-x| - |y-w| \geq |x-y| - 20(t/(3M))^{1/\alpha}$. Hence, if $|x-y| \leq 21(t/(3M))^{1/\alpha}$, then

$$z_d \wedge w_d \geq 5(\frac{t}{3M})^{1/\alpha} \geq \frac{5}{41}\big((\frac{t}{3M})^{1/\alpha} \vee |z-w|\big), \quad (z,w) \in W_x(\frac{t}{3M}) \times W_y(\frac{t}{3M}). \tag{6.6}$$

If $|x-y| > 21(t/(3M))^{1/\alpha}$, then using **(A3)**(I) in the first inequality below, $|z-w| \leq 2|x-y|$ and Lemma 10.9 in the second and (6.3) in the third, we get

$$\begin{aligned} &\inf_{(z,w)\in W_x(t/(3M))\times W_y(t/(3M))} J(z,w) \geq C_2^{-1} \inf_{(z,w)\in W_x(t/(3M))\times W_y(t/(3M))} \frac{B_{\beta_1,\beta_2,\beta_3,\beta_4}(z,w)}{|z-w|^{d+\alpha}} \\ &\geq c_3 \frac{B_{\beta_1,\beta_2,\beta_3,\beta_4}(x+5(t/(3M))^{1/\alpha}\mathbf{e}_d, y+5(t/(3M))^{1/\alpha}\mathbf{e}_d)}{|x-y|^{d+\alpha}} \geq c_4 \frac{A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)}{|x-y|^{d+\alpha}}. \end{aligned} \tag{6.7}$$

Now, by Propositions 3.2 and 4.5, (6.6), (6.7) and (6.4), we have

$$\begin{aligned} &\inf_{(z,w)\in W_x(t/(3M))\times W_y(t/(3M))} p(t/3,z,w) \\ &\geq c_5 \begin{cases} t^{-d/\alpha} & \text{if } |x-y| \leq 21(t/(3M))^{1/\alpha}, \\ tA_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)|x-y|^{-d-\alpha} & \text{if } |x-y| > 21(t/(3M))^{1/\alpha} \end{cases} \\ &\geq c_6 A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \end{aligned} \tag{6.8}$$

Combining (6.5) and (6.8), we get the desired result. □

Corollary 6.3. *It holds that for any $t>0$ and $x \in \mathbb{R}^d_+$,*

$$\mathbb{P}_x(\zeta > t) \asymp \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q.$$

Proof. Using Proposition 6.2 and (6.4), we get

$$\begin{aligned} \mathbb{P}_x(\zeta > t) &\geq \int_{B(x,2t^{1/\alpha}):y_d\geq t^{1/\alpha}} p(t,x,y)dy \geq c_1\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q t^{-d/\alpha} \int_{B(x,2t^{1/\alpha}):y_d\geq t^{1/\alpha}} dy \\ &\geq c_2\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q. \end{aligned}$$

Combining the above with Corollary 5.2, we arrive at the result. □

Lemma 6.4. *There exists a constant $C>0$ such that for all $t>0$ and $x,y \in \mathbb{R}^d_+$,*

$$\begin{aligned} p(t,x,y) &\geq C\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \\ &\times \left(|x-y|^\alpha \wedge t\right) \int_{(y_d\vee t^{1/\alpha})\wedge(|x-y|/4)}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+r\mathbf{e}_d,y)\frac{dr}{r^{\alpha+1}}. \end{aligned}$$

Proof. Case (i): $(x_d \wedge y_d) \vee t^{1/\alpha} \geq |x-y|/4$. In this case, we see from (6.4) that $A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)$ is bounded below by a positive constant. Moreover, since $\sup_{s>0,\, z,w\in\mathbb{R}^d_+} A_{\beta_1,\beta_2,\beta_3,\beta_4}(s,z,w) \leq c_1 < \infty$, we get

$$\begin{aligned} &|x-y|^\alpha \int_{(y_d\vee t^{1/\alpha})\wedge(|x-y|/4)}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+r\mathbf{e}_d,y)\frac{dr}{r^{\alpha+1}} \\ &\leq c_1^2|x-y|^\alpha \int_{|x-y|/4}^\infty \frac{dr}{r^{\alpha+1}} = 4^\alpha c_1^2/\alpha. \end{aligned}$$

Therefore, we get the result from Proposition 6.2 in this case.

Case (ii): $(x_d \wedge y_d) \vee t^{1/\alpha} < |x-y|/4$. For $r>0$, set $x(r) := x + r\mathbf{e}_d$ and

$$K(r) := \left\{ z = (\widetilde{z}, z_d) \in \mathbb{R}^d_+ : |\widetilde{z} - \widetilde{x}| < \frac{r}{2},\ z_d = x_d + r \right\}.$$

Let

$$K := \left\{ z \in \mathbb{R}^d_+ : z \in K(r) \text{ for some } (y_d \vee t^{1/\alpha}) \wedge \frac{|x-y|}{4} < r < \frac{|x-y|}{2} \right\}.$$

For any $z = (\widetilde{z}, x_d + r) \in K$, since

$$r \le |x-z| \le \frac{\sqrt{5}}{2} r \le \frac{\sqrt{5}}{4}|x-y|, \qquad (1 - \frac{\sqrt{5}}{4})|x-y| \le |y-z| \le (1+\frac{\sqrt{5}}{4})|x-y|, \tag{6.9}$$

we see that

$$\begin{aligned} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,z) &\asymp \left(1 \wedge \frac{x_d \vee t^{1/\alpha}}{r}\right)^{\beta_1} \left(1 \wedge \frac{(x_d + r) \vee t^{1/\alpha}}{r}\right)^{\beta_2} \\ &\quad \times \log^{\beta_3}\left(e + \frac{((x_d+r)\vee t^{1/\alpha}) \wedge r}{(x_d \vee t^{1/\alpha}) \wedge r}\right) \log^{\beta_4}\left(e + \frac{r}{((x_d+r)\vee t^{1/\alpha}) \wedge r}\right) \\ &\asymp \left(1 \wedge \frac{x_d \vee t^{1/\alpha}}{r}\right)^{\beta_1} \log^{\beta_3}\left(e + \frac{r}{(x_d \vee t^{1/\alpha}) \wedge r}\right) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x(r)) \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} &A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,z,y) \\ &\asymp \left(1 \wedge \frac{y_d \vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1} \left(1 \wedge \frac{x_d + r}{|x-y|}\right)^{\beta_2} \log^{\beta_3}\left(e + \frac{x_d + r}{y_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{|x-y|}{x_d + r}\right) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x(r),y). \end{aligned} \tag{6.11}$$

Thus, using the semigroup property, Proposition 6.2 and (6.9), we get

$$\begin{aligned} p(t,x,y) &\ge \int_K p(t/2,x,z)p(t/2,z,y)dz \\ &\ge c_2^2 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \int_K \frac{tA_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,z)}{|x-z|^{d+\alpha}} \frac{tA_{\beta_1,\beta_2,\beta_3,\beta_4}(t,z,y)}{|y-z|^{d+\alpha}} dz \\ &\ge c_3 t^2 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \\ &\quad \times \int_{(y_d \vee t^{1/\alpha}) \wedge (|x-y|/4)}^{|x-y|/2} \frac{A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x(r))}{r^{d+\alpha}} \frac{A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x(r),y)}{|x-y|^{d+\alpha}} \int_{K(r)} d\widetilde{z} dr \\ &\ge c_4 \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \frac{t}{|x-y|^{d+\alpha}} \\ &\quad \times t \int_{(y_d \vee t^{1/\alpha}) \wedge (|x-y|/4)}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x(r))\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x(r),y) \frac{dr}{r^{\alpha+1}}. \end{aligned}$$

□

Note that, for any $x, y \in \mathbb{R}^d_+$ and $|x-y|/4 \le r \le |x-y|/2$,

$$\begin{aligned} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d) &\asymp \left(1 \wedge \frac{x_d \vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1} \log^{\beta_3}\left(e + \frac{|x-y|}{(x_d \vee t^{1/\alpha}) \wedge |x-y|}\right) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x + \frac{|x-y|}{2}\mathbf{e}_d) \end{aligned} \tag{6.12}$$

and, since $y_d \le x_d + 4r$,

$$\begin{aligned} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x + r\mathbf{e}_d, y) &\asymp \left(1 \wedge \frac{y_d \vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1} \log^{\beta_3}\left(e + \frac{|x-y|}{(y_d \vee t^{1/\alpha}) \wedge |x-y|}\right) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x + \frac{|x-y|}{2}\mathbf{e}_d, y) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, y, y + \frac{|x-y|}{2}\mathbf{e}_d). \end{aligned} \tag{6.13}$$

In particular, we have

$$\begin{aligned} &A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, x + \frac{|x-y|}{2}\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x + \frac{|x-y|}{2}\mathbf{e}_d, y) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, x + \frac{|x-y|}{2}\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, y, y + \frac{|x-y|}{2}\mathbf{e}_d) \\ &\asymp A_{\beta_1,\beta_1,0,\beta_3}(t, x, y) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) \vee t^{1/\alpha}) \wedge |x-y|}\right). \end{aligned} \tag{6.14}$$

By (6.12)–(6.14), we have that for all $t > 0$ and $x, y \in \mathbb{R}^d_+$,

$$\begin{aligned} &\int_{|x-y|/4}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, x + r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x + r\mathbf{e}_d, y) \frac{dr}{r^{\alpha+1}} \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, x + \frac{|x-y|}{2}\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x + \frac{|x-y|}{2}\mathbf{e}_d, y) \int_{|x-y|/4}^{|x-y|/2} \frac{dr}{r^{\alpha+1}} \\ &\asymp |x-y|^{-\alpha} A_{\beta_1,\beta_1,0,\beta_3}(t, x, y) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) \vee t^{1/\alpha}) \wedge |x-y|}\right). \end{aligned} \tag{6.15}$$

Remark 6.5. Here we give a proof of the comparability of (1.6) and (1.7). Let $t > 0$ and $x, y \in \overline{\mathbb{R}}^d_+$ be such that $|x-y| > 6t^{1/\alpha}$. For any $z \in B(x + 2^{-1}|x-y|\mathbf{e}_d, 4^{-1}|x-y|)$, by using the triangle inequality several times, we have

$$z_d - t^{1/\alpha} \asymp z_d \asymp x_d \wedge y_d + |x-y| \asymp x_d \vee y_d + |x-y| \tag{6.16}$$

and

$$|x + t^{1/\alpha}\mathbf{e}_d - z| \asymp |y + t^{1/\alpha}\mathbf{e}_d - z| \asymp |x-z| \asymp |y-z| \asymp |x-y|. \tag{6.17}$$

For any $t > 0$ and $x, y \in \overline{\mathbb{R}}^d_+$ with $|x-y| > 6t^{1/\alpha}$, if $x_d \wedge y_d \ge |x-y|/4$, then by **(A3)**, (6.3), (6.4), (6.16) and (6.17),

$$\begin{aligned} &t|x-y|^{d+\alpha} \int_{B(x+2^{-1}|x-y|\mathbf{e}_d,\, 4^{-1}|x-y|)} J(x + t^{1/\alpha}\mathbf{e}_d, z)\, J(z, y + t^{1/\alpha}\mathbf{e}_d) dz \\ &\asymp \frac{t}{|x-y|^{d+\alpha}} \int_{B(x+2^{-1}|x-y|\mathbf{e}_d,\, 4^{-1}|x-y|)} dz \asymp \frac{t}{|x-y|^{\alpha}} \\ &\asymp \left(1 \wedge \frac{t}{|x-y|^{\alpha}}\right) B_{\beta_1,\beta_1,0,\beta_3}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) + t^{1/\alpha}) \wedge |x-y|}\right), \end{aligned}$$

Otherwise, if $x_d \wedge y_d < |x-y|/4$, then $x_d \vee y_d \le x_d \wedge y_d + |x-y| < 5|x-y|/4$ so that

$$z_d - t^{1/\alpha} \asymp z_d \asymp |x-y| \quad \text{for } z \in B(x + 2^{-1}|x-y|\mathbf{e}_d,\, 4^{-1}|x-y|)$$

by (6.16). Using this, (6.17) and (6.3), we get that for $z \in B(x + 2^{-1}|x-y|\mathbf{e}_d,\, 4^{-1}|x-y|)$,

$$\begin{aligned} &B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, z) \asymp B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, z + t^{1/\alpha}\mathbf{e}_d) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, z) \asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, x + 2^{-1}|x-y|\mathbf{e}_d) \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} &B_{\beta_1,\beta_2,\beta_3,\beta_4}(z, y+t^{1/\alpha}\mathbf{e}_d) \asymp B_{\beta_1,\beta_2,\beta_3,\beta_4}(z+t^{1/\alpha}\mathbf{e}_d, y+t^{1/\alpha}\mathbf{e}_d) \\ &\asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,z,z) \asymp A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x+2^{-1}|x-y|\mathbf{e}_d, y). \end{aligned} \tag{6.19}$$

By **(A3)**, (6.17), (6.18), (6.19) and (6.14), we arrive at

$$\begin{aligned} &t|x-y|^{d+\alpha}\int_{B(x+2^{-1}|x-y|\mathbf{e}_d,\,4^{-1}|x-y|)} J(x+t^{1/\alpha}\mathbf{e}_d, z)\, J(z, y+t^{1/\alpha}\mathbf{e}_d)dz \\ &\asymp \frac{tA_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+2^{-1}|x-y|\mathbf{e}_d)A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+2^{-1}|x-y|\mathbf{e}_d,y)}{|x-y|^{d+\alpha}}\int_{B(x+2^{-1}|x-y|\mathbf{e}_d,\,4^{-1}|x-y|)} dz \\ &\asymp \frac{t}{|x-y|^{\alpha}}A_{\beta_1,\beta_1,0,\beta_3}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right) \\ &\asymp \left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)B_{\beta_1,\beta_1,0,\beta_3}(x+t^{1/\alpha}\mathbf{e}_d, y+t^{1/\alpha}\mathbf{e}_d)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)+t^{1/\alpha})\wedge|x-y|}\right). \end{aligned}$$

Hence, (1.6) and (1.7) are comparable.

Combining (6.15), Proposition 6.2 and Lemma 6.4, we get the following result.

Proposition 6.6. *There exists a constant $C>0$ such that for all $t>0$ and $x,y\in\mathbb{R}^d_+$,*

$$\begin{aligned} p(t,x,y) &\geq C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\Big[A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y) \\ &\quad+\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+\frac{|x-y|}{2}\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+\frac{|x-y|}{2}\mathbf{e}_d,y)\Big] \\ &\asymp\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\Big[A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y) \\ &\quad+\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)A_{\beta_1,\beta_1,0,\beta_3}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right)\Big]. \end{aligned}$$

Remark 6.7. If $\beta_2<\alpha+\beta_1$, then there is a constant $C>0$ such that for all $t>0$ and $x,y\in\mathbb{R}^d$,

$$\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)A_{\beta_1,\beta_1,0,\beta_3}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right)\leq CA_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y).$$

Indeed, for $\varepsilon:=(\alpha+\beta_1-\beta_2)/2>0$, using (10.1), we get that for all $t>0$ and $x,y\in\mathbb{R}^d$,

$$\begin{aligned} &\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)A_{\beta_1,\beta_1,0,\beta_3}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right) \\ &\leq c_1\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)\left(1\wedge\frac{x_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1-\varepsilon}\left(1\wedge\frac{y_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1-\varepsilon} \\ &= c_1\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{\varepsilon}\left(1\wedge\frac{x_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1-\varepsilon}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{-\beta_1+\beta_2+\varepsilon}\left(1\wedge\frac{y_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1-\varepsilon} \\ &\leq c_1\left(1\wedge\frac{x_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1}\left(1\wedge\frac{y_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_2}\leq c_1A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y). \end{aligned}$$

Therefore, in view of Proposition 6.2, Proposition 6.6 is relevant only if $\beta_2\geq\alpha+\beta_1$.

Lemma 6.8. *Suppose that $\beta_2=\alpha+\beta_1$. Then for all $t>0$ and $x,y\in\mathbb{R}^d_+$,*

$$\begin{aligned} &(|x-y|^{\alpha}\wedge t)\int_{(x_d\vee y_d\vee t^{1/\alpha})\wedge(|x-y|/4)}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+r\mathbf{e}_d,y)\frac{dr}{r^{\alpha+1}} \\ &\asymp\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)A_{\beta_1,\beta_1,0,\beta_3+\beta_4+1}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right). \end{aligned}$$

Proof. Without loss of generality, we assume $x_d \le y_d$.

Assume first that $y_d \vee t^{1/\alpha} < |x-y|/4$. Using $\beta_2 = \alpha+\beta_1$ and (6.10)–(6.11), since $x_d \le y_d \vee t^{1/\alpha}$, we get

$$\begin{aligned}
&\int_{y_d\vee t^{1/\alpha}}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+r\mathbf{e}_d,y)\frac{dr}{r^{\alpha+1}}\\
&\asymp \int_{y_d\vee t^{1/\alpha}}^{|x-y|/2} \left(\frac{x_d\vee t^{1/\alpha}}{r}\right)^{\beta_1}\left(\frac{y_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1}\left(\frac{r}{|x-y|}\right)^{\alpha+\beta_1}\\
&\qquad\times \log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{|x-y|}{r}\right)\frac{dr}{r^{\alpha+1}}\\
&\asymp \left(\frac{x_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1}\left(\frac{y_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1}|x-y|^{-\alpha}\\
&\qquad\times \int_{y_d\vee t^{1/\alpha}}^{|x-y|/2}\log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{|x-y|}{r}\right)\frac{dr}{r}.
\end{aligned}\tag{6.20}$$

By a change of the variables and Lemma 10.11, we see that

$$\begin{aligned}
&\int_{y_d\vee t^{1/\alpha}}^{|x-y|/2}\log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{|x-y|}{r}\right)\frac{dr}{r}\\
&=\int_{4(y_d\vee t^{1/\alpha})/|x-y|}^{2}\log^{\beta_3}\left(e+\frac{|x-y|s}{4(x_d\vee t^{1/\alpha})}\right)\log^{\beta_3}\left(e+\frac{|x-y|s}{4(y_d\vee t^{1/\alpha})}\right)\log^{\beta_4}\left(e+\frac{4}{s}\right)\frac{ds}{s}\\
&\asymp \log^{\beta_3}\left(e+\frac{|x-y|}{4(x_d\vee t^{1/\alpha})}\right)\log^{\beta_3+\beta_4+1}\left(e+\frac{|x-y|}{4(y_d\vee t^{1/\alpha})}\right).
\end{aligned}\tag{6.21}$$

By (6.20)–(6.21), since $y_d \vee t^{1/\alpha} < |x-y|/4$, we obtain

$$\begin{aligned}
&\big(|x-y|^\alpha\wedge t\big)\int_{y_d\vee t^{1/\alpha}}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+r\mathbf{e}_d,y)\frac{dr}{r^{\alpha+1}}\\
&\asymp \left(1\wedge\frac{t}{|x-y|^\alpha}\right)\left(\frac{x_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1}\left(\frac{y_d\vee t^{1/\alpha}}{|x-y|}\right)^{\beta_1}\\
&\qquad\times\log^{\beta_3}\left(e+\frac{|x-y|}{4(x_d\vee t^{1/\alpha})}\right)\log^{\beta_3+\beta_4+1}\left(e+\frac{|x-y|}{4(y_d\vee t^{1/\alpha})}\right).
\end{aligned}\tag{6.22}$$

If $y_d\vee t^{1/\alpha}\ge |x-y|/4$, then $\log\big(e+\frac{|x-y|}{(y_d\vee t^{1/\alpha})\wedge|x-y|}\big)\asymp 1$. Thus, by (6.15),

$$\begin{aligned}
&\big(|x-y|^\alpha\wedge t\big)\int_{|x-y|/4}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,x+r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x+r\mathbf{e}_d,y)\frac{dr}{r^{\alpha+1}}\\
&\asymp\left(1\wedge\frac{t}{|x-y|^\alpha}\right)A_{\beta_1,\beta_1,0,\beta_3+\beta_4+1}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{(x_d\vee t^{1/\alpha})\wedge|x-y|}\right).
\end{aligned}\tag{6.23}$$

The proof is now complete. □

Combining Proposition 6.2, Lemma 6.4 and Lemma 6.8. we get the following

Proposition 6.9. *Suppose that $\beta_2=\alpha+\beta_1$. There exists a constant $C>0$ such that for all $t>0$ and $x,y\in\mathbb{R}^d_+$,*

$$\begin{aligned}
p(t,x,y)\ge C&\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\Bigg[A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)\\
&+\left(1\wedge\frac{t}{|x-y|^\alpha}\right)A_{\beta_1,\beta_1,0,\beta_3+\beta_4+1}(t,x,y)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)\vee t^{1/\alpha})\wedge|x-y|}\right)\Bigg].
\end{aligned}$$

Remark 6.10. In this section, for $\overline{Y}$, **(A3)**(II) is only used to get **(IUBS)**. Therefore, the results of Propositions 6.6 and 6.9 hold under weaker assumptions **(A1)**, **(A3)**(I), **(A4)** and **(IUBS)**.

7. Sharp upper bounds of heat kernels

In this section we prove the sharp upper bounds of heat kernels. The key results are Theorem 7.5 and its Corollary 7.9 which deals with the case $\beta_2 < \alpha + \beta_1$, and Theorem 7.10 which deals with the case $\beta_2 \ge \alpha + \beta_1$.

Recall that $U(r) = \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : |\widetilde{x}| < r/2, 0 \le x_d < r/2\}$ for $r > 0$, $d \ge 2$ and $U(r) = [0, r/2)$ for $d = 1$, and that the function $A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)$ is defined by (6.2). We also recall that we use $p(t,x,y)$ for both $\overline{p}(t,x,y)$ and $p^\kappa(t,x,y)$. We remind the readers that, for an open set $D \subset \overline{\mathbb{R}}^d_+$ relative to the topology on $\overline{\mathbb{R}}^d_+$, $\tau_D = \overline{\tau}_D = \inf\{t > 0 : \overline{Y}_t \notin D\}$ when we consider $\overline{Y}$, and $\tau_D = \tau^\kappa_D = \inf\{t > 0 : Y^\kappa_t \notin D \cap \mathbb{R}^d_+\}$ when we consider Y^κ.

Lemma 7.1. *Let $b_1, b_3 \ge 0$ be constants with $b_1 > 0$ if $b_3 > 0$. Suppose that there exists a constant $C_0 > 0$ such that for all $t > 0$ and $z, y \in \mathbb{R}^d_+$,*

$$p(t,z,y) \le C_0 \left(1 \wedge \frac{z_d}{t^{1/\alpha}}\right)^q A_{b_1,0,b_3,0}(t,z,y) \left(t^{-d/\alpha} \wedge \frac{t}{|z-y|^{d+\alpha}}\right). \tag{7.1}$$

Then there exists a constant $C = C(C_0) > 0$ such that for all $t > 0$ and $x = (\widetilde{0}, x_d) \in \mathbb{R}^d_+$ with $x_d \le 2^{-5}$,

$$\mathbb{P}_x(\tau_{U(1)} < t < \zeta) \le Ct \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q (x_d \vee t^{1/\alpha})^{b_1} \log^{b_3}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right). \tag{7.2}$$

Proof. We first note that (7.1) implies that for all $t > 0$ and $y \in \mathbb{R}^d_+$,

$$\begin{aligned}
&\mathbb{P}_y(|Y_t - y| > 2^{-3}, t < \zeta) = \int_{z \in \mathbb{R}^d_+, |z-y|>2^{-3}} p(t,y,z)dz \\
&\le c_1 t \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q \int_{z \in \mathbb{R}^d_+, |z-y|>2^{-3}} \left(\frac{y_d \vee t^{1/\alpha}}{|z-y|} \wedge 1\right)^{b_1} \log^{b_3}\left(e + \frac{|z-y|}{y_d \vee t^{1/\alpha}}\right) \frac{dz}{|z-y|^{d+\alpha}} \\
&\le c_2 t \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q (y_d \vee t^{1/\alpha})^{b_1} \log^{b_3}\left(e + \frac{2^{-3}}{y_d \vee t^{1/\alpha}}\right) \int_{z \in \mathbb{R}^d_+, |z-y|>2^{-3}} \frac{dz}{|z-y|^{d+\alpha/2+b_1}} \\
&\le c_3 t \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q (y_d \vee t^{1/\alpha})^{b_1} \log^{b_3}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right),
\end{aligned} \tag{7.3}$$

where in the first inequality above we used (7.1) and Lemma 10.2, and in the second we used (10.2).

By Proposition 2.7 (see also Remark 3.12), we have

$$\sup_{s \le t, y \in \mathbb{R}^d_+} \mathbb{P}_y\big(|Y_s - y| \ge 2^{-2}, s < \zeta\big) \le c_1 \sup_{s \le t, y \in \mathbb{R}^d_+} \int_{z \in \mathbb{R}^d_+, |z-y| \ge 2^{-2}} \frac{s}{|z-y|^{d+\alpha}} dz \le c_2 t. \tag{7.4}$$

If $t \ge 1/(2c_2)$, then (7.2) follows from Corollary 5.2.

Let $t < 1/(2c_2)$. By the strong Markov property and (7.4), we have

$$\begin{aligned}
&\mathbb{P}_x\big(\tau_{U(1)} < t < \zeta, |Y_t - Y_{\tau_{U(1)}}| \ge 2^{-2}\big) = \mathbb{E}_x\left[\mathbb{P}_{Y_{\tau_{U(1)}}}\left(|Y_{t-\tau_{U(1)}} - Y_0| \ge 2^{-2}\right) : \tau_{U(1)} < t < \zeta\right] \\
&\le \mathbb{P}_x(\tau_{U(1)} < t < \zeta) \sup_{s \le t, y \in \mathbb{R}^d_+} \mathbb{P}_y(|Y_s - y| \ge 2^{-2}, s < \zeta) \le 2^{-1}\mathbb{P}_x(\tau_{U(1)} < t < \zeta).
\end{aligned}$$

Thus,

$$\mathbb{P}_x(\tau_{U(1)} < t < \zeta) = 2\big(\mathbb{P}_x(\tau_{U(1)} < t < \zeta) - 2^{-1}\mathbb{P}_x(\tau_{U(1)} < t < \zeta)\big)$$

$$\leq 2\big(\mathbb{P}_x(\tau_{U(1)} < t < \zeta) - \mathbb{P}_x(\tau_{U(1)} < t < \zeta,\, |Y_t - Y_{\tau_{U(1)}}| \geq 2^{-2})\big)$$
$$= 2\mathbb{P}_x(\tau_{U(1)} < t < \zeta,\, |Y_t - Y_{\tau_{U(1)}}| < 2^{-2}). \tag{7.5}$$

Note that by the triangle inequality, for any $y \in \mathbb{R}^d_+ \setminus U(1)$ and $z \in B(y, 2^{-2})$, we have $|z - x| \geq |y - x| - |y - z| > 15/32 - 1/4 > 2^{-3}$. Therefore using (7.3) and (7.5), we have

$$\mathbb{P}_x(\tau_{U(1)} < t < \zeta) \leq 2\mathbb{P}_x(\tau_{U(1)} < t < \zeta,\, |Y_t - Y_{\tau_{U(1)}}| < 2^{-2})$$
$$\leq 2\mathbb{P}_x(|Y_t - x| > 2^{-3},\, t < \zeta) \leq c_4 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q (x_d \vee t^{1/\alpha})^{b_1} \log^{b_3}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right).$$

The proof is complete. □

Note that for any $t, k, r > 0$ and $a \geq 0$,

$$\left(1 \wedge \frac{r}{t^{1/\alpha}}\right)^a (r \vee t^{1/\alpha})^k = r^k \left(1 \wedge \frac{r}{t^{1/\alpha}}\right)^{a-k}. \tag{7.6}$$

Lemma 7.2. *There exists a constant $C > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d_+$,*

$$p(t, x, y) \leq C \left(1 \wedge \frac{x_d \wedge y_d}{t^{1/\alpha}}\right)^q A_{\beta_1,0,\beta_3,0}(t, x, y) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right).$$

Proof. By Proposition 5.1, the lemma holds for $\beta_1 = 0$.

We assume $\beta_1 > 0$ and set $a_n = \beta_1 \wedge \frac{n\alpha}{2}$ for $n \geq 0$. Below, we prove by induction that for any $n \geq 0$, there exists a constant $C > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d_+$,

$$p(t, x, y) \leq C \left(1 \wedge \frac{x_d \wedge y_d}{t^{1/\alpha}}\right)^q A_{a_n,0,\beta_3,0}(t, x, y) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right). \tag{7.7}$$

The lemma is a direct consequence of (7.7).

By Proposition 5.1 and the fact that the logarithmic term in $A_{\beta_1,0,\beta_3,0}(t, x, y)$ is always larger than 1, (7.7) holds for $n = 0$. Suppose (7.7) holds for $n - 1$. By symmetry and (3.30), we can assume without loss of generality that $x_d \leq y_d$, $\widetilde{x} = 0$ and $|x - y| = 5$. If $t > 1$ or $x_d > 2^{-5}$, then (7.7) follows from Proposition 5.1 and (6.4).

Let $t \leq 1$ and $x_d \leq 2^{-5}$. Then $y_d \leq x_d + |x - y| \leq 4 + 2^{-5}$ by the triangle inequality. Our goal is to show that there exists a constant $c_1 > 0$ independent of t, x, y such that

$$p(t, x, y) \leq c_1 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q (x_d \vee t^{1/\alpha})^{a_n} \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right). \tag{7.8}$$

Set $V_1 = U(1)$, $V_3 = B(y, 2) \cap \overline{\mathbb{R}}^d_+$ and $V_2 = \overline{\mathbb{R}}^d_+ \setminus (V_1 \cup V_3)$. Similarly to (5.10) and (5.11), we get from Proposition 2.7 and the triangle inequality that

$$\sup_{s \leq t,\, z \in V_2} p(s, z, y) \leq c_2 \sup_{s \leq t,\, z \in \mathbb{R}^d_+, |z-y| \geq 2} \frac{s}{|z - y|^{d+\alpha}} \leq 2^{-d-\alpha} c_2 t \tag{7.9}$$

and

$$\mathrm{dist}(V_1, V_3) \geq \sup_{u \in V_1,\, w \in V_3} (4 - |x - u| - |y - w|) \geq 1. \tag{7.10}$$

By the induction hypothesis, condition (7.1) in Lemma 7.1 holds with $b_1 = a_{n-1}$ and $b_3 = \beta_3$. Thus, since $a_n - a_{n-1} \leq \alpha/2$, we get from Lemma 7.1 and (7.9) that

$$\mathbb{P}_x(\tau_{V_1} < t < \zeta) \sup_{s \leq t,\, z \in V_2} p(s, z, y)$$
$$\leq c_3 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q (x_d \vee t^{1/\alpha})^{a_{n-1}} (t^{1/\alpha})^{\alpha/2} t^{1/2} \log^{\beta_3}\left(e + \frac{1}{t^{1/\alpha}}\right)$$
$$\leq c_3 t \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q (x_d \vee t^{1/\alpha})^{a_{n-1}} (t^{1/\alpha})^{a_n - a_{n-1}} \left(\sup_{s \leq 1} s^{1/2} \log^{\beta_3}\left(e + \frac{1}{s^{1/\alpha}}\right)\right)$$

$$\leq c_4 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^q (x_d\vee t^{1/\alpha})^{a_n}. \tag{7.11}$$

In order to apply Lemma 3.15 and get the desired result, it remains to bound $\int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds$. We consider the following two cases separately.

(Case 1) $q\geq \alpha+a_n$ and $10x_d<t^{1/\alpha}$.

Pick $\varepsilon\in(0,\beta_1)$ such that $q<\alpha+\beta_1-\varepsilon$. Using **(A3)**(II), Lemmas 10.1(i)–(ii), 10.2 (see Remark 10.3), and (7.10), we see that for all $u\in V_1$ and $w\in V_3$,

$$\mathcal{B}(u,w)\leq c_5 B_{\beta_1-\varepsilon,0,0,0}(u,w)\leq c_6\left(\frac{u_d}{|u-w|}\right)^{\beta_1-\varepsilon}\leq c_6 u_d^{\beta_1-\varepsilon}. \tag{7.12}$$

By (7.12) and Lemma 5.12, we have

$$\begin{aligned}&\int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds\\ &\leq c_6\int_0^t\int_{V_1} p^{V_1}(t-s,x,u)u_d^{\beta_1-\varepsilon}du\int_{V_3}p(s,y,w)dwds\leq c_6\int_0^\infty\int_{V_1}p^{V_1}(s,x,u)u_d^{\beta_1-\varepsilon}duds\\ &\leq c_7x_d^q=c_7t\left(\frac{x_d}{t^{1/\alpha}}\right)^q(t^{1/\alpha})^{q-\alpha}\leq c_8t\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q(x_d\vee t^{1/\alpha})^{a_n}. \end{aligned}\tag{7.13}$$

In the last inequality above, we used the facts that $q-\alpha\geq a_n$, $10x_d<t^{1/\alpha}$ and $x_d\vee t^{1/\alpha}\leq 1$. Now, using Lemma 3.15, (7.11) and (7.13), we get (7.8) in this case.

(Case 2) $q<\alpha+a_n$ or $10x_d\geq t^{1/\alpha}$.

By **(A3)**(II), (7.10), Lemma 10.1(ii) and Lemma 10.2 (see Remark 10.3), it holds that for all $u\in V_1$ and $w\in V_3$,

$$\begin{aligned}\mathcal{B}(u,w)&\leq c_9B_{\beta_1,0,\beta_3,0}(u,w)\leq c_{10}\left(\frac{u_d}{|u-w|}\right)^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d\wedge|u-w|}{u_d\wedge|u-w|}\right)\\ &\leq c_{10}u_d^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right).\end{aligned}\tag{7.14}$$

By (10.2) (or using that $u\mapsto u^{\beta_1}\log^{\beta_3}(e+t/u)$ is almost increasing) and Corollary 5.2, since $\beta_1>0$ and $a_n\leq\beta_1$, we get that for any $0<s<t$ and $w\in V_3$,

$$\begin{aligned}&\int_{u\in V_1:u_d<x_d}p(s,x,u)u_d^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right)du\leq\int_{u\in V_1:u_d<x_d\vee s^{1/\alpha}}p(s,x,u)u_d^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right)du\\ &\leq c_{11}(x_d\vee s^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{x_d\vee s^{1/\alpha}}\right)\int_{u\in V_1:u_d<x_d\vee s^{1/\alpha}}p(s,x,u)du\\ &\leq c_{11}\mathbb{P}_x(\zeta>s)(x_d\vee s^{1/\alpha})^{a_n}\log^{\beta_3}\left(e+\frac{w_d}{x_d\vee s^{1/\alpha}}\right)\\ &\leq c_{12}\left(1\wedge\frac{x_d}{s^{1/\alpha}}\right)^q(x_d\vee s^{1/\alpha})^{a_n}\log^{\beta_3}\left(e+\frac{w_d}{x_d\vee s^{1/\alpha}}\right).\end{aligned}\tag{7.15}$$

Next, using the induction hypothesis and Lemma 10.10, since $a_n\leq\beta_1$ and $a_n<\alpha+a_{n-1}$, we get that for any $0<s<t$ and $w\in V_3$,

$$\begin{aligned}&\int_{u\in V_1:u_d\geq x_d}p(s,x,u)u_d^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right)du\\ &\leq c_{13}\left(1\wedge\frac{x_d}{s^{1/\alpha}}\right)^q\int_{u\in V_1:u_d\geq x_d}\left(\frac{x_d\vee s^{1/\alpha}}{|x-u|}\wedge 1\right)^{a_{n-1}}\log^{\beta_3}\left(e+\frac{|x-u|}{(x_d\vee s^{1/\alpha})\wedge|x-u|}\right)\\ &\qquad\qquad\times\left(s^{-d/\alpha}\wedge\frac{s}{|x-u|^{d+\alpha}}\right)u_d^{a_n}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right)du\end{aligned}$$

$$\leq c_{14}\left(1\wedge\frac{x_d}{s^{1/\alpha}}\right)^q (x_d\vee s^{1/\alpha})^{a_n}\log^{\beta_3}\left(e+\frac{w_d}{x_d\vee s^{1/\alpha}}\right). \tag{7.16}$$

Similarly, again splitting the integration into two parts $w_d < y_d$ and $w_d \geq y_d$, and using the induction hypothesis and Lemma 10.10 again, we also get that for any $0 < s < t$,

$$\int_{V_3} p(s,y,w)\log^{\beta_3}\left(e+\frac{w_d}{x_d\vee(t-s)^{1/\alpha}}\right)dw \leq c_{15}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^q\log^{\beta_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\right)$$
$$\leq c_{15}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\right). \tag{7.17}$$

By (7.14)–(7.17) and (7.6), we have

$$\int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)dudwds$$
$$\leq c_{10}\int_0^t\int_{V_3} p(s,y,w)\int_{V_1} p(t-s,x,u)u_d^{\beta_1}\log^{\beta_3}\left(e+\frac{w_d}{u_d}\right)dudwds$$
$$\leq c_{16}\int_0^t\left(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\right)^q (x_d\vee(t-s)^{1/\alpha})^{a_n}\int_{V_3} p(s,y,w)\log^{\beta_3}\left(e+\frac{w_d}{x_d\vee(t-s)^{1/\alpha}}\right)dwds$$
$$\leq c_{17}x_d^{a_n}\int_0^t\left(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\right)^{q-a_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\right)ds =: I. \tag{7.18}$$

When $q < \alpha + a_n$, we get from Lemma 10.4 that

$$I \leq c_{18}tx_d^{a_n}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-a_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right). \tag{7.19}$$

When $10x_d \geq t^{1/\alpha}$, we also get from Lemma 10.4 that

$$I \leq c_{17}x_d^{a_n}\int_0^t\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\right)ds \leq c_{19}tx_d^{a_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)$$
$$\leq 10^{q-a_n}c_{19}tx_d^{a_n}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-a_n}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right). \tag{7.20}$$

Now, (7.8) follows from (7.6), (7.11), (7.18), (7.20) and Lemma 3.15. The proof is complete. □

Now we use Lemma 7.2 to improve the bound in (7.11).

Lemma 7.3. *Let $0 < t \leq 1$ and $x, y \in \mathbb{R}^d_+$ be such that $\widetilde{x} = \widetilde{0}$, $x_d \leq 2^{-5}$ and $|x-y| = 5$. Set $V_1 = U(1)$, $V_3 = B(y,2)\cap\overline{\mathbb{R}}^d_+$ and $V_2 = \overline{\mathbb{R}}^d_+ \setminus (V_1\cup V_3)$. There exists a constant $C > 0$ independent of t, x and y such that*

$$\mathbb{P}_x(\tau_{V_1} < t < \zeta)\sup_{s\leq t,\, z\in V_2} p(s,z,y) \leq Ct^2\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q (x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\beta_1}$$
$$\times\log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).$$

Proof. Note that, since $y_d \leq x_d + |x-y| \leq 5 + 2^{-5} < 6$, for any $0 < s \leq 1$ and $z \in \mathbb{R}^d_+$ with $|z-y| \geq 2$,

$$(y_d\vee s^{1/\alpha})\wedge|z-y| \geq \frac{1}{3}(y_d\vee s^{1/\alpha}). \tag{7.21}$$

By Lemmas 7.2 and 10.2 and applying (7.21),

$$\sup_{s\leq t,\, z\in V_2} p(s,z,y) \leq c_1\sup_{s\leq t,\, z\in\mathbb{R}^d_+, |z-y|\geq 2}\left(1\wedge\frac{z_d\wedge y_d}{s^{1/\alpha}}\right)^q A_{\beta_1,0,\beta_3,0}(s,z,y)\frac{s}{|z-y|^{d+\alpha}}$$

$$\le c_1 \sup_{s\le t,\, z\in\mathbb{R}^d_+, |z-y|\ge 2} \left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^q \left(\frac{y_d\vee s^{1/\alpha}}{|z-y|}\right)^{\beta_1} \log^{\beta_3}\left(e+\frac{3|z-y|}{y_d\vee s^{1/\alpha}}\right)\frac{s}{|z-y|^{d+\alpha}}. \tag{7.22}$$

Using that $u\mapsto u^{\beta_1}\log^{\beta_3}(e+1/u)$ is almost increasing, we have that for any $0<s\le t\le 1$ and $z\in\mathbb{R}^d_+$ with $|z-y|\ge 2$,

$$\begin{aligned}
&\left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^q \left(\frac{y_d\vee s^{1/\alpha}}{|z-y|}\right)^{\beta_1} \log^{\beta_3}\left(e+\frac{3|z-y|}{y_d\vee s^{1/\alpha}}\right)\frac{s}{|z-y|^{d+\alpha}}\\
&\le c_2 \left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^q (y_d\vee s^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)\frac{s}{|z-y|^{d+\alpha}}\\
&\le 2^{-d-\alpha}c_2 s\left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^q (y_d\vee s^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right).
\end{aligned} \tag{7.23}$$

Let $\varepsilon>0$ be such that $q<\alpha+\beta_1-\varepsilon$. Using (7.6), we see that

$$\begin{aligned}
&s\left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^q (y_d\vee s^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)\\
&= s y_d^{\beta_1-\varepsilon}\left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^{q-\beta_1+\varepsilon} (y_d\vee s^{1/\alpha})^{\varepsilon}\log^{\beta_3}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)\\
&= s^{(\alpha+\beta_1-q-\varepsilon)/\alpha} y_d^{\beta_1-\varepsilon}(y_d\wedge s^{1/\alpha})^{q-\beta_1+\varepsilon}(y_d\vee s^{1/\alpha})^{\varepsilon}\log^{\beta_3}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right).
\end{aligned}$$

Thus, the map $s\mapsto s\big(1\wedge \frac{y_d}{s^{1/\alpha}}\big)^q(y_d\vee s^{1/\alpha})^{\beta_1}\log^{\beta_3}\big(e+\frac{1}{y_d\vee s^{1/\alpha}}\big)$ is almost increasing on $(0,t]$ by (10.2). Using this and (7.22)–(7.23), we get that

$$\begin{aligned}
\sup_{s\le t,\, z\in V_2} p(s,z,y) &\le c_3 \sup_{s\le t} s\left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^q (y_d\vee s^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)\\
&\le c_4 t\left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^q (y_d\vee t^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).
\end{aligned} \tag{7.24}$$

Note that (7.1) is satisfied with $a_1=\beta_1$ and $a_3=\beta_3$ by Lemma 7.2. Thus, by Lemma 7.1 and (7.24) we obtain

$$\begin{aligned}
\mathbb{P}_x(\tau_{V_1}<t<\zeta)\sup_{s\le t,\, z\in V_2} p(s,z,y) &\le c_4 t^2\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^q (x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\beta_1}\\
&\quad\times \log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).
\end{aligned}$$

□

Lemma 7.4. *Let $\eta_1,\eta_2,\gamma\ge 0$. There exists a constant $C>0$ such that for any $x\in\mathbb{R}^d_+$ and any $s,k,l>0$,*

$$\begin{aligned}
&\int_{B_+(x,2)} p(s,x,z) z_d^{\gamma}\log^{\eta_1}\left(e+\frac{k}{z_d}\right)\log^{\eta_2}\left(e+\frac{z_d}{l}\right)dz\\
&\le C x_d^{\gamma}\left(1\wedge \frac{x_d}{s^{1/\alpha}}\right)^{q-\gamma}\log^{\eta_1}\left(e+\frac{k}{x_d\vee s^{1/\alpha}}\right)\log^{\eta_2}\left(e+\frac{x_d\vee s^{1/\alpha}}{l}\right)\\
&\quad + C\mathbf{1}_{\{\gamma>\alpha+\beta_1\}} s x_d^{\beta_1}\left(1\wedge \frac{x_d}{s^{1/\alpha}}\right)^{q-\beta_1}\log^{\beta_3}\left(e+\frac{2}{x_d\vee s^{1/\alpha}}\right)\log^{\eta_1}(e+k)\log^{\eta_2}\left(e+\frac{1}{l}\right)\\
&\quad + C\mathbf{1}_{\{\gamma=\alpha+\beta_1,\, x_d\vee s^{1/\alpha}<2\}} s x_d^{\beta_1}\left(1\wedge \frac{x_d}{s^{1/\alpha}}\right)^{q-\beta_1}\\
&\qquad \times \int_{x_d\vee s^{1/\alpha}}^2 \log^{\beta_3}\left(e+\frac{r}{x_d\vee s^{1/\alpha}}\right)\log^{\eta_1}\left(e+\frac{k}{r}\right)\log^{\eta_2}\left(e+\frac{r}{l}\right)\frac{dr}{r}.
\end{aligned}$$

Proof. For any $x, z \in \mathbb{R}^d_+$ and $s > 0$, by Lemmas 7.2 and 10.2,

$$p(s,x,z) \le c_1 \left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^q A_{\beta_1,0,\beta_3,0}(s,x,z) \left(s^{-d/\alpha} \wedge \frac{s}{|x-z|^{d+\alpha}}\right)$$
$$\le c_2 \left(1 \wedge \frac{x_d}{s^{1/\alpha}}\right)^q \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{\beta_1} \log^{\beta_3}\left(e + \frac{|x-z|}{(x_d \vee s^{1/\alpha}) \wedge |x-z|}\right) \left(s^{-d/\alpha} \wedge \frac{s}{|x-z|^{d+\alpha}}\right).$$

Now combining (7.6) and Lemma 10.10, we get the desired result. □

We now state the first main result of this section.

Theorem 7.5. *For any $\varepsilon \in (0, \alpha/2]$, there exists a constant $C > 0$ such that*

$$p(t,x,y) \le C \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^q A_{\beta_1,\beta_2 \wedge (\alpha+\beta_1-\varepsilon),\beta_3,\beta_4}(t,x,y) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

for all $t > 0$ and $x, y \in \mathbb{R}^d_+$.

This theorem will proved by using several lemmas. We first introduce some additional notation: For $b_1, b_2, b_3, b_4 \ge 0$, $t > 0$ and $x, y \in \mathbb{R}^d_+$, define

$$\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4) := \int_0^{t/2} \int_{B_+(y,2)} \int_{B_+(x,2)} \mathbf{1}_{\{u_d \le w_d\}} p(t-s,x,u) p(s,y,w) \times u_d^{b_1} w_d^{b_2} \log^{b_3}\left(e + \frac{w_d}{u_d}\right) \log^{b_4}\left(e + \frac{1}{w_d}\right) du\, dw\, ds, \tag{7.25}$$

$$\mathcal{I}_{t,2}(x,y;b_1,b_2,b_3,b_4) := \int_0^{t/2} \int_{B_+(y,2)} \int_{B_+(x,2)} \mathbf{1}_{\{w_d \le u_d\}} p(t-s,x,u) p(s,y,w) \times u_d^{b_2} w_d^{b_1} \log^{b_3}\left(e + \frac{u_d}{w_d}\right) \log^{b_4}\left(e + \frac{1}{u_d}\right) du\, dw\, ds. \tag{7.26}$$

Lemma 7.6. *Let $b_1, b_2, b_3, b_4 \ge 0$. Let $b_1' \in [0, b_1]$ and $b_2' := b_1 + b_2 - b_1'$. Then*

$$\mathcal{I}_{t,i}(x,y;b_1,b_2,b_3,b_4) \le \mathcal{I}_{t,i}(x,y;b_1',b_2',b_3,b_4), \quad i = 1, 2.$$

Proof. If $u_d \le w_d$, then $u_d^{b_1} w_d^{b_2} \le u_d^{b_1'} w_d^{b_2'}$, implying that $\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4) \le \mathcal{I}_{t,1}(x,y;b_1',b_2',b_3,b_4)$. The other inequality is analogous. □

Lemma 7.7. *Let $b_1, b_2, b_3, b_4 \ge 0$ with $b_1 \vee b_2 < \alpha + \beta_1$, $x, y \in \mathbb{R}^d_+$ with $|x - y| = 5$, and $t \in (0, 1]$.*
(i) If $b_2 > q - \alpha$ or $y_d \ge t^{1/\alpha}$, then there exists a constant $C > 0$ independent of t, x, y such that

$$\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4) \le C t x_d^{b_1} y_d^{b_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right).$$

(ii) If $b_1 > q - \alpha$ or $y_d \ge t^{1/\alpha}$, then there exists a constant $C > 0$ independent of t, x, y such that

$$\mathcal{I}_{t,2}(x,y;b_1,b_2,b_3,b_4) \le C t x_d^{b_2} y_d^{b_1} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_2} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_1} \log^{b_3}\left(e + \frac{x_d \vee t^{1/\alpha}}{y_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right).$$

Proof. We give the proof for (i). (ii) can be proved similarly.

By using Lemma 7.4 together with the fact that $t - s \asymp t$ if $0 \le s \le t/2$, we see that for $0 \le s \le t/2$,

$$\int_{B_+(x,2)} p(t-s,x,u) u_d^{b_1} \log^{b_3}\left(e + \frac{w_d}{u_d}\right) du \le c_1 x_d^{b_1} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \log^{b_3}\left(e + \frac{w_d}{x_d \vee t^{1/\alpha}}\right).$$

Thus, using Lemma 7.4 again, we get that for $0 \le s \le t/2$,

$$\begin{aligned}&\int_{B_+(y,2)}\int_{B_+(x,2)} p(t-s,x,u)p(s,y,w)u_d^{b_1}w_d^{b_2}\log^{b_3}\left(e+\frac{w_d}{u_d}\right)\log^{b_4}\left(e+\frac{1}{w_d}\right)du\,dw\\ &\le c_1x_d^{b_1}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\int_{B_+(y,2)} p(s,y,w)w_d^{b_2}\log^{b_3}\left(e+\frac{w_d}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{w_d}\right)dw\\ &\le c_2x_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)\end{aligned}$$

From this and Lemma 10.5, we get that

$$\begin{aligned}&\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4)\\ &\le c_2x_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\int_0^{\frac{t}{2}}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds\\ &\le c_2x_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\int_0^{t}\left(1\wedge\frac{y_d}{s^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee (t-s)^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right)ds\\ &\le c_3tx_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).\end{aligned}$$

□

Lemma 7.8. *Let $b_1, b_2, b_3, b_4 \ge 0$ be such that $b_1 > q-\alpha$ and $b_1 \vee b_2 < \alpha+\beta_1$. Assume that $b_2 > 0$ if $b_4 > 0$. Then, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d_+$ with $x_d \le y_d$ and $|x-y| = 5$, and all $t \in (0,1]$,*

$$\begin{aligned}&\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4)+\mathcal{I}_{t,2}(x,y;b_1,b_2,b_3,b_4)\\ &\le Ctx_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)\end{aligned}\tag{7.27}$$

and

$$\begin{aligned}&\mathcal{I}_{t,1}(y,x;b_1,b_2,b_3,b_4)+\mathcal{I}_{t,2}(y,x;b_1,b_2,b_3,b_4)\\ &\le Ctx_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).\end{aligned}\tag{7.28}$$

Proof. Let $\delta \in (0,1)$ be such that

$$b_1 \vee b_2 + (1-\delta)(b_1 \wedge b_2) < \alpha + \beta_1, \tag{7.29}$$

and let $b_1' := \delta(b_1 \wedge b_2)$ and $b_2' := b_1 + b_2 - b_1'$. Then we see that $b_1' \in [0, b_1 \wedge b_2]$, $b_2' \ge b_1 > q-\alpha$ and

$$b_1' \le b_2' = b_1 \vee b_2 + (1-\delta)(b_1 \wedge b_2) < \alpha + \beta_1 \tag{7.30}$$

by (7.29). Moreover, since $x_d \le y_d$ and $b_2' - b_1 = b_2 - b_1'$, we see that

$$(x_d \vee t^{1/\alpha})^{b_2'-b_1}(y_d \vee t^{1/\alpha})^{b_1'-b_2}\log^{b_4}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{-b_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right) \le c_1. \tag{7.31}$$

Indeed, when $b_4 = 0$, (7.31) clearly holds with $c_1 = 1$. If $b_4 > 0$, then $b_2 > 0$ so that $b_2 > \delta b_2 \ge b_1'$. Hence, we get (7.31) from (10.2).

(i) We first prove (7.27). For this, we distinguish between two cases: $y_d \ge t^{1/\alpha}$ and $y_d < t^{1/\alpha}$.

Assume first that $y_d \ge t^{1/\alpha}$. The desired bound for $\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4)$ follows from Lemma 7.7(i). On the other hand, by using Lemma 7.6 in the first inequality, Lemma 7.7(ii) in the second inequality (which uses (7.30) and $y_d \ge t^{1/\alpha}$), (7.6) in the equality, and (7.31) in the last inequality, we get that

$$\begin{aligned}
&\mathcal{I}_{t,2}(x,y;b_1,b_2,b_3,b_4) \le \mathcal{I}_{t,2}(x,y;b_1',b_2',b_3,b_4)\\
&\le c_2 t x_d^{b_2'} y_d^{b_1'} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_2'} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_1'} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right)\\
&= c_2 t x_d^{b_1} y_d^{b_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right)\\
&\quad \times (x_d \vee t^{1/\alpha})^{b_2'-b_1} (y_d \vee t^{1/\alpha})^{b_1'-b_2} \log^{b_4}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right) \log^{-b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right)\\
&\le c_1 c_2 t x_d^{b_1} y_d^{b_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right).
\end{aligned}$$

Assume now that $y_d < t^{1/\alpha}$. Using Lemma 7.6 and Lemma 7.7(i) (which uses $b_2' > q-\alpha$), we get

$$\begin{aligned}
&\mathcal{I}_{t,1}(x,y;b_1,b_2,b_3,b_4) \le \mathcal{I}_{t,1}(x,y;b_1',b_2',b_3,b_4)\\
&\le c_3 t x_d^{b_1'} y_d^{b_2'} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1'} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2'} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right). \quad (7.32)
\end{aligned}$$

Also, since $b_1 > q-\alpha$, we get from Lemma 7.7(ii) that

$$\begin{aligned}
&\mathcal{I}_{t,2}(x,y;b_1,b_2,b_3,b_4)\\
&\le c_4 t x_d^{b_2} y_d^{b_1} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_2} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_1} \log^{b_3}\left(e + \frac{x_d \vee t^{1/\alpha}}{y_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right). \quad (7.33)
\end{aligned}$$

Since $x_d \le y_d < t^{1/\alpha}$ and $b_1' + b_2' = b_1 + b_2$, it holds that $x_d \vee t^{1/\alpha} = y_d \vee t^{1/\alpha} = t^{1/\alpha}$ and

$$\begin{aligned}
&x_d^{b_1'} y_d^{b_2'} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1'} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2'} = x_d^{b_2} y_d^{b_1} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_2} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_1}\\
&= \frac{x_d^q y_d^q}{t^{(2q-b_1-b_2)/\alpha}} = x_d^{b_1} y_d^{b_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2}.
\end{aligned}$$

Thus, (7.27) follows from (7.32) and (7.33).

(ii) Now, we prove (7.28). By using Lemma 7.6 in the first inequality, Lemma 7.7(i) in the second inequality (which uses (7.30) and $b_2' > q-\alpha$), (7.6) together with the fact that $x_d \le y_d$ in the third inequality, (7.31) in the last inequality, we obtain

$$\begin{aligned}
&\mathcal{I}_{t,1}(y,x;b_1,b_2,b_3,b_4) \le \mathcal{I}_{t,1}(y,x;b_1',b_2',b_3,b_4)\\
&\le c_5 t x_d^{b_2'} y_d^{b_1'} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_2'} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_1'} \log^{b_3}\left(e + \frac{x_d \vee t^{1/\alpha}}{y_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right)\\
&\le c_5 t x_d^{b_1} y_d^{b_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right)\\
&\quad \times (x_d \vee t^{1/\alpha})^{b_2'-b_1} (y_d \vee t^{1/\alpha})^{b_1'-b_2} \log^{b_4}\left(e + \frac{1}{x_d \vee t^{1/\alpha}}\right) \log^{-b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right)\\
&\le c_1 c_5 t x_d^{b_1} y_d^{b_2} \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{b_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right).
\end{aligned}$$

On the other hand, by Lemma 7.7(ii), it holds that

$$\mathcal{I}_{t,2}(y,x;b_1,b_2,b_3,b_4)$$

$$\le c_3 t x_d^{b_1} y_d^{b_2}\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q-b_2}\log^{b_3}\Big(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{y_d\vee t^{1/\alpha}}\Big).$$

The proof is complete. □

Proof of Theorem 7.5. Set $\widehat{\beta}_2 := \beta_2\wedge(\alpha+\beta_1-\varepsilon)$. As in the proof of Lemma 7.2, by symmetry, Proposition 5.1, (3.30), and (6.4), we can assume without loss of generality that $x_d \le y_d\wedge 2^{-5}$, $\widetilde{x}=0$ and $|x-y|=5$, and then it is enough to show that there exists a constant $c_1>0$ independent of x and y such that for any $t\le 1$,

$$\begin{aligned} p(t,x,y) \le c_1 t\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q}(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\widehat{\beta}_2}\\ \times\log^{\beta_3}\Big(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\Big)\log^{\beta_4}\Big(e+\frac{1}{y_d\vee t^{1/\alpha}}\Big). \end{aligned} \tag{7.34}$$

Let $t\le 1$. Set $V_1=U(1)$, $V_3=B(y,2)\cap\overline{\mathbb{R}}^d_+$ and $V_2=\overline{\mathbb{R}}^d_+\setminus(V_1\cup V_3)$. By Lemma 7.3,

$$\begin{aligned} &\mathbb{P}_x(\tau_{V_1}<t<\zeta)\sup_{s\le t,\, z\in V_2} p(s,z,y)\\ &\le c_2 t^2\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q}(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\beta_1}\log^{2\beta_3}\Big(e+\frac{1}{t^{1/\alpha}}\Big)\\ &= c_2 t\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q}(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\alpha+\beta_1-\varepsilon}(y_d\vee t^{1/\alpha})^{-\alpha+\varepsilon}t\log^{2\beta_3}\Big(e+\frac{1}{t^{1/\alpha}}\Big)\\ &\le c_2 t\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q}(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\widehat{\beta}_2}t^{\varepsilon/\alpha}\log^{2\beta_3}\Big(e+\frac{1}{t^{1/\alpha}}\Big)\\ &\le c_3 t\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q}(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\widehat{\beta}_2}, \end{aligned} \tag{7.35}$$

where in the last inequality above we used (10.1).

Next, we show that there exists a constant $C'>0$ such that

$$\begin{aligned} I &:= \int_0^t\int_{V_3}\int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)\,du\,dw\,ds\\ &\le C' t x_d^{\beta_1} y_d^{\widehat{\beta}_2}\Big(1\wedge \frac{x_d}{t^{1/\alpha}}\Big)^{q-\beta_1}\Big(1\wedge \frac{y_d}{t^{1/\alpha}}\Big)^{q-\widehat{\beta}_2}\log^{\beta_3}\Big(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\Big)\log^{\beta_4}\Big(e+\frac{1}{y_d\vee t^{1/\alpha}}\Big). \end{aligned} \tag{7.36}$$

Once we get (7.36), by (7.6) and (7.35), we can apply Lemma 3.15 to get (7.34) and finish the proof.

By **(A3)**(II), since $|u-w|\asymp 1$ and $u_d\vee w_d\le y_d+2\le 7+2^{-5}$ for $u\in V_1$ and $w\in V_3$, we have

$$\begin{aligned} I &\le c_4\int_0^t\int_{V_3}\int_{V_1}\mathbf{1}_{\{u_d\le w_d\}}p(t-s,x,u)p(s,y,w)u_d^{\beta_1}w_d^{\widehat{\beta}_2}\log^{\beta_3}\Big(e+\frac{w_d}{u_d}\Big)\log^{\beta_4}\Big(e+\frac{1}{w_d}\Big)du\,dw\,ds\\ &\quad + c_4\int_0^t\int_{V_3}\int_{V_1}\mathbf{1}_{\{w_d\le u_d\}}p(t-s,x,u)p(s,y,w)u_d^{\widehat{\beta}_2}w_d^{\beta_1}\log^{\beta_3}\Big(e+\frac{u_d}{w_d}\Big)\log^{\beta_4}\Big(e+\frac{1}{u_d}\Big)du\,dw\,ds\\ &= c_4\Big(\big(\int_0^{t/2}+\int_{t/2}^t\big)\int_{V_3}\int_{V_1}\mathbf{1}_{\{u_d\le w_d\}}\cdots+\big(\int_0^{t/2}+\int_{t/2}^t\big)\int_{V_3}\int_{V_1}\mathbf{1}_{\{w_d\le u_d\}}\cdots\Big). \end{aligned}$$

Thus, by the change of variable $\widetilde{s}=t-s$ in integrals $\int_{t/2}^t$,

$$I\le c_4\sum_{i=1}^2\mathcal{I}_{t,i}(x,y;\beta_1,\widehat{\beta}_2,\beta_3,\beta_4)+c_4\sum_{i=1}^2\mathcal{I}_{t,i}(y,x;\beta_1,\widehat{\beta}_2,\beta_3,\beta_4),$$

where the functions $\mathcal{I}_{t,i}(x,y;\beta_1,\widehat{\beta}_2,\beta_3,\beta_4)$, $1\le i\le 2$, are defined in (7.25)–(7.26). Then by using Lemma 7.6(i) and Lemma 7.8, we conclude that (7.36) holds. The proof is complete. □

As an immediate consequence of Theorem 7.5, we obtain the following sharp upper bound of the heat kernel for $\beta_2<\alpha+\beta_1$.

Corollary 7.9. *Suppose that $\beta_2<\alpha+\beta_1$. There exists a constant $C>0$ such that*

$$p(t,x,y)\le C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right),$$

for all $t>0$ and $x,y\in\mathbb{R}^d_+$.

Here is the second main result of the section.

Theorem 7.10. *Suppose that $\beta_2\ge\alpha+\beta_1$. There exists a constant $C>0$ such that*

$$\begin{aligned}p(t,x,y)\le C\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\bigg[A_{\beta_1,\beta_2,\beta_3,\beta_4}(t,x,y)\\+\left(1\wedge\frac{t}{|x-y|^{\alpha}}\right)\log^{\beta_3}\left(e+\frac{|x-y|}{((x_d\wedge y_d)+t^{1/\alpha})\wedge|x-y|}\right)\\\times\left(\mathbf{1}_{\{\beta_2>\alpha+\beta_1\}}A_{\beta_1,\beta_1,0,\beta_3}(t,x,y)+\mathbf{1}_{\{\beta_2=\alpha+\beta_1\}}A_{\beta_1,\beta_1,0,\beta_3+\beta_4+1}(t,x,y)\right)\bigg]\end{aligned}$$

for all $t>0$ and $x,y\in\mathbb{R}^d_+$.

Again, we first introduce additional notation and prove a lemma. For $b_1,b_2,b_3,b_4\ge0$, $t>0$ and $x,y\in\mathbb{R}^d_+$, define

$$\begin{aligned}\mathcal{I}_t(x,y;b_1,b_2,b_3,b_4):=\int_0^t\int_{B_+(y,2)}\int_{B_+(x,2)}p(t-s,x,u)p(s,y,w)\\\times u_d^{b_1}w_d^{b_2}\log^{b_3}\left(e+\frac{w_d}{u_d}\right)\log^{b_4}\left(e+\frac{1}{w_d}\right)du\,dw\,ds.\end{aligned}\tag{7.37}$$

Lemma 7.11. *Let $b_1,b_2,b_3,b_4\ge0$ be such that $b_1\wedge b_2>q-\alpha$ and $b_1<\alpha+\beta_1$. There exists a constant $C>0$ such that for all $x,y\in\mathbb{R}^d_+$ with $|x-y|=5$, and all $t\in(0,1]$,*

$$\begin{aligned}&\mathcal{I}_t(x,y;b_1,b_2,b_3,b_4)\\&\le Ctx_d^{b_1}y_d^{b_2}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{q-b_2}\log^{b_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)\\&+\mathbf{1}_{\{b_2>\alpha+\beta_1\}}Ct^2x_d^{b_1}y_d^{\beta_1}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{q-\beta_1}\log^{b_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{b_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)\\&+\mathbf{1}_{\{b_2=\alpha+\beta_1,y_d<2\}}Ctx_d^{b_1}y_d^{\beta_1}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^{q-\beta_1}\\&\quad\times\int_{y_d}^2(r^\alpha\wedge t)\log^{b_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{b_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}.\end{aligned}$$

Proof. By using Lemma 7.4 in the first inequality (integration with respect to u; note that $b_1<\alpha+\beta_1$) and the second inequality (integration with respect to w), we get

$$\begin{aligned}\mathcal{I}_t(x,y;b_1,b_2,b_3,b_4)\le c_1x_d^{b_1}\int_0^t\left(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\right)^{q-b_1}\\\times\int_{B_+(y,2)}p(s,y,w)w_d^{b_2}\log^{b_3}\left(e+\frac{w_d}{x_d\vee(t-s)^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{w_d}\right)dwds\end{aligned}$$

$$\le c_2 \left(x_d^{b_1} y_d^{b_2} I_1 + \mathbf{1}_{\{b_2>\alpha+\beta_1\}} x_d^{b_1} y_d^{\beta_1} I_2 + \mathbf{1}_{\{b_2=\alpha+\beta_1, y_d<2\}} x_d^{b_1} y_d^{\beta_1} I_3 \right),$$

where

$$I_1 := \int_0^t \left(1\wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{q-b_1} \left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee (t-s)^{1/\alpha}}\right) \log^{b_4}\left(e+\frac{1}{y_d\vee s^{1/\alpha}}\right) ds,$$

$$I_2 := \int_0^t s\left(1\wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{q-b_1} \left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^{q-\beta_1} \log^{b_3}\left(e+\frac{2}{y_d\vee s^{1/\alpha}}\right) \log^{b_3}\left(e+\frac{1}{x_d\vee (t-s)^{1/\alpha}}\right) ds,$$

and

$$\begin{aligned} I_3 := &\int_0^t s\left(1\wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{q-b_1} \left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^{q-\beta_1} \\ &\times \int_{y_d\vee s^{1/\alpha}}^2 \log^{b_3}\left(e+\frac{r}{y_d\vee s^{1/\alpha}}\right) \log^{b_3}\left(e+\frac{r}{x_d\vee (t-s)^{1/\alpha}}\right) \log^{b_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}\, ds. \end{aligned}$$

Applying Lemma 10.5 to I_1 and I_2, Lemma 10.6 to I_3, and (10.2), wee see that

$$I_1 \le c_3 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2} \log^{b_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right) \log^{b_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right),$$

$$I_2 \le c_3 t^2\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-\beta_1} \log^{b_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right) \log^{b_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right),$$

and

$$\begin{aligned} I_2 \le &c_3 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1} \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-\beta_1} \\ &\times \int_{y_d}^2 (r^\alpha\wedge t) \log^{b_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right) \log^{b_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right) \log^{b_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}. \end{aligned}$$

This proves the lemma. □

Proof of Theorem 7.10. As in the proof of Lemma 7.2, by symmetry, (6.14), Proposition 5.1, (3.30) and (6.4), we can assume without loss of generality that $x_d \le y_d \wedge 2^{-5}$, $\widetilde{x} = 0$ and $|x-y| = 5$, and then it is enough to show that there exists a constant $c_1 > 0$ independent of x and y such that for any $t \le 1$,

$$\begin{aligned} p(t,x,y) \le &c_1 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^q (x_d\vee t^{1/\alpha})^{\beta_1} \\ &\times \left[(y_d\vee t^{1/\alpha})^{\beta_2} \log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right) \right. \qquad (7.38) \\ &+ \mathbf{1}_{\{\beta_2>\alpha+\beta_1\}} t (y_d\vee t^{1/\alpha})^{\beta_1} \log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right) \log^{\beta_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right) \\ &\left. + \mathbf{1}_{\{\beta_2=\alpha+\beta_1\}} t (y_d\vee t^{1/\alpha})^{\beta_1} \log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right) \log^{\beta_3+\beta_4+1}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right) \right]. \qquad (7.39) \end{aligned}$$

Let $t \le 1$. Set $V_1 = U(1)$, $V_3 = B(y,2) \cap \overline{\mathbb{R}}^d_+$ and $V_2 = \overline{\mathbb{R}}^d_+ \setminus (V_1 \cup V_3)$. By Lemmas 3.15 and 7.3 it remains to prove that $I := \int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)\,du\,dw\,ds$ is bounded above by the right-hand side of (7.39).

By **(A3)**(II), since $|u-w| \asymp 1$ for $u \in V_1$ and $w \in V_3$, using the change of variables $\widetilde{s} = t-s$ we have

$$I \le c_2 \left(\mathcal{I}_t(x,y;\beta_1,\beta_2,\beta_3,\beta_4) + \mathcal{I}_t(y,x;\beta_1,\beta_2,\beta_3,\beta_4) \right), \qquad (7.40)$$

where the functions $\mathcal{I}_t(x,y;\beta_1,\beta_2,\beta_3,\beta_4)$ is defined in (7.37). By Lemma 7.11 and (7.6), the right hand side of (7.40) is less than or equal to $c_3 t(1\wedge \frac{x_d}{t^{1/\alpha}})^q(1\wedge \frac{y_d}{t^{1/\alpha}})^q$ times

$$(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\beta_2}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right) \tag{7.41}$$

$$+(x_d\vee t^{1/\alpha})^{\beta_2}(y_d\vee t^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{x_d\vee t^{1/\alpha}}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right) \tag{7.42}$$

$$+\mathbf{1}_{\{\beta_2>\alpha+\beta_1\}}t(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right) \tag{7.43}$$

$$+\mathbf{1}_{\{\beta_2=\alpha+\beta_1\}}(x_d\vee t^{1/\alpha})^{\beta_1}(y_d\vee t^{1/\alpha})^{\beta_1}$$
$$\times\int_{x_d}^2 (r^\alpha\wedge t)\log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}. \tag{7.44}$$

In the last line we used $x_d\le y_d\wedge 2$ to get that $\mathbf{1}_{\{y_d<2\}}\int_{y_d}^2\le\int_{x_d}^2$.

Since $\beta_2>\beta_1$ and $x_d\le y_d$, clearly

$$(x_d\vee t^{1/\alpha})^{\beta_2}\log^{\beta_3}\left(e+\frac{x_d\vee t^{1/\alpha}}{y_d\vee t^{1/\alpha}}\right)\le (x_d\vee t^{1/\alpha})^{\beta_1}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right), \tag{7.45}$$

and by using (10.2) (with $\varepsilon=(\beta_2-\beta_1)/\beta_4)$), we get

$$(y_d\vee t^{1/\alpha})^{\beta_1}\log^{\beta_4}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\le c_4(y_d\vee t^{1/\alpha})^{\beta_2}\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right). \tag{7.46}$$

Applying (7.45) and (7.46) to (7.42) and combining it with (7.41) and (7.43), we arrive at the result in case $\beta_2>\alpha+\beta_1$.

Assume now that $\beta_2=\alpha+\beta_1$. From the above argument, to prove the result, in view of (7.38), (7.39) and (7.44), it suffices to show that

$$\begin{aligned}&\int_{x_d}^2 (r^\alpha\wedge t)\log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}\\&\le c_5 t\log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3+\beta_4+1}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right)\\&\quad + c_5(y_d\vee t^{1/\alpha})^{\beta_2-\beta_1}\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).\end{aligned} \tag{7.47}$$

By Lemma 10.11 (with $b_1=\beta_3$, $b_2=\beta_4$, $k=x_d\vee t^{1/\alpha}$ and $l=y_d\vee t^{1/\alpha}$), it holds that

$$\begin{aligned}&\int_{y_d\vee t^{1/\alpha}}^2 (r^\alpha\wedge t)\log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}\\&\le t\int_{y_d\vee t^{1/\alpha}}^2 \log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}\\&\le c_6 t\log^{\beta_3}\left(e+\frac{1}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3+\beta_4+1}\left(e+\frac{1}{y_d\vee t^{1/\alpha}}\right).\end{aligned} \tag{7.48}$$

On the other hand, using (10.2), we see that

$$\begin{aligned}&\int_{x_d}^{y_d\vee t^{1/\alpha}} (r^\alpha\wedge t)\log^{\beta_3}\left(e+\frac{r}{x_d\vee t^{1/\alpha}}\right)\log^{\beta_3}\left(e+\frac{r}{y_d\vee t^{1/\alpha}}\right)\log^{\beta_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}\\&\le \log^{\beta_3}(e+1)\log^{\beta_3}\left(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\right)\int_{x_d}^{y_d\vee t^{1/\alpha}}\log^{\beta_4}\left(e+\frac{1}{r}\right)\frac{dr}{r^{1-\alpha}}\end{aligned}$$

$$\le c_7 \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right) \int_{x_d}^{y_d \vee t^{1/\alpha}} \left(\frac{y_d \vee t^{1/\alpha}}{r}\right)^{\alpha/2} \frac{dr}{r^{1-\alpha}}$$
$$\le c_8 (y_d \vee t^{1/\alpha})^{\alpha} \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right)$$
$$= c_8 (y_d \vee t^{1/\alpha})^{\beta_2 - \beta_1} \log^{\beta_3}\left(e + \frac{y_d \vee t^{1/\alpha}}{x_d \vee t^{1/\alpha}}\right) \log^{\beta_4}\left(e + \frac{1}{y_d \vee t^{1/\alpha}}\right). \tag{7.49}$$

Combining (7.48)–(7.49), we show that (7.47) holds true. The proof is complete. □

8. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Since $\overline{p}(t, x, y)$ is jointly continuous (see Remark 3.12), it suffices to prove that (1.2)–(1.5) hold for $(t, x, y) \in (0, \infty) \times \mathbb{R}^d_+ \times \mathbb{R}^d_+$.

We first note that by **(A3)** and (6.4),

$$t^{-d/\alpha} \wedge \big(tJ(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)\big) \asymp t^{-d/\alpha} \wedge \frac{tB_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d)}{|x-y|^{d+\alpha}}$$
$$\asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) B_{\beta_1,\beta_2,\beta_3,\beta_4}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d), \tag{8.1}$$

which implies the second comparison in (1.3).

(i) Using (6.3), we get the lower heat kernel estimate in the first comparison in (1.3) from Proposition 6.2 and the upper heat kernel estimate from Corollary 7.9.

(ii) For (1.4), using (6.3), we get the lower heat kernel estimate from Proposition 6.6 (see Remark 6.7) and the upper heat kernel estimate from Theorem 7.10.

(iii) Using (6.3), we get the lower heat kernel estimate in (1.5) from Proposition 6.9. The upper heat kernel estimate in (1.5) follows from Theorem 7.10.

From the comparisons in (i)-(iii) and (6.4), we have that $\overline{p}(t, x, y) \asymp t^{-d/\alpha}$ when $t^{1/\alpha} \ge |x-y|/8$ and this implies that (1.2) holds for $t^{1/\alpha} \ge |x-y|/8$. Moreover, (1.2) for $\beta_2 < \alpha + \beta_1$ follows from (1.3), (6.3) and (8.1).

By **(A3)** and (6.3), we have that when $t^{1/\alpha} < |x-y|/8$,

$$t \int_{(x_d \vee y_d \vee t^{1/\alpha}) \wedge (|x-y|/4)}^{|x-y|/2} J(x + t^{1/\alpha}\mathbf{e}_d, x + r\mathbf{e}_d) J(x + r\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) r^{d-1} dr$$
$$\asymp \frac{t}{|x-y|^{d+\alpha}} \int_{(x_d \vee y_d \vee t^{1/\alpha}) \wedge (|x-y|/4)}^{|x-y|/2} A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x, x + r\mathbf{e}_d)\, A_{\beta_1,\beta_2,\beta_3,\beta_4}(t, x + r\mathbf{e}_d, y) \frac{dr}{r^{\alpha+1}}. \tag{8.2}$$

Thus, we see that, for $\beta_2 = \alpha + \beta_1$ and $t^{1/\alpha} < |x-y|/8$, (1.2) follows from (1.5), (8.2) and Lemma 6.8, and that, for $\beta_2 > \alpha + \beta_1$ and $t^{1/\alpha} < |x-y|/8$, the lower bound in (1.2) follows from (8.2) and Proposition 6.2 and Lemma 6.4.

We have from (8.2), (6.15) and (6.3) that, when $t^{1/\alpha} < |x-y|/8$,

$$t \int_{(x_d \vee y_d \vee t^{1/\alpha}) \wedge (|x-y|/4)}^{|x-y|/2} J(x + t^{1/\alpha}\mathbf{e}_d, x + r\mathbf{e}_d) J(x + r\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) r^{d-1} dr$$
$$\ge \frac{c_1 t}{|x-y|^{d+2\alpha}} B_{\beta_1,\beta_1,0,\beta_3}(x + t^{1/\alpha}\mathbf{e}_d, y + t^{1/\alpha}\mathbf{e}_d) \log^{\beta_3}\left(e + \frac{|x-y|}{((x_d \wedge y_d) \vee t^{1/\alpha}) \wedge |x-y|}\right). \tag{8.3}$$

Now, for $\beta_2 > \alpha + \beta_1$ and $t^{1/\alpha} < |x-y|/8$, the upper bound in (1.2) follows from (8.2), (8.3) and the upper bound in (1.4).

Finally, from the joint continuity of $\overline{p}(t, x, y)$ and upper heat kernel estimates, we deduce that $\overline{Y}$ is a Feller process and finish the proof by Remark 3.12. □

Proof of Theorem 1.2. The second comparison in (1.9) follows from Corollary 6.3. By (6.3), Theorem 1.1 and Propositions 6.2, 6.6 and 6.9, $p^\kappa(t,x,y) \ge c_1\big(1\wedge \frac{x_d}{t^{1/\alpha}}\big)^{q_\kappa}\big(1\wedge \frac{y_d}{t^{1/\alpha}}\big)^{q_\kappa}\overline{p}(t,x,y)$ for all $(t,x,y)\in(0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$. On the other hand, by (6.3), Theorem 1.1, Corollary 7.9, and Theorem 7.10, $p^\kappa(t,x,y) \le c_2\big(1\wedge \frac{x_d}{t^{1/\alpha}}\big)^{q_\kappa}\big(1\wedge \frac{y_d}{t^{1/\alpha}}\big)^{q_\kappa}\overline{p}(t,x,y)$ for all $(t,x,y)\in(0,\infty)\times\mathbb{R}^d_+\times\mathbb{R}^d_+$ and hence (1.9) holds true.

Note that for each $(t,y)\in(0,\infty)\times\mathbb{R}^d_+$, the map $x\mapsto p^\kappa(t,x,y)$ vanishes continuously on $\partial\mathbb{R}^d_+$. Hence, using the joint continuity of $p^\kappa(t,x,y)$ and upper heat kernel estimates, we deduce that Y^κ is a Feller process. By Remark 3.12, the proof is complete. □

9. Green function estimates

In this section, we give proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. When $d>\alpha$, we get the upper bound of (1.11) from Corollary 3.13. On the other hand, by Lemma 3.7 and Remark 3.12, we have

$$\overline{G}(x,y) \ge \int_{|x-y|^\alpha}^\infty \overline{p}(t,x,y)dt \ge c_1\int_{|x-y|^\alpha}^\infty t^{-d/\alpha}dt = \begin{cases} \frac{c_1\alpha}{d-\alpha}|x-y|^{-d+\alpha} & \text{if } d>\alpha;\\ \infty & \text{if } d\le\alpha.\end{cases}$$

The proof is complete. □

In the remainder of this section, we assume the setting of Theorem 1.4 holds and denote by q the constant q_κ in (1.8), which is strictly positive.

Let $x,y\in\mathbb{R}^d_+$ be such that $x_d\le y_d$ and $|x-y|=1$. From Theorem 7.5, Proposition 6.2 and (6.4), we have for $t\le 1$,

$$\begin{aligned} p^\kappa(t,x,y) &\le Ct\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\big((x_d\vee t^{1/\alpha})\wedge 1\big)^{\beta_1}\big((y_d\vee t^{1/\alpha})\wedge 1\big)^{\beta_2\wedge(\alpha/2+\beta_1)}\\ &\quad\times\log^{\beta_3}\left(e+\frac{(y_d\vee t^{1/\alpha})\wedge 1}{(x_d\vee t^{1/\alpha})\wedge 1}\right)\log^{\beta_4}\left(e+\frac{1}{(y_d\vee t^{1/\alpha})\wedge 1}\right), \end{aligned} \tag{9.1}$$

$$\begin{aligned} p^\kappa(t,x,y) &\ge ct\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q\big((x_d\vee t^{1/\alpha})\wedge 1\big)^{\beta_1}\big((y_d\vee t^{1/\alpha})\wedge 1\big)^{\beta_2}\\ &\quad\times\log^{\beta_3}\left(e+\frac{(y_d\vee t^{1/\alpha})\wedge 1}{(x_d\vee t^{1/\alpha})\wedge 1}\right)\log^{\beta_4}\left(e+\frac{1}{(y_d\vee t^{1/\alpha})\wedge 1}\right), \end{aligned} \tag{9.2}$$

and for $t>1$,

$$p^\kappa(t,x,y)\asymp t^{-d/\alpha}\left(1\wedge\frac{x_d}{t^{1/\alpha}}\right)^q\left(1\wedge\frac{y_d}{t^{1/\alpha}}\right)^q. \tag{9.3}$$

Lemma 9.1. *Let $x,y\in\mathbb{R}^d_+$ be such that $x_d\le y_d$ and $|x-y|=1$. Set $\widehat{q}:=2\alpha+\beta_1+\beta_2-q$. Then we have*

$$\int_0^1 p^\kappa(t,x,y)dt \asymp \begin{cases} (x_d\wedge 1)^q(y_d\wedge 1)^q & \text{if } q<\widehat{q},\\ (x_d\wedge 1)^q(y_d\wedge 1)^q\log^{\beta_4+1}\left(e+\dfrac{1}{y_d\wedge 1}\right) & \text{if } q=\widehat{q},\\ (x_d\wedge 1)^q(y_d\wedge 1)^{\widehat{q}}\log^{\beta_4}\left(e+\dfrac{1}{y_d\wedge 1}\right) & \text{if } q>\widehat{q},\end{cases}$$

where the comparison constant is independent of x,y.

Proof. Set $G_1:=\int_0^1 p^\kappa(t,x,y)dt$ and $\widehat{\beta}_2:=\beta_2\wedge(\alpha/2+\beta_1)$.

(Case 1) $x_d\ge 1$: We have from (9.1) and (9.2) that $G_1\asymp\int_0^1 tdt\asymp 1$.

(Case 2) $y_d \ge 1 > x_d$: By (9.2) and (7.6),

$$G_1 \ge c_1 x_d^{\beta_1} \int_{1/2}^1 t\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-\beta_1} dt \ge 2^{-1} c_1 x_d^{\beta_1} \int_{1/2}^1 (1\wedge x_d)^{q-\beta_1} dt = 2^{-2} c_1 x_d^q.$$

Besides, since $\alpha + \beta_1 - q > 0$, we get from (9.1) and (7.6) that

$$\begin{aligned} G_1 &\le c_2 x_d^{\beta_1} \int_0^1 t\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-\beta_1} \log^{\beta_3}\left(e + \frac{1}{t^{1/\alpha}}\right) dt \\ &\le c_3 x_d^q \int_0^1 t^{(\alpha+\beta_1-q)/\alpha} \log^{\beta_3}\left(e+\frac{1}{t^{1/\alpha}}\right) dt \le c_4 x_d^q. \end{aligned}$$

(Case 3) $1 > y_d \ge x_d$: Note that

$$\int_{y_d^\alpha}^1 t^{-1+(\widehat{q}-q)/\alpha} \log^{\beta_4}\left(e+\frac{1}{t^{1/\alpha}}\right) dt \asymp \begin{cases} 1 & \text{if } q < \widehat{q}; \\ \log^{\beta_4+1}(e+1/y_d) & \text{if } q = \widehat{q}; \\ y_d^{\widehat{q}-q} \log^{\beta_4}(e+1/y_d) & \text{if } q > \widehat{q}. \end{cases} \tag{9.4}$$

For the lower bound, we get from (9.2) and (9.4) that

$$\begin{aligned} G_1 &\ge c_5 x_d^q y_d^q \int_{y_d^\alpha}^1 t^{(\alpha+\beta_1+\beta_2-2q)/\alpha} \log^{\beta_4}\left(e+\frac{1}{t^{1/\alpha}}\right) dt \\ &= c_5 x_d^q y_d^q \int_{y_d^\alpha}^1 t^{-1+(\widehat{q}-q)/\alpha} \log^{\beta_4}\left(e+\frac{1}{t^{1/\alpha}}\right) dt \\ &\asymp x_d^q y_d^q \times \begin{cases} 1 & \text{if } q < \widehat{q}; \\ \log^{\beta_4+1}(e+1/y_d) & \text{if } q = \widehat{q}; \\ y_d^{\widehat{q}-q}\log^{\beta_4}(e+1/y_d) & \text{if } q > \widehat{q}. \end{cases} \end{aligned}$$

For the upper bound, we see from (9.1) that

$$\begin{aligned} G_1 \le\; & c_6 x_d^{\beta_1} y_d^{\widehat{\beta}_2} \log^{\beta_3}\left(e+\frac{y_d}{x_d}\right) \log^{\beta_4}\left(e+\frac{1}{y_d}\right) \int_0^{x_d^\alpha} t dt \\ & + c_6 x_d^q y_d^{\widehat{\beta}_2} \log^{\beta_4}\left(e+\frac{1}{y_d}\right) \int_{x_d^\alpha}^{y_d^\alpha} t^{(\alpha+\beta_1-q)/\alpha} \log^{\beta_3}\left(e+\frac{y_d}{t^{1/\alpha}}\right) dt \\ & + c_6 x_d^q y_d^q \int_{y_d^\alpha}^1 t^{(\alpha+\beta_1+\widehat{\beta}_2-2q)/\alpha} \log^{\beta_4}\left(e+\frac{1}{t^{1/\alpha}}\right) dt \\ =:\; & c_6(I + II + III). \end{aligned}$$

Using (10.1), we obtain

$$I \le c_7 x_d^{2\alpha+\beta_1} y_d^{\widehat{\beta}_2} \left(\frac{y_d}{x_d}\right)^{2\alpha+\beta_1-q} \log^{\beta_4}\left(e+\frac{1}{y_d}\right) = c_7 x_d^q y_d^{2\alpha+\beta_1+\widehat{\beta}_2-q} \log^{\beta_4}\left(e+\frac{1}{y_d}\right). \tag{9.5}$$

Since $\alpha+\beta_1 > q$, the map $t \mapsto t^{(\alpha+\beta_1-q)/\alpha}\log^{\beta_3}(e+y_d/t^{1/\alpha})$ is almost increasing. Therefore,

$$II \le c_8 x_d^q y_d^{\alpha+\beta_1+\widehat{\beta}_2-q} \log^{\beta_4}\left(e+\frac{1}{y_d}\right) \int_{x_d^\alpha}^{y_d^\alpha} dt \le c_8 x_d^q y_d^{2\alpha+\beta_1+\widehat{\beta}_2-q} \log^{\beta_4}\left(e+\frac{1}{y_d}\right). \tag{9.6}$$

We consider the cases $q < \widehat{q}$ and $q \ge \widehat{q}$ separately. First, suppose that $q < \widehat{q}$, which is equivalent to $q < \alpha + (\beta_1+\beta_2)/2$. Since $q < \alpha+\beta_1$, it follows that $q < \alpha + (\beta_1 + \widehat{\beta}_2)/2$. Using (10.1), since $y_d < 1$, we see from (9.5)–(9.6) that

$$I + II \le c_9 x_d^q y_d^q y_d^{2\alpha+\beta_1+\widehat{\beta}_2-2q} \log^{\beta_4}\left(e+\frac{1}{y_d}\right) \le c_{10} x_d^q y_d^q.$$

Moreover, since $(\alpha+\beta_1+\widehat{\beta}_2-2q)/\alpha>-1$, we get

$$III \le c_7 x_d^q y_d^q \int_0^1 t^{(\alpha+\beta_1+\widehat{\beta}_2-2q)/\alpha} \log^{\beta_4}\left(e+\frac{1}{t^{1/\alpha}}\right) dt \le c_{11} x_d^q y_d^q.$$

Therefore, we arrive at the result in this case.

Suppose that $q \ge \widehat{q}$. In this case, we have $2\alpha+\beta_1+\beta_2 = q+\widehat{q} \le 2q < 2\alpha+2\beta_1$. Thus, $\beta_2<\beta_1$ and $\widehat{\beta}_2=\beta_2$. Then we deduce the desired upper bound from (9.4)–(9.6). The proof is complete. □

Lemma 9.2. *Let $x, y \in \mathbb{R}^d_+$ be such that $x_d \le y_d$ and $|x-y|=1$. Then we have*

$$\int_1^\infty p^\kappa(t,x,y)dt \asymp \begin{cases} (x_d\wedge 1)^q (y_d \wedge 1)^q & \text{if } d>\alpha, \\ (x_d\wedge 1)^q (y_d \wedge 1)^q \log\big(e+(x_d\vee 1)\big) & \text{if } d=1=\alpha, \\ (x_d\wedge 1)^q (y_d \wedge 1)^q (x_d\vee 1)^{\alpha-1} & \text{if } d=1<\alpha, \end{cases}$$

where the comparison constant is independent of x, y.

Proof. By (9.3), we have

$$\int_1^\infty p^\kappa(t,x,y)dt \asymp \int_1^\infty t^{-d/\alpha}\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^q \left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^q dt =: G_2.$$

If $d>\alpha$, then by Lemma 10.12, $G_2 \asymp (x_d\wedge 1)^q (y_d\wedge 1)^q$.

If $d=1=\alpha$, then using Lemma 10.12, we get

$$\begin{aligned} G_2 &\asymp \int_1^{x_d\vee 1} t^{-1}dt + x_d^q \int_{x_d\vee 1}^\infty t^{-1-q}\left(1\wedge\frac{y_d}{t}\right)^q dt \\ &\asymp \mathbf{1}_{\{x_d>1\}}\log(x_d) + x_d^q (x_d\vee 1)^{-q}\left(1\wedge \frac{y_d}{x_d\vee 1}\right)^q \\ &\asymp \begin{cases} \log(e+x_d) & \text{if } x_d>1 \\ x_d^q (y_d\wedge 1)^q & \text{if } x_d \le 1 \end{cases} \asymp (x_d\wedge 1)^q (y_d\wedge 1)^q \log(e+(x_d\vee 1)). \end{aligned}$$

If $d=1<\alpha$, then since $y_d>2$ implies $x_d \ge y_d - |x-y| \ge y_d/2 > 1$, and $\alpha-q-1\le 0$, we get

$$\begin{aligned} G_2 &\asymp \mathbf{1}_{\{y_d>2\}}\left(\int_1^{x_d^\alpha} t^{-1/\alpha}dt + x_d^q \int_{x_d^\alpha}^{(2y_d)^\alpha} t^{-(q+1)/\alpha}dt + x_d^q y_d^q \int_{(2y_d)^\alpha}^\infty t^{-(2q+1)/\alpha}dt\right) \\ &\quad + \mathbf{1}_{\{y_d\le 2\}} x_d^q y_d^q \int_1^\infty t^{-(2q+1)/\alpha}dt \\ &\asymp \mathbf{1}_{\{y_d>2\}}\left(x_d^{\alpha-1} + x_d^{q+\alpha-(q+1)} + x_d^q y_d^{\alpha-q-1}\right) + \mathbf{1}_{\{y_d\le 2\}} x_d^q y_d^q \\ &\asymp \mathbf{1}_{\{y_d>2\}} x_d^{\alpha-1} + \mathbf{1}_{\{y_d\le 2\}} x_d^q y_d^q \asymp (x_d\wedge 1)^q (y_d\wedge 1)^q (x_d\vee 1)^{\alpha-1}. \end{aligned}$$

The proof is complete. □

Proof of Theorem 1.4. From scaling property (3.30), we obtain

$$G^\kappa(x,y) = |x-y|^{-d+\alpha}\, G^\kappa(x/|x-y|, y/|x-y|), \quad x,y\in\mathbb{R}^d_+. \tag{9.7}$$

Using (9.7), symmetry and Lemmas 9.1 and 9.2, we deduce the desired result. □

10. Appendix

Note that for any $\varepsilon > 0$,

$$\log(e+r) < (2+\varepsilon^{-1})r^{\varepsilon} \quad \text{for all } r \ge 1, \tag{10.1}$$

$$\frac{\log(e+ar)}{\log(e+r)} < (1+\varepsilon^{-1})a^{\varepsilon} \qquad \text{for all } a \ge 1 \text{ and } r > 0. \tag{10.2}$$

Recall the definition of $A_{b_1,b_2,b_3,b_4}(t,x,y)$ from (6.2).

Lemma 10.1. *Let $b_1, b_2, b_3, b_4 \ge 0$.*
(i) If $b_1 > 0$, then for any $\varepsilon \in (0, b_1]$, there exists $c_1 > 0$ such that

$$A_{b_1,b_2,b_3,b_4}(t,x,y) \le c_1 A_{b_1-\varepsilon,b_2,0,b_4}(t,x,y) \quad \text{for all } t \ge 0,\ x,y \in \mathbb{R}^d_+.$$

(ii) If $b_2 > 0$, then for any $\varepsilon \in (0, b_2]$, there exists $c_2 > 0$ such that

$$A_{b_1,b_2,b_3,b_4}(t,x,y) \le c_2 A_{b_1,b_2-\varepsilon,b_3,0}(t,x,y) \quad \text{for all } t \ge 0,\ x,y \in \mathbb{R}^d_+.$$

Proof. The results follow from (10.1). □

Lemma 10.2. *Let $b_1, b_3 \ge 0$. Suppose that $b_1 > 0$ if $b_3 > 0$. Then there exists a constant $C > 0$ such that for all $t \ge 0$ and $x, y \in \mathbb{R}^d_+$,*

$$A_{b_1,0,b_3,0}(t,x,y) \le C\left(\frac{x_d \vee t^{1/\alpha}}{|x-y|} \wedge 1\right)^{b_1} \log^{b_3}\left(e + \frac{(y_d \vee t^{1/\alpha}) \wedge |x-y|}{(x_d \vee t^{1/\alpha}) \wedge |x-y|}\right).$$

Proof. When $x_d \le y_d$, the result is clear. Assume that $x_d > y_d$. Set $r = \frac{x_d \vee t^{1/\alpha}}{|x-y|} \wedge 1$ and $s = \frac{y_d \vee t^{1/\alpha}}{|x-y|} \wedge 1$. Then $0 < s \le r \le 1$ and the desired inequality is equivalent to

$$\log^{b_3}(e + r/s)\log^{-b_3}(e + s/r) \le C(r/s)^{b_1}. \tag{10.3}$$

If $b_3 = 0$, then (10.3) clearly holds with $C = 1$. If $b_3 > 0$ and $b_1 > 0$, then we get from (10.1) that

$$\log^{b_3}(e + r/s)\log^{-b_3}(e + s/r) \le \log^{b_3}(e + r/s) \le c_1(r/s)^{b_1}.$$

This proves the lemma. □

Remark 10.3. Recall that $B_{b_1,b_2,b_3,b_4}(x,y) = A_{b_1,b_2,b_3,b_4}(0,x,y)$ for $x, y \in \mathbb{R}^d_+$. Therefore, the results of Lemmas 10.1 and 10.2 hold true with $B_{b_1,0,b_3,0}(x,y)$ instead of $A_{b_1,0,b_3,0}(t,x,y)$.

Lemma 10.4. *Let $\gamma \in \mathbb{R}$, $b \ge 0$ and $t, k, l > 0$. Suppose that either $\gamma < \alpha$ or $k \ge t^{1/\alpha}$. Then we have*

$$\int_0^t \left(1 \wedge \frac{k}{s^{1/\alpha}}\right)^{\gamma} \log^b\left(e + \frac{l}{k \vee s^{1/\alpha}}\right) ds \le Ct\left(1 \wedge \frac{k}{t^{1/\alpha}}\right)^{\gamma} \log^b\left(e + \frac{l}{k \vee t^{1/\alpha}}\right),$$

where $C > 0$ is a constant which depends only on γ and b.

Proof. If $k \ge t^{1/\alpha}$, then the desired inequality holds since the left hand side is $t\log^b(e + l/k)$.

Suppose that $k < t^{1/\alpha}$ and $\gamma < \alpha$. Let $\varepsilon > 0$ be such that $\gamma + b\varepsilon < \alpha$. Then the left hand side is

$$\begin{aligned}
&\log^b\left(e + \frac{l}{k}\right)\int_0^{k^\alpha} ds + k^{\gamma}\int_{k^\alpha}^t \frac{1}{s^{\gamma/\alpha}} \log^b\left(e + \frac{l}{s^{1/\alpha}}\right) ds \\
&\le k^{\alpha}\log^b\left(e + \frac{l}{k}\right) + c_1 k^{\gamma}\log^b\left(e + \frac{l}{t^{1/\alpha}}\right)\int_{k^\alpha}^t \frac{t^{b\varepsilon/\alpha}}{s^{\gamma/\alpha + b\varepsilon/\alpha}} ds \\
&\le c_2 k^{\alpha}\left(\frac{t^{1/\alpha}}{k}\right)^{\alpha-\gamma}\log^b\left(e + \frac{l}{t^{1/\alpha}}\right) + c_2 k^{\gamma} t^{1-\gamma/\alpha}\log^b\left(e + \frac{l}{t^{1/\alpha}}\right)
\end{aligned}$$

$$= 2c_2 t\Big(\frac{k}{t^{1/\alpha}}\Big)^\gamma \log^b\Big(e+\frac{l}{t^{1/\alpha}}\Big) = 2c_2 t\Big(1\wedge\frac{k}{t^{1/\alpha}}\Big)^\gamma \log^b\Big(e+\frac{l}{t^{1/\alpha}}\Big).$$

We used (10.2) in both inequalities above. □

Lemma 10.5. *Let $b_1, b_2 \in \mathbb{R}$, $b_3, b_4 \ge 0$ and $t, x_d, y_d > 0$. Suppose that (1) either $b_1 > q-\alpha$ or $x_d \ge t^{1/\alpha}$, and (2) either $b_2 > q-\alpha$ or $y_d \ge t^{1/\alpha}$. Then we have*

$$\int_0^t \Big(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}\log^{b_3}\Big(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{y_d\vee s^{1/\alpha}}\Big)ds$$
$$\le Ct\Big(1\wedge\frac{x_d}{t^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{t^{1/\alpha}}\Big)^{q-b_2}\log^{b_3}\Big(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{y_d\vee t^{1/\alpha}}\Big)$$

where $C>0$ is a constant which depends only on b_1, b_2, b_3 and b_4.

Proof. Observe that

$$\int_0^{\frac{t}{2}} \Big(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}\log^{b_3}\Big(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{y_d\vee s^{1/\alpha}}\Big)ds$$
$$\le c_1\Big(1\wedge\frac{x_d}{t^{1/\alpha}}\Big)^{q-b_1}\log^{b_3}\Big(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee t^{1/\alpha}}\Big)\int_0^{\frac{t}{2}}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}\log^{b_4}\Big(e+\frac{1}{y_d\vee s^{1/\alpha}}\Big)ds$$

and

$$\int_{\frac{t}{2}}^t \Big(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}\log^{b_3}\Big(e+\frac{y_d\vee s^{1/\alpha}}{x_d\vee(t-s)^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{y_d\vee s^{1/\alpha}}\Big)ds$$
$$\le c_2\Big(1\wedge\frac{y_d}{t^{1/\alpha}}\Big)^{q-b_2}\log^{b_4}\Big(e+\frac{1}{y_d\vee t^{1/\alpha}}\Big)\int_0^{\frac{t}{2}}\Big(1\wedge\frac{x_d}{s^{1/\alpha}}\Big)^{q-b_1}\log^{b_3}\Big(e+\frac{y_d\vee t^{1/\alpha}}{x_d\vee s^{1/\alpha}}\Big)ds.$$

Using Lemma 10.4 twice, we arrive at the result. □

Lemma 10.6. *Let $b_1, b_2 \in \mathbb{R}$, $b_3, b_4 \ge 0$ and $t, x_d, y_d > 0$. Suppose that (1) either $b_1 > q-\alpha$ or $x_d \ge t^{1/\alpha}$, and (2) either $b_2 > q-\alpha$ or $y_d \ge t^{1/\alpha}$. Then we have that for $y_d\vee t^{1/\alpha} < 2$,*

$$\int_0^t s\Big(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}$$
$$\times\int_{y_d\vee s^{1/\alpha}}^2 \log^{b_3}\Big(e+\frac{r}{y_d\vee s^{1/\alpha}}\Big)\log^{b_3}\Big(e+\frac{r}{x_d\vee(t-s)^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{r}\Big)\frac{dr}{r}\,ds$$
$$\le Ct\Big(1\wedge\frac{x_d}{t^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{t^{1/\alpha}}\Big)^{q-b_2}$$
$$\times\int_{y_d}^2 (r^\alpha\wedge t)\log^{b_3}\Big(e+\frac{r}{x_d\vee t^{1/\alpha}}\Big)\log^{b_3}\Big(e+\frac{r}{y_d\vee t^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{r}\Big)\frac{dr}{r},$$

where $C>0$ is a constant which depends only on b_1, b_2, b_3 and b_4.

Proof. Using Fubini's theorem and Lemma 10.4 twice as in the proof of Lemma 10.5, we get

$$\int_0^t s\Big(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}$$
$$\times\int_{y_d\vee s^{1/\alpha}}^2 \log^{b_3}\Big(e+\frac{r}{y_d\vee s^{1/\alpha}}\Big)\log^{b_3}\Big(e+\frac{r}{x_d\vee(t-s)^{1/\alpha}}\Big)\log^{b_4}\Big(e+\frac{1}{r}\Big)\frac{dr}{r}\,ds$$
$$=\int_{y_d}^2\int_0^{r^\alpha\wedge t} s\Big(1\wedge\frac{x_d}{(t-s)^{1/\alpha}}\Big)^{q-b_1}\Big(1\wedge\frac{y_d}{s^{1/\alpha}}\Big)^{q-b_2}$$

$$\times \log^{b_3}\left(e+\frac{r}{y_d \vee s^{1/\alpha}}\right)\log^{b_3}\left(e+\frac{r}{x_d \vee (t-s)^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{r}\right)ds\frac{dr}{r}$$
$$\leq \int_{y_d}^{2}(r^\alpha \wedge t)\int_0^t \left(1\wedge \frac{x_d}{(t-s)^{1/\alpha}}\right)^{q-b_1}\left(1\wedge \frac{y_d}{s^{1/\alpha}}\right)^{q-b_2}$$
$$\times \log^{b_3}\left(e+\frac{r}{y_d \vee s^{1/\alpha}}\right)\log^{b_3}\left(e+\frac{r}{x_d \vee (t-s)^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{r}\right)ds\frac{dr}{r}$$
$$\leq Ct\left(1\wedge \frac{x_d}{t^{1/\alpha}}\right)^{q-b_1}\left(1\wedge \frac{y_d}{t^{1/\alpha}}\right)^{q-b_2}$$
$$\times \int_{y_d}^{2}(r^\alpha \wedge t)\log^{b_3}\left(e+\frac{r}{x_d \vee t^{1/\alpha}}\right)\log^{b_3}\left(e+\frac{r}{y_d \vee t^{1/\alpha}}\right)\log^{b_4}\left(e+\frac{1}{r}\right)\frac{dr}{r}.$$

□

Lemma 10.7. *There is a constant $C>0$ such that for all $x\in\mathbb{R}^d_+$ and $A>0$,*

$$\int_{B_+(x,A)} z_d^{-1/2}dz \leq CA^d(x_d\vee A)^{-1/2}.$$

Proof. We have

$$\int_{B_+(x,A)} z_d^{-1/2}dz \leq \int_{\widetilde{z}\in\mathbb{R}^{d-1},\,|\widetilde{x}-\widetilde{z}|<A} d\widetilde{z}\int_{(x_d-A)\vee 0}^{x_d+A} z_d^{-1/2}dz_d \leq c_1A^{d-1}\int_{(x_d-A)\vee 0}^{x_d+A} z_d^{-1/2}dz_d. \quad (10.4)$$

If $x_d\geq 2A$, then

$$\int_{(x_d-A)\vee 0}^{x_d+A} z_d^{-1/2}dz_d \leq \frac{1}{(x_d/2)^{1/2}}\int_{x_d-A}^{x_d+A}dz_d = 2^{3/2}Ax_d^{-1/2}. \quad (10.5)$$

If $x_d<2A$, then

$$\int_{(x_d-A)\vee 0}^{x_d+A} z_d^{-1/2}dz_d \leq \int_0^{4A} z_d^{-1/2}dz_d = 4A^{1/2}. \quad (10.6)$$

Combining (10.4) with (10.5)–(10.6), we arrive at the result. □

Lemma 10.8. *(i) There is a constant $C>0$ such that for all $x\in\mathbb{R}^d_+$ and $0<A<x_d$,*

$$\int_{z\in\mathbb{R}^d_+,\,x_d\geq|x-z|>A}\frac{dz}{z_d^{1/2}|x-z|^{d+\alpha}} \leq Cx_d^{-1/2}A^{-\alpha}.$$

(ii) Let $\varepsilon\in(0,1)$ and $\delta>0$. There is a constant $C'>0$ such that for all $x\in\mathbb{R}^d_+$ and $A\geq x_d$,

$$\int_{z\in\mathbb{R}^d_+,\,|x-z|>A}\frac{dz}{z_d^{\varepsilon}|x-z|^{d+\delta}} \leq C'A^{-\varepsilon-\delta}.$$

Proof. Without loss of generality, we assume $x=(\widetilde{0},x_d)$.

(i) Note that

$$\int_{z\in\mathbb{R}^d_+,\,x_d\geq|x-z|>A}\frac{dz}{z_d^{1/2}|x-z|^{d+\alpha}}$$
$$=\int_{z\in\mathbb{R}^d_+,\,x_d\geq|x-z|>A,\,|\widetilde{z}|\leq|x_d-z_d|}\frac{dz}{z_d^{1/2}|x-z|^{d+\alpha}}+\int_{z\in\mathbb{R}^d_+,\,x_d\geq|x-z|>A,\,|\widetilde{z}|>|x_d-z_d|}\frac{dz}{z_d^{1/2}|x-z|^{d+\alpha}}$$
$$=: I+II.$$

First, we have

$$
\begin{aligned}
I &\le \int_{x_d \ge |x_d - z_d| > \frac{A}{2}} \frac{1}{z_d^{1/2} |x_d - z_d|^{d+\alpha}} \int_{\widetilde{z} \in \mathbb{R}^{d-1},\, |\widetilde{z}| \le |x_d - z_d|} d\widetilde{z} dz_d \\
&\le c_1 \bigg(\int_0^{\frac{x_d}{2}} \frac{dz_d}{z_d^{1/2} |x_d - z_d|^{1+\alpha}} + \int_{\frac{x_d}{2}}^{x_d - \frac{A}{2}} \frac{dz_d}{z_d^{1/2} |x_d - z_d|^{1+\alpha}} + \int_{x_d + \frac{A}{2}}^{2x_d} \frac{dz_d}{z_d^{1/2} |x_d - z_d|^{1+\alpha}} \bigg) \\
&\le c_1 \bigg(\frac{2^{1+\alpha}}{x_d^{1+\alpha}} \int_0^{\frac{x_d}{2}} z_d^{-1/2} dz_d + \frac{1}{(x_d/2)^{1/2}} \int_{\frac{x_d}{2}}^{x_d - \frac{A}{2}} \frac{dz_d}{|x_d - z_d|^{1+\alpha}} + x_d^{-1/2} \int_{x_d + \frac{A}{2}}^{2x_d} \frac{dz_d}{|x_d - z_d|^{1+\alpha}} \bigg) \\
&\le c_2 \left(x_d^{-1/2-\alpha} + x_d^{-1/2} A^{-\alpha} + x_d^{-1/2} A^{-\alpha} \right) \le c_3 x_d^{-1/2} A^{-\alpha}.
\end{aligned}
$$

On the other hand, we see that for any $z \in \mathbb{R}^d_+$ with $x_d \ge |x - z| > A$ and $|\widetilde{z}| > |x_d - z_d|$,

$$
z_d \ge x_d - |x_d - z_d| \ge x_d - \frac{1}{2}(|\widetilde{z}| + |x_d - z_d|) \ge x_d - \frac{1}{\sqrt{2}}|x - z| \ge \left(1 - \frac{1}{\sqrt{2}}\right) x_d.
$$

Hence, we also have that

$$
II \le c_4 x_d^{-1/2} \int_{z \in \mathbb{R}^d_+,\, |x-z| > A} \frac{dz}{|x - z|^{d+\alpha}} \le c_5 x_d^{-1/2} A^{-\alpha}.
$$

(ii) Observe that

$$
\begin{aligned}
&\int_{z \in \mathbb{R}^d_+,\, |x-z| > A} \frac{dz}{z_d^{\varepsilon}\, |x - z|^{d+\delta}} \\
&\le \int_{z \in \mathbb{R}^d_+,\, |x_d - z_d| \ge |\widetilde{z}| \vee \frac{A}{2}} \frac{d\widetilde{z}\, dz_d}{z_d^{\varepsilon}\, |x_d - z_d|^{d+\delta}} + \int_{z \in \mathbb{R}^d_+,\, |\widetilde{z}| \ge |x_d - z_d| \vee \frac{A}{2}} \frac{d\widetilde{z}\, dz_d}{z_d^{\varepsilon}\, |\widetilde{z}|^{d+\delta}} =: I + II.
\end{aligned}
$$

Using Fubini's theorem, since $A \ge x_d$, we see that

$$
\begin{aligned}
II &\le \int_{\widetilde{z} \in \mathbb{R}^{d-1},\, |\widetilde{z}| \ge \frac{A}{2}} \frac{1}{|\widetilde{z}|^{d+\delta}} \int_0^{|\widetilde{z}| + x_d} z_d^{-\varepsilon} dz_d\, d\widetilde{z} \le c_1 \int_{\widetilde{z} \in \mathbb{R}^{d-1},\, |\widetilde{z}| \ge \frac{A}{2}} \frac{(|\widetilde{z}| + A)^{1-\varepsilon}}{|\widetilde{z}|^{d+\delta}} d\widetilde{z} \\
&\le c_2 \int_{\widetilde{z} \in \mathbb{R}^{d-1},\, |\widetilde{z}| \ge \frac{A}{2}} \frac{d\widetilde{z}}{|\widetilde{z}|^{d-1+\varepsilon+\delta}} \le c_3 A^{-\varepsilon-\delta}.
\end{aligned}
$$

On the other hand, using the fact that $\int_{\widetilde{z} \in \mathbb{R}^{d-1},\, |\widetilde{z}| \le |x_d - z_d|} d\widetilde{z} \le c_4 |x_d - z_d|^{d-1}$, we also see that

$$
\begin{aligned}
I &\le c_5 \bigg(\int_0^{(x_d - \frac{A}{2}) \vee 0} \frac{dz_d}{z_d^{\varepsilon}\, |x_d - z_d|^{1+\delta}} + \int_{x_d + \frac{A}{2}}^{\infty} \frac{dz_d}{z_d^{\varepsilon}\, |x_d - z_d|^{1+\delta}} \bigg) \\
&\le c_6 \bigg(\frac{1}{A^{1+\delta}} \int_0^{(x_d - \frac{A}{2}) \vee 0} z_d^{-\varepsilon} dz_d + \frac{1}{A^{\varepsilon}} \int_{x_d + \frac{A}{2}}^{\infty} \frac{dz_d}{|x_d - z_d|^{1+\delta}} \bigg) \\
&\le c_7 \bigg(\frac{1}{A^{1+\delta}} \Big(\Big(x_d - \frac{A}{2}\Big) \vee 0 \Big)^{1-\varepsilon} + \frac{1}{A^{\varepsilon+\delta}} \bigg) \le \frac{c_6}{A^{\varepsilon+\delta}},
\end{aligned}
$$

where we used the fact that $A \ge x_d$ in the last inequality. The proof is complete. □

For $\gamma, \eta_1, \eta_2 \ge 0$ and $k, l > 0$, define

$$
f_{\gamma,\eta_1,\eta_2,k,l}(r) := r^{\gamma} \log^{\eta_1}\left(e + \frac{k}{r}\right) \log^{\eta_2}\left(e + \frac{r}{l}\right).
$$

Lemma 10.9. *Let $\gamma, \eta_1, \eta_2 \ge 0$.*
(i) For any $\varepsilon > 0$, there exist constants $C, C' > 0$ such that for any $k, l, r > 0$ and any $a \ge 1$,

$$
C a^{\gamma-\varepsilon} \le \frac{f_{\gamma,\eta_1,\eta_2,k,l}(ar)}{f_{\gamma,\eta_1,\eta_2,k,l}(r)} \le C' a^{\gamma+\varepsilon}. \tag{10.7}
$$

(ii) Assume that $\gamma > 0$. Then there exists a constant $C > 0$ such that for any $k, l, r > 0$ and any $a \ge 1$,

$$\frac{f_{\gamma,\eta_1,\eta_2,k,l}(ar)}{f_{\gamma,\eta_1,\eta_2,k,l}(r)} \ge C.$$

Proof. (i) If $\eta_2 = 0$, the second inequality in (10.7) is true with any $\varepsilon \ge 0$. In case $\eta_2 > 0$, for any given $\varepsilon > 0$, let $\varepsilon' := \varepsilon/\eta_2$. We get from (10.2) that for all $a \ge 1$ and $r > 0$,

$$\frac{f_{\gamma,\eta_1,\eta_2,k,l}(ar)}{f_{\gamma,\eta_1,\eta_2,k,l}(r)} \le a^\gamma \left(\frac{\log(e + ar/l)}{\log(e + r/l)} \right)^{\eta_2} \le a^\gamma \big((1 + 1/\varepsilon') a^{\varepsilon'} \big)^{\eta_2} = c(\varepsilon, \eta_2) a^{\gamma+\varepsilon}.$$

The first inequality can be proved by a similar argument.
(ii) The desired result follows from the first inequality in (10.7) with $\varepsilon = \gamma$. □

Lemma 10.10. *Let $b_1, b_2, \eta_1, \eta_2, \gamma \ge 0$. There exists a constant $C > 0$ such that for any $x \in \mathbb{R}^d_+$ and $s, k, l > 0$,*

$$\begin{aligned}
&\int_{B_+(x,2)} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2}\left(e + \frac{|x-z|}{(x_d \vee s^{1/\alpha}) \wedge |x-z|}\right) \\
&\qquad \times \left(s^{-d/\alpha} \wedge \frac{s}{|x-z|^{d+\alpha}}\right) z_d^\gamma \log^{\eta_1}\left(e + \frac{k}{z_d}\right) \log^{\eta_2}\left(e + \frac{z_d}{l}\right) dz \\
&\le C (x_d \vee s^{1/\alpha})^\gamma \log^{\eta_1}\left(e + \frac{k}{x_d \vee s^{1/\alpha}}\right) \log^{\eta_2}\left(e + \frac{x_d \vee s^{1/\alpha}}{l}\right) \\
&\quad + C \mathbf{1}_{\{x_d \vee s^{1/\alpha} < 2\}} s (x_d \vee s^{1/\alpha})^{b_1} \bigg[\mathbf{1}_{\{\gamma > \alpha + b_1\}} \log^{b_2}\left(e + \frac{2}{x_d \vee s^{1/\alpha}}\right) \log^{\eta_1}(e+k) \log^{\eta_2}\left(e + \frac{1}{l}\right) \\
&\qquad\qquad + \mathbf{1}_{\{\gamma = \alpha + b_1\}} \int_{x_d \vee s^{1/\alpha}}^2 \log^{b_2}\left(e + \frac{r}{x_d \vee s^{1/\alpha}}\right) \log^{\eta_1}\left(e + \frac{k}{r}\right) \log^{\eta_2}\left(e + \frac{r}{l}\right) \frac{dr}{r} \bigg].
\end{aligned}$$

Proof. Using the triangle inequality, we see that for any $z \in \mathbb{R}^d_+$,

$$z_d \le x_d + |x-z| \le 2(x_d \vee |x-z|). \tag{10.8}$$

Therefore, using Lemma 10.9(i)-(ii) and Lemma 10.7, we get that

$$\begin{aligned}
&\int_{B_+(x,s^{1/\alpha})} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2}\left(e + \frac{|x-z|}{(x_d \vee s^{1/\alpha}) \wedge |x-z|}\right) s^{-d/\alpha} f_{\gamma,\eta_1,\eta_2,k,l}(z_d) dz \\
&= (\log^{b_2}(e+1)) s^{-d/\alpha} \int_{B_+(x,s^{1/\alpha})} z_d^{-1/2} f_{\gamma+\frac12,\eta_1,\eta_2,k,l}(z_d) dz \\
&\le c_1 s^{-d/\alpha} f_{\gamma+\frac12,\eta_1,\eta_2,k,l}(2(x_d \vee s^{1/\alpha})) \int_{B_+(x,s^{1/\alpha})} z_d^{-1/2} dz \\
&\le c_2 (x_d \vee s^{1/\alpha})^{-1/2} f_{\gamma+\frac12,\eta_1,\eta_2,k,l}(x_d \vee s^{1/\alpha}) = c_2 f_{\gamma,\eta_1,\eta_2,k,l}(x_d \vee s^{1/\alpha}).
\end{aligned}$$

When $x_d > s^{1/\alpha}$, we get from (10.8), Lemma 10.9(i)-(ii) and Lemma 10.8(i) that

$$\begin{aligned}
&s \int_{z \in \mathbb{R}^d_+,\, x_d \ge |x-z| > s^{1/\alpha}} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2}\left(e + \frac{|x-z|}{(x_d \vee s^{1/\alpha}) \wedge |x-z|}\right) \frac{f_{\gamma,\eta_1,\eta_2,k,l}(z_d)}{|x-z|^{d+\alpha}} dz \\
&= (\log^{b_2}(e+1)) s \int_{z \in \mathbb{R}^d_+,\, x_d \ge |x-z| > s^{1/\alpha}} \frac{f_{\gamma+\frac12,\eta_1,\eta_2,k,l}(z_d)}{z_d^{1/2} |x-z|^{d+\alpha}} dz \\
&\le c_3 s f_{\gamma+\frac12,\eta_1,\eta_2,k,l}(2x_d) \int_{z \in \mathbb{R}^d_+,\, x_d \ge |x-z| > s^{1/\alpha}} \frac{1}{z_d^{1/2} |x-z|^{d+\alpha}} dz
\end{aligned}$$

$$\le c_4 x_d^{-1/2} f_{\gamma+\frac{1}{2},\eta_1,\eta_2,k,l}(x_d) = c_4 f_{\gamma,\eta_1,\eta_2,k,l}(x_d \vee s^{1/\alpha}).$$

It remains to bound the integral over $\{z \in \mathbb{R}^d_+ : x_d \vee s^{1/\alpha} < |x-z| < 2\}$ under the assumption $x_d \vee s^{1/\alpha} < 2$. For this, we consider the following three cases separately.

(i) Case $\gamma < \alpha + b_1$: Fix $\varepsilon \in (0,1)$ such that $\gamma + 3\varepsilon < \alpha + b_1$. Using (10.8), (10.1), Lemma 10.9(i)-(ii) and Lemma 10.8(ii), we get

$$\begin{aligned}
& s \int_{z\in\mathbb{R}^d_+,\, |x-z|>x_d\vee s^{1/\alpha}} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2}\left(e + \frac{|x-z|}{(x_d\vee s^{1/\alpha})\wedge|x-z|}\right) \frac{f_{\gamma,\eta_1,\eta_2,k,l}(z_d)}{|x-z|^{d+\alpha}} dz \\
&= s(x_d \vee s^{1/\alpha})^{b_1} \int_{z\in\mathbb{R}^d_+,\, |x-z|>x_d\vee s^{1/\alpha}} \log^{b_2}\left(e + \frac{|x-z|}{x_d\vee s^{1/\alpha}}\right) \frac{f_{\gamma+\varepsilon,\eta_1,\eta_2,k,l}(z_d)}{z_d^{\varepsilon}|x-z|^{d+\alpha+b_1}} dz \\
&\le c_5 s(x_d \vee s^{1/\alpha})^{b_1} \int_{z\in\mathbb{R}^d_+,\, |x-z|>x_d\vee s^{1/\alpha}} \log^{b_2}\left(e + \frac{|x-z|}{x_d\vee s^{1/\alpha}}\right) \frac{f_{\gamma+\varepsilon,\eta_1,\eta_2,k,l}(2|x-z|)}{z_d^{\varepsilon}|x-z|^{d+\alpha+b_1}} dz \\
&\le c_6 s(x_d \vee s^{1/\alpha})^{b_1} \int_{z\in\mathbb{R}^d_+,\, |x-z|>x_d\vee s^{1/\alpha}} \left(\frac{|x-z|}{x_d\vee s^{1/\alpha}}\right)^{\varepsilon+\gamma+2\varepsilon} \frac{f_{\gamma+\varepsilon,\eta_1,\eta_2,k,l}(x_d\vee s^{1/\alpha})}{z_d^{\varepsilon}|x-z|^{d+\alpha+b_1}} dz \\
&= c_6 s(x_d \vee s^{1/\alpha})^{b_1-\gamma-3\varepsilon} f_{\gamma+\varepsilon,\eta_1,\eta_2,k,l}(x_d\vee s^{1/\alpha}) \int_{z\in\mathbb{R}^d_+,\, |x-z|>x_d\vee s^{1/\alpha}} \frac{1}{z_d^{\varepsilon}|x-z|^{d+\alpha+b_1-\gamma-3\varepsilon}} dz \\
&\le c_7 s(x_d \vee s^{1/\alpha})^{-\alpha}(x_d \vee s^{1/\alpha})^{-\varepsilon} f_{\gamma+\varepsilon,\eta_1,\eta_2,k,l}(x_d\vee s^{1/\alpha}) \le c_7 f_{\gamma,\eta_1,\eta_2,k,l}(x_d\vee s^{1/\alpha}).
\end{aligned}$$

(ii) Case $\gamma > \alpha + b_1$: Fix $\varepsilon > 0$ such that $\gamma - \varepsilon > \alpha + b_1$. Using (10.8), Lemma 10.9(i)-(ii) and (10.2), we get

$$\begin{aligned}
& s \int_{z\in\mathbb{R}^d_+,\, x_d\vee s^{1/\alpha}<|x-z|<2} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2}\left(e + \frac{|x-z|}{(x_d\vee s^{1/\alpha})\wedge|x-z|}\right) \frac{f_{\gamma,\eta_1,\eta_2,k,l}(z_d)}{|x-z|^{d+\alpha}} dz \\
&\le c_8 s(x_d \vee s^{1/\alpha})^{b_1} \int_{z\in\mathbb{R}^d_+,\, x_d\vee s^{1/\alpha}<|x-z|<2} \log^{b_2}\left(e + \frac{|x-z|}{x_d\vee s^{1/\alpha}}\right) \frac{f_{\gamma,\eta_1,\eta_2,k,l}(2|x-z|)}{|x-z|^{d+\alpha+b_1}} dz \\
&\le c_9 s(x_d \vee s^{1/\alpha})^{b_1} f_{\gamma,\eta_1,\eta_2,k,l}(4) \log^{b_2}\left(e + \frac{2}{x_d\vee s^{1/\alpha}}\right) \int_{z\in\mathbb{R}^d_+,\, |x-z|<2} \frac{dz}{|x-z|^{d+\alpha+b_1-\gamma+\varepsilon}} \\
&\le c_{10} s(x_d \vee s^{1/\alpha})^{b_1} f_{\gamma,\eta_1,\eta_2,k,l}(1) \log^{b_2}\left(e + \frac{2}{x_d\vee s^{1/\alpha}}\right).
\end{aligned}$$

(iii) Case $\gamma = \alpha + b_1$: In this case, we see that

$$\begin{aligned}
& s \int_{z\in\mathbb{R}^d_+,\, x_d\vee s^{1/\alpha}<|x-z|<2} \left(1 \wedge \frac{x_d \vee s^{1/\alpha}}{|x-z|}\right)^{b_1} \log^{b_2}\left(e + \frac{|x-z|}{(x_d\vee s^{1/\alpha})\wedge|x-z|}\right) \frac{f_{\gamma,\eta_1,\eta_2,k,l}(z_d)}{|x-z|^{d+\alpha}} dz \\
&\le c_{11} s(x_d \vee s^{1/\alpha})^{b_1} \int_{z\in\mathbb{R}^d_+,\, x_d\vee s^{1/\alpha}<|x-z|<2} \log^{b_2}\left(e + \frac{|x-z|}{x_d\vee s^{1/\alpha}}\right) \frac{f_{\gamma,\eta_1,\eta_2,k,l}(|x-z|)}{|x-z|^{d+\alpha+b_1}} dz \\
&= c_{11} s(x_d \vee s^{1/\alpha})^{b_1} \\
&\times \int_{z\in\mathbb{R}^d_+,\, x_d\vee s^{1/\alpha}<|x-z|<2} \log^{b_2}\left(e + \frac{|x-z|}{x_d\vee s^{1/\alpha}}\right) \log^{\eta_1}\left(e + \frac{k}{|x-z|}\right) \log^{\eta_2}\left(e + \frac{|x-z|}{l}\right) \frac{dz}{|x-z|^d} \\
&\le c_{12} s(x_d \vee s^{1/\alpha})^{b_1} \int_{x_d\vee s^{1/\alpha}}^2 \log^{b_2}\left(e + \frac{r}{x_d\vee s^{1/\alpha}}\right) \log^{\eta_1}\left(e + \frac{k}{r}\right) \log^{\eta_2}\left(e + \frac{r}{l}\right) \frac{dr}{r}.
\end{aligned}$$

The proof is complete. □

Lemma 10.11. *Let $b_1, b_2 \geq 0$. For any $0 < k \leq l < 1$,*

$$\int_l^2 \log^{b_1}\left(e+\frac{r}{k}\right)\log^{b_1}\left(e+\frac{r}{l}\right)\log^{b_2}\left(e+\frac{1}{r}\right)\frac{dr}{r} \asymp \log^{b_1}\left(e+\frac{1}{k}\right)\log^{b_1+b_2+1}\left(e+\frac{1}{l}\right),$$

with comparison constants independent of k and l.

Proof. Note that

$$\log\left(e+\frac{1}{r}\right) \asymp \log\left(\frac{2e}{r}\right) \asymp \log\left(e+\frac{1}{\sqrt{r}}\right), \quad 0<r<2. \tag{10.9}$$

Hence, we get

$$\begin{aligned}
&\int_l^2 \log^{b_1}\left(e+\frac{r}{k}\right)\log^{b_1}\left(e+\frac{r}{l}\right)\log^{b_2}\left(e+\frac{1}{r}\right)\frac{dr}{r}\\
&\geq c_1\int_{\sqrt{l}}^2 \log^{b_1}\left(\frac{2er}{k}\right)\log^{b_1}\left(\frac{2er}{l}\right)\log^{b_2}\left(\frac{2e}{r}\right)\frac{dr}{r}\\
&\geq c_1\log^{b_1}\left(\frac{2e}{\sqrt{k}}\right)\log^{b_1}\left(\frac{2e}{\sqrt{l}}\right)\int_{\sqrt{l}}^2 \log^{b_2}\left(\frac{2e}{r}\right)\frac{dr}{r}\\
&= \frac{c_1}{b_2+1}\log^{b_1}\left(\frac{2e}{\sqrt{k}}\right)\log^{b_1}\left(\frac{2e}{\sqrt{l}}\right)\left(\log^{b_2+1}\left(\frac{2e}{\sqrt{l}}\right)-1\right)\\
&\geq c_2\log^{b_1}\left(e+\frac{1}{k}\right)\log^{b_1+b_2+1}\left(e+\frac{1}{l}\right).
\end{aligned}$$

On the other hand, using (10.9) and (10.2), we get

$$\begin{aligned}
&\int_l^2 \log^{b_1}\left(e+\frac{r}{k}\right)\log^{b_1}\left(e+\frac{r}{l}\right)\log^{b_2}\left(e+\frac{1}{r}\right)\frac{dr}{r}\\
&\leq c_3\log^{b_1}\left(e+\frac{2}{k}\right)\log^{b_1}\left(e+\frac{2}{l}\right)\int_l^2 \log^{b_2}\left(\frac{2e}{r}\right)\frac{dr}{r} \leq c_4\log^{b_1}\left(e+\frac{1}{k}\right)\log^{b_1+b_2+1}\left(e+\frac{1}{l}\right).
\end{aligned}$$

□

Lemma 10.12. *Let $\gamma > 1$ and $b_1, b_2 \geq 0$. For any $a, k, l > 0$,*

$$\int_a^\infty t^{-\gamma}\left(1\wedge\frac{k}{t}\right)^{b_1}\left(1\wedge\frac{l}{t}\right)^{b_2}dt \asymp a^{1-\gamma}\left(1\wedge\frac{k}{a}\right)^{b_1}\left(1\wedge\frac{l}{a}\right)^{b_2},$$

with comparison constants independent of a, k and l.

Proof. We have

$$\int_a^\infty t^{-\gamma}\left(1\wedge\frac{k}{t}\right)^{b_1}\left(1\wedge\frac{l}{t}\right)^{b_2}dt \leq \left(1\wedge\frac{k}{a}\right)^{b_1}\left(1\wedge\frac{l}{a}\right)^{b_2}\int_a^\infty t^{-\gamma}dt \leq \frac{a^{1-\gamma}}{\gamma-1}\left(1\wedge\frac{k}{a}\right)^{b_1}\left(1\wedge\frac{l}{a}\right)^{b_2}$$

and

$$\begin{aligned}
\int_a^\infty t^{-\gamma}\left(1\wedge\frac{k}{t}\right)^{b_1}\left(1\wedge\frac{l}{t}\right)^{b_2}dt &\geq \left(1\wedge\frac{k}{2a}\right)^{b_1}\left(1\wedge\frac{l}{2a}\right)^{b_2}\int_a^{2a} t^{-\gamma}dt\\
&\geq \frac{(1-2^{1-\gamma})a^{1-\gamma}}{2^{b_1+b_2}(\gamma-1)}\left(1\wedge\frac{k}{a}\right)^{b_1}\left(1\wedge\frac{l}{a}\right)^{b_2}.
\end{aligned}$$

□

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