The problem-solving legacy of Paul Erdős

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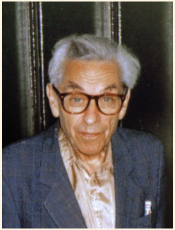
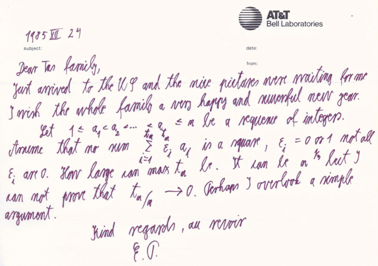
Paul Erdős



Paul Erdős (1913–1996)

- Published around 1500 mathematical papers, with many coauthors.
- Contributed to: *combinatorics*, *graph theory*, *number theory*, *geometry*, *probability*, *set theory*, etc.
- Had no permanent address, constantly traveled, carried all his belongings in a suitcase.

Paul Erdős and Terence Tao in 1985



AT&T Bell Laboratories

1985 XII 24

subject:

date:

from:

Dear Tao family,

Just arrived to the US and the nice pictures were waiting for me. I wish the whole family a very happy and wonderful new year.

Let $1 \le a_1 < a_2 < \dots < a_{t_n} \le n$ be a sequence of integers. Assume that no sum $\sum_{i=1}^{t_n} \varepsilon_i a_i$ is a square, $\varepsilon_i = 0$ or 1 not all ε_i are 0. How large can max t_n be. It can be $n/3$ but I can not prove that $t_n/n \to 0$. Perhaps I overlook a simple argument.

Kind regards, au revoir

E. P.

A letter to the Tao family

The greatest problem poser

The following question occurred to me a few weeks ago: Let S_1 and S_2 be two sets of distinct points $x_1,\dots,x_n$; $y_1,\dots,y_n$. The sets S_1 and S_2 do not have to be disjoint. Denote by $d(S_1;S_2)$ the number of distinct distances $d(x_i, y_j)$. Is it true that

$$(5) \qquad \min_{S_1,S_2} \ d(S_1,S_2)/f_2(n) \to 0$$

as n tends to infinity? (5) is perhaps of interest for the following reason: By a well known remark of Lenz, (5) certainly holds in 4-dimensions, since there $\min d(S_1,S_2) = 1$ for every n. For 2 or 3 dimensions (5) is open and quite possibly the answer is negative.

Here is one final problem of this type: Let $x_1,\dots,x_n$ be n points in the plane, no four on a circle and every circle whose center is one of the x_i contains at most two of our points. Clearly for every x_i we then have

$$d(x_i) \geq \frac{n-1}{2}.$$

Is it true that there is an absolute constant c so that

$$(6) \qquad \max_{1 \leq i \leq n} d(x_i) > (1+c)\frac{n}{2} \ ?$$

I offer 25 dollars for a solution.

We need the assumption that no four of our points are on a circle since otherwise the regular polygon gives a counterexample. Perhaps in fact

$$\sum_{i=1}^{n} d(x_i) > (1+c)\frac{n^2}{2}$$

also holds. It might be of some interest to try to deduce (6) from as weak an assumption as possible. It should certainly hold if we only assume that no k of our points are on a circle where k is independent of n, perhaps this assumption can be weakened further. We also assume that not too many of our points are on a line.

Let S be a set of n points in the plane no three on a line, no four on a circle. Denote by h(n) the largest integer for which such a set determines at least h(n) distinct distances. Pach just told me that $h(2^n) \leq 3^n$. The projection of the n-dimensional cube shows this. Perhaps $h(n)/n \to \infty$, but as far as I know this is still open.

P. Erdős, *Some old and new problems in combinatorial geometry*, Applications of discrete mathematics (Clemson, 1986), pp. 32–37, SIAM, Philadelphia, 1988

Erdős on Graphs (mathweb.ucsd.edu/~erdosproblems)

- Attempts to document problems posed by Paul Erdős in graph theory (and related areas).
- Grew out of a 1997 survey paper by Fan Chung, and a 1998 book *Erdős on Graphs* by Fan Chung and Ronald Graham.
- Launched by students of Fan Chung in 2010.
- Contains 170 problems.

Problem

Any set of $2^{n-2}+1$ points in the plane in general position contains a convex n-gon

Let $g(n)$ be the smallest number such that any set of n points in general position contains a convex n-gon. Paul Erdösand George Szekeres proved upper [1] and lower [2] bounds:

$$2^{n-2}+1 \le g(n) \le \binom{2n-4}{n-2}+1.$$

The upper bound has been improved several times, but the lower bound has not budged. It is correct for $n \le 5$, and is conjectured to be exact.

Fan Chung and Ron Graham [1] made the first (albeit minor) improvement to the upper bound, showing $g(n) \le \binom{2n-4}{n-2}$.

Next, Kleitman and Pachter [3] improved this to $g(n) \le \binom{2n-4}{n-2}+7-2n$.

Then Tóth and Valtr [4] improved this by about a factor of 2, showing $g(n) \le \binom{2n-5}{n-2}+2$, which they later improved slightly [5] to $g(n) \le \binom{2n-5}{n-2}+1$.

Erdős Problems (www.erdosproblems.com)

- Launched by Thomas Bloom about 2 years ago.
- Currently contains 946 problems posed by Paul Erdős.
- 333 of those are fully solved.
- Browseable by tags: *irrationality*, *primes*, *arithmetic progressions*, *polynomials*, etc.

OPEN - $500

If $A \subseteq \{1, \ldots, N\}$ with $|A| = n$ is such that the subset sums $\sum_{a \in S} a$ are distinct for all $S \subseteq A$ then

$$N \gg 2^n.$$

#1: [Er56][Er57][Er59][Er61][Er69][Er70c][Er73][ErSp74][ErGr80][Er81][Er82e][Er85c][Er90][Er91][Er92b][Er95][Er97c][Er98] number theory, additive combinatorics

Erdős called this 'perhaps my first serious problem' (in [Er98] he dates it to 1931). The powers of 2 show that 2^n would be best possible here. The trivial lower bound is $N \gg 2^n/n$, since all 2^n distinct subset sums must lie in $[0, Nn)$. Erdős and Moser [Er56] proved

$$N \geq (\tfrac{1}{4} - o(1)) \frac{2^n}{\sqrt{n}}.$$

(In [Er85c] Erdős offered $100 for any improvement of the constant $1/4$ here.)

A number of improvements of the constant have been given (see [St23] for a history), with the current record $\sqrt{2/\pi}$ first proved in unpublished work of Elkies and Gleason. Two proofs achieving this constant are provided by Dubroff, Fox, and Xu [DFX21], who in fact prove the exact bound $N \geq \binom{n}{\lfloor n/2 \rfloor}$.

In [Er73] and [ErGr80] the generalisation where $A \subseteq (0, N]$ is a set of real numbers such that the subset sums all differ by at least 1 is proposed, with the same conjectured bound. (The second proof of [DFX21] applies also to this generalisation.)

This problem appears in Erdős' book with Spencer [ErSp74] in the final chapter titled 'The kitchen sink'. As Ruzsa writes in [Ru99] "it is a rich kitchen where such things go to the sink".

The sequence of minimal N for a given n is A276661 in the OEIS.

See also [350].

Additional thanks to: Zach Hunter and Ralf Stephan

Paul Erdős, 1990s: *A well chosen problem can isolate an essential difficulty in a particular area, serving as a benchmark against which progress in this area can be measured. An innocent looking problem often gives no hint as to its true nature. It might be like a 'marshmallow,' serving as a tasty tidbit supplying a few moments of fleeting enjoyment. Or it might be like an 'acorn,' requiring deep and subtle new insights from which a mighty oak can develop.*

Are they difficult?

Christoph Thiele, 2000s (paraphrased): *Every unsolved problem is infinitely difficult. Every solved problem is trivial — just read the solution.*

Terence Tao, 2024: *Every so often, I have taken a look at a random problem from the site [Erdős Problems] for fun. A few times, I was able to make progress on one of the problems, leading to a couple papers; but the more common outcome is that I play around with the problem for a while, see why the problem is difficult, and then eventually give up and do something else.*

Are they modern? Obviously no.

Melvyn Nathanson, 1996: *He [Paul Erdős] learned mathematics in the 1930's in Hungary and England, and England at that time was a kind of mathematical backwater. For the rest of his life he concentrated on the fields that he had learned as a boy. Elementary and analytic number theory (...), graph theory, set theory, probability theory, and classical analysis. In these fields he was an absolute master, a virtuoso.*

At the same time, it is extraordinary to think of the parts of mathematics he never learned. Much of contemporary number theory, for example. (...)

A few months ago, on his last visit to New Jersey, I was telling Erdős something about p-adic analysis. Erdős was not interested. "You know," he said about the p-adic numbers, "they don't really exist."

Are they modern? But they have some charm.

Melvyn Nathanson, 2018: *None of this seemed to matter to Erdős, who was content to prove and conjecture and publish more than 1500 papers.*
Not because of politicking, but because of computer science and because his mathematics was always beautiful, in the past decade the reputation of Erdős and the respect paid to discrete mathematics have increased exponentially. (…) Fields Medals are awarded to mathematicians who solve Erdős-type problems. Science has changed.
Time has proved the fertility and richness of Erdős's work.

Are they good for starting a career? Probably no.

Terence Tao, 2007: ***Don't prematurely obsess on a single "big problem" or "big theory"***
(…) I would strongly advocate a more balanced, patient, and flexible approach instead: one can certainly keep the big problems and theories in mind, and tinker with them occasionally, but spend most of your time on more feasible "low-hanging fruit," which will build up your experience, mathematical power, and credibility for when you are ready to tackle the more ambitious projects.

Why are problem databases useful?

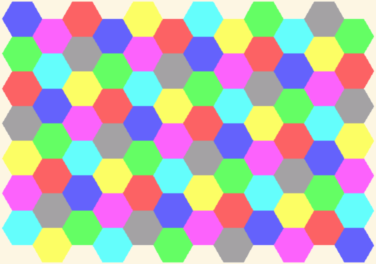
> Terence Tao, 2024: *(...) one of the values added by having databases of problems [is] lowering the "friction" costs of locating interesting questions and finding their most recent status, it facilitates a different way of making progress in a mathematical field, by scanning large numbers of problems for "quick wins," which is a largely orthogonal approach to the more traditional method of thinking long and hard on one (or a very few) difficult problems at a time. In the near and medium term, I would expect the traditional approach to still be dominant, but I also expect it to be increasingly complemented by these other approaches.*

Erdős problems list 58 preprints (partially) addressing the problems since January 1, 2024.

Case study of Erdős problems

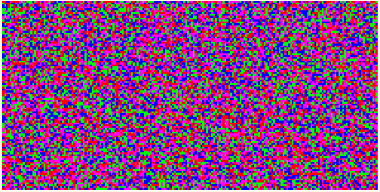
1. Euclidean Ramsey theory
2. Euclidean measure theory
3. Egyptian fractions
4. Irrationality problems
5. Arithmetic progressions

Identifies monochromatic configurations present in every finite coloring of $\mathbb{R}^d$.



Systematic study initiated by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus (1970s).

A *finite coloring* of $S \subseteq \mathbb{R}^n$
= any partition of S into finitely many *color-classes* $\mathscr{C}_1, \ldots, \mathscr{C}_r$.



A coloring is *measurable* if $\mathscr{C}_j$ are Lebesgue-measurable.

A coloring is *Jordan-measurable* if $\mathscr{C}_j$ have boundaries of measure 0.

Ronald L. Graham (1935–2020)

Graham, 1979

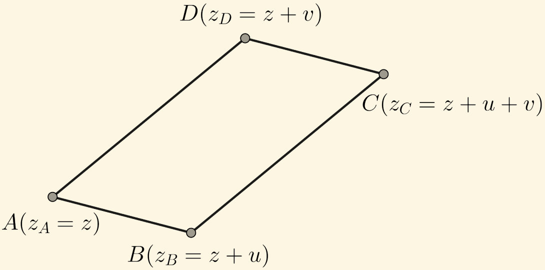
For all finite colorings of the plane some color-class contains, for every $\alpha > 0$, vertices of a right-angled triangle of area α.

Erdős and Graham, 1979: *The question is: Is this also true for rectangles?* (EP #189)

Theorem (K., 2023)

There exists a Jordan-measurable coloring of the plane in 25 colors such that no color-class contains the vertices of a rectangle of area 1.

Proof. Relax a rectangle of area 1 to a parallelogram $\mathcal{P}$ with $|AB| \cdot |AD| = 1$.



Define a complex "invariant" quantity:

$$\mathscr{I}(\mathcal{P}) := z_A^2 - z_B^2 + z_C^2 - z_D^2 = 2uv.$$

On the one hand, for a parallelogram with $|AB| \cdot |AD| = 1$,

$$|\mathscr{I}(\mathcal{P})| = 2|u||v| = 2.$$

For each pair $(j, k) \in \{0, 1, 2, 3, 4\}^2$ define a color-class $\mathscr{C}_{j,k}$ as

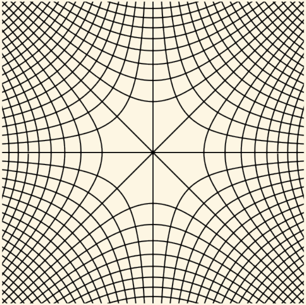
$$\mathscr{C}_{j,k} := \Big\{ z \in \mathbb{C} \,:\, z^2 \in \frac{10}{3}\Big(\mathbb{Z} + \mathbb{i}\mathbb{Z} + \frac{j + \mathbb{i}k}{5} + \Big[0, \frac{1}{5}\Big) + \mathbb{i}\Big[0, \frac{1}{5}\Big)\Big)\Big\}.$$

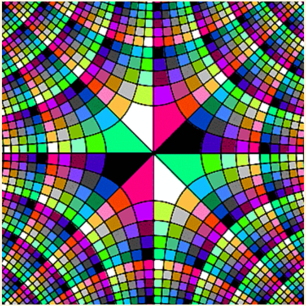
One the other hand, for a monochromatic parallelogram,

$$\mathscr{I}(\mathcal{P}) \in \frac{10}{3}\Big(\mathbb{Z} + \mathbb{i}\mathbb{Z} + \Big(-\frac{2}{5}, \frac{2}{5}\Big) + \mathbb{i}\Big(-\frac{2}{5}, \frac{2}{5}\Big)\Big),$$

which is never $= 2$ in the absolute value. $\square$

The coloring $(\mathscr{C}_{j,k})$ of $\mathbb{R}^2$:





Aftermath:

- A general negative result for m-dimensional rectangular boxes in $\mathbb{R}^d$, $d \geqslant m$ (K., 2025).
- A positive result for measurable colorings and m-dimensional rectangular boxes with sufficiently large volumes α (depending on the coloring) in $\mathbb{R}^d$, $d \geqslant m + 1$ (K., 2025).
- Certain positive/negative results for simplices and parallelotopes (K., 2025).
- Positive results use the tools from multilinear harmonic analysis, which I was developing over the previous several years.

- Relates to: the hyperbolic Hadwiger–Nelson problem (Bardestani and Mallahi-Karai, 2017; Davies, 2024) and isometric embeddings of hypercube graphs (K. and Predojević, 2023).
- Relates to a lot of previous work on configurations in sets of positive upper density (Bourgain; Lyall and Magyar; K. and Durcik; Falconer, K., and Yavicoli, etc.).

Tip:

- Try to relate a problem that you solve to your previuos research (your “comfort zone”).

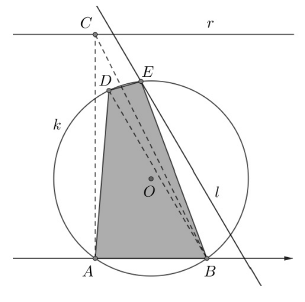
In the proceedings of the conference *Measure Theory*, held at Oberwolfach in 1983 Erdős asked:

> *Let S have infinite planar measure, consider all sets of 4 points, x_1, x_2, x_3, x_4, so that the area of the convex hull is 1. Can one find 4 such points in S if we insist that they have some regularity conditions?*
> *(...) can we assume that (x_1, x_2, x_3, x_4) is inscribed in a circle?*
> *Is it true that, for n large enough, we have a convex polygon $(x_1, x_2, \ldots, x_n)$ of area 1, [such that] $x_i \in S$ and all sides (x_i, x_{i+1}) are equal?* (parts of EP #353)

Theorem (K. and Predojević, 2024)

Every measurable planar set of infinite Lebesgue measure contains the four vertices of a cyclic quadrilateral of area 1.

Idea of proof.



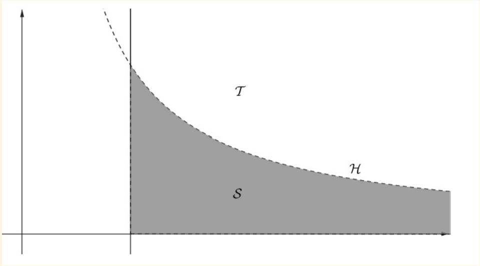
Consider the map $D \mapsto E$, $\Phi(D, E) = 0$, locally near C.

$$\Phi(x_1, y_1, x_2, y_2) = \left(y_1 x_2 + (c - x_1) y_2 - 2,\ x_2^2 + y_2^2 - c x_2 + \frac{c x_1 - x_1^2 - y_1^2}{y_1} y_2\right)$$

Theorem (K. and Predojević, 2024)

There exists a planar set of infinite Lebesgue measure such that every convex polygon with congruent sides and all vertices in that set has area strictly less than 1.

Idea of proof.



$$\mathcal{S} := \{(x, y) \in \mathbb{R}^2 \, : \, x > 1, \, y > 0, \, 4xy < 1\}$$

Aftermath:

- Junnosuke Koizumi solved all 3 remaining subproblems from EP #353 building on our ideas.
- Probably not a part of a bigger research program.

Ancient Egyptians preferred to write positive rational numbers as

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n}$$

with positive integers $m_1 < m_2 < \cdots < m_n$.

Rhind Mathematical Papyrus (around 1550 B.C.), British Museum

Ronald Graham, 2013: *There are various explanations as to why the Egyptians chose to use such representations, but perhaps the most compelling is that given to the author some years ago by the legendary mathematician André Weil. When I asked him why he thought the Egyptians used this method for representing fractions, he thought for a moment and then said, "It is easy to explain. They took a wrong turn!".*

A rational number $q \in [0, \infty)$ is an *n-term Egyptian underapproximation* of a real number $x \in (0, \infty)$ if

$$q = \sum_{k=1}^{n} \frac{1}{m_k} < x, \quad m_1 < m_2 < \cdots < m_n.$$

Consider the number $x = 11/24 = 0.45833\ldots$.
Its greedy two-term Egyptian underapp. is $1/3 + 1/9 = 0.44\ldots$.
Its best two-term Egyptian underapp. is $1/4 + 1/5 = 0.45$.

For some numbers greedy and best underapp. coincide:
$x = 1/b$ (Erdős, 1950);
$x = a/b,\ a \mid b+1$ (Nathanson, 2023);
$\ldots$ (Chu, 2024).

Erdős and Graham, 1980, claimed that every positive rational has eventually greedy best Egyptian underapproximations, but gave no proof or a reference.

(Graham retracted this 33 years later and this is still open!)

Then they commented:

> *It is not difficult to construct irrationals for which the result fails. Conceivably, however, it holds for almost all reals.*
>
> (EP #206)

Theorem (K., 2024)

The set of positive real numbers with eventually greedy best Egyptian underapproximations has Lebesgue measure zero.

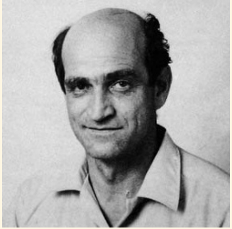
Idea of proof. Greedy underapproximations have denominators from $(1/i_j, 1/(i_j - 1)]$, $j = 1, 2, 3, \ldots$, where $i_{j+1} \geqslant i_j^2 - i_j + 1$.

For every integer $i \geqslant 1000$, at least 1‰ of the numbers in the interval $(1/i, 1/(i-1)]$ have non-greedy best two-term Egyptian underapproximations.
(Quantifies the results of Nathanson and Chu.)

Then iterate to squeeze the numbers with eventually greedy underapp. inside a Cantor-type set of measure 0.

Aftermath:

- People continued more intensive / quantitative study of greedy vs. best two-term Egyptian underapproximations (Shiliaev, 2025).
- Egyptian fractions have been studied a lot recently (Liu and Sawhney, 2024; Conlon, Fox, He, Mubayi, Pham, Suk, and Verstraëte, 2024), but …
  this is probably not a part of a bigger research program.
  (It is about real, not rational numbers.)
- Richard Green discussed the paper and its background in the popular Substack newsletter *A Piece of the Pi: mathematics explained*.



Ernst G. Straus (1922–1983)

Photo: AMS Mathematical Reviews

The Seated Scribe (2500 B.C.), Louvre

Photo by Rama, CC BY-SA 3.0 fr

In the 1960s Erdős and Straus used the term *Ahmes series* for

$$\sum_{k=1}^{\infty} \frac{1}{a_k}, \qquad a_1 < a_2 < a_3 < \cdots \text{ positive integers.}$$

The main question: Is the sum $\in \mathbb{Q}$ or $\notin \mathbb{Q}$?

Ahmes was an Egyptian scribe who (re)wrote the Rhind Mathematical Papyrus.

Heuristic

It is much more likely for $\sum_{k=1}^{\infty} \frac{1}{a_k}$ to be $\in \mathbb{R} \setminus \mathbb{Q}$ than $\in \mathbb{Q}$.

Rigorous probabilistic sense

$A, B \subseteq \mathbb{N}$, $\sum_{n \in A \cup B} \frac{1}{n} < \infty$, B infinite, $B' \subseteq B$ random, $A' := A \Delta B'$

$$\mathbb{P}\Big(\sum_{n \in A'} \frac{1}{n} \in \mathbb{Q} \Big) = 0.$$

Rigorous topological sense

$B \in \mathbb{N}$ infinite, $\sum_{n \in B} \frac{1}{n} < \infty$

$$\Big\{ A \subseteq B : \sum_{n \in A} \frac{1}{n} \in \mathbb{Q} \Big\} \text{ is of the first category in } \{0, 1\}^B.$$

Joint work with Terence Tao:

- **One-dimensional results**
  E.g., (ir)rationality of certain "perturbations" of $\sum_k \frac{1}{a_k}$.
- **Higher-dimensional results**
  E.g., simultaneous rationality of
  $\left(\sum_k \frac{1}{a_k}, \sum_k \frac{1}{a_k+1}, \ldots, \sum_k \frac{1}{a_k+d-1} \right) \in \mathbb{Q}^d$.
- **Infinite-dimensional results**
  E.g., simultaneous rationality of
  $\left(\sum_k \frac{1}{a_k+t} : t \in \mathbb{N} \right) \in \mathbb{Q}^{\mathbb{N}}$.

Erdős and Graham gave a possible definition of an *irrationality sequence* $a_1 < a_2 < a_3 < \cdots \in \mathbb{N}$.

(This was the third one appearing in the literature.)

Definition

We require that

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n} \notin \mathbb{Q}$$

for every bounded $(b_n)_{n=1}^{\infty}$ such that $b_n \in \mathbb{Z} \setminus \{0\},\ a_n + b_n \neq 0$.

Erdős and Graham, 1980: 2^{2^n} *is an irrationality sequence although we do not know about* 2^n *or* $n!$. (EP #264)

Erdős, 1986: *Is there an irrationality sequence* a_n *of this type which increases exponentially? It is not hard to show that it cannot increase slower than exponentially.*

Theorem: a negative result (K. and Tao, 2024)

$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty \implies (a_n)_{n=1}^{\infty}$ is not an irrationality sequence.

In particular,

$$a_n = 2^n$$

and sequences

$$a_n \sim \theta^n, \quad \theta > 1$$

are **not** irrationality sequences.

The condition can be weakened to

$$\liminf_{n\to\infty} \left(a_n^2 \sum_{k=n+1}^{\infty} \frac{1}{a_k^2} \right) > 0.$$

Theorem: a negative result — K. and Tao, 2024

$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty \implies (a_n)_{n=1}^{\infty}$ is not an irrationality sequence.

Proof.
$$\frac{1}{a_n^2} \leqslant C \sum_{k=n+1}^{\infty} \frac{1}{a_k^2}, \qquad a_n \geqslant 4C+1$$
$$J_n := \{a_n+1, a_n+2, a_n+3, \ldots, a_n+4C+1\}$$
$$I_n := \left[\sum_{k=n}^{\infty} \frac{1}{\max J_k}, \sum_{k=n}^{\infty} \frac{1}{\min J_k}\right] \subset (0,\infty)$$

We first claim that $I_n = \left\{\frac{1}{j} : j \in J_n\right\} + I_{n+1}$.

The gaps $\frac{1}{j} - \frac{1}{j+1} < \frac{1}{a_n^2}$ are smaller than the length of I_{n+1},

$$\sum_{k=n+1}^{\infty} \left(\frac{1}{a_n+1} - \frac{1}{a_n+4C+1}\right) \geqslant \sum_{k=n+1}^{\infty} \frac{C}{a_k^2}.$$

Thus, we have $I_n = \left\{ \frac{1}{j} \; : \; j \in J_n \right\} + I_{n+1}$ for all $n \geqslant m$.

Now we claim that the tail sums

$$\left\{ \sum_{k=m}^{\infty} \frac{1}{x_k} \; : \; x_k \in J_k \text{ for every } k \geqslant m \right\}$$

fill in the whole segment I_m.

Fix some $x \in I_m$ and inductively construct $x_k \in J_k$ for $k \geqslant m$ s.t.

$$x \in \sum_{k=m}^{n-1} \frac{1}{x_k} + I_n \quad \Longrightarrow \quad x = \sum_{k=m}^{\infty} \frac{1}{x_k}.$$

Taking

$$x \in I_m \cap \mathbb{Q}$$

and writing

$$x_n = a_n + b_n$$

we conclude:
there exists $(b_n)_{n=1}^{\infty}$ in $[1, 4C + 1]^{\mathbb{N}}$ s.t.

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n} \in \mathbb{Q}.$$

□

Corollary

$a_n = 2^n$ is not an irrationality sequence.

Note that the representations are not explicit, e.g.,

$$\frac{3}{4} = \sum_{n=1}^{\infty} \frac{1}{2^n + b_n}$$

with $1 \leqslant b_n \leqslant 5$ for every n.

Theorem: a positive result — K. and Tao, 2024

For $F\colon \mathbb{N} \to (0,\infty)$, $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \infty$ there exists an irrationality sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \sim F(n)$.

In particular, there exists an irrationality sequence with, say,

$$a_n \sim 2^{n \log_2 \log_2 \log_2 n}.$$

The proof gives more: there exists an irrationality sequence with, say,

$$a_n = n! + O(\log_2 \log_2 n).$$

It is still open if $\sum_{n=2}^{\infty} \frac{1}{n!-1} \notin \mathbb{Q}$. (EP #68).

Theorem: a positive result (K. and Tao, 2024)

For $F\colon \mathbb{N} \to (0,\infty)$, $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \infty$ there exists an irrationality sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \sim F(n)$.

Idea of proof. Choose $(c_n)_{n=1}^{\infty}$ with "much slower" growth than F.

Consider a sequence $(a_n)_{n=1}^{\infty}$ constructed randomly with

$$a_n \in \lfloor F(n) \rfloor + \{1, 2, 3, \ldots, c_n\}$$

uniformly and independently for each $n \geqslant n_0$.

Then $a_n \sim F(n)$ and even $a_n = F(n) + O(c_n)$ and one can prove:

$$\mathbb{P}\Big((a_n)_{n=1}^{\infty} \text{ is an irrationality sequence}\Big) = 1.$$

4. Irrationality problems — $\mathbb{R}^d$ results

Erdős, 1986: *Once I asked: Assume that $\sum \frac{1}{n_k}$ and $\sum \frac{1}{n_k - 1}$ are both rational. How fast can n_k tend to infinity? I was (and am) sure that $n_k^{1/k} \to \infty$ is possible but $n_k^{1/2^k}$ must tend to 1. Unfortunately almost nothing is known. David Cantor observed that*

$$\sum_{k=3}^{\infty} \frac{1}{\binom{k}{2}} \quad \textit{and} \quad \sum_{k=3}^{\infty} \frac{1}{\binom{k}{2} - 1}$$

are both rational and we do not know any sequence with this property which tends to infinity faster than polynomially. (EP #265)

Erdős, 1983: *(…) and we could never decide if n_k can increase exponentially or even faster.*

Theorem (K. and Tao, 2024)

For every $d \in \mathbb{N}$ there exists $\beta > 1$ such that

$$\left\{ \left(\sum_{k=1}^{\infty} \frac{1}{a_k}, \sum_{k=1}^{\infty} \frac{1}{a_k+1}, \dots, \sum_{k=1}^{\infty} \frac{1}{a_k+d-1} \right) \\ : (a_k)_{k=1}^{\infty}, \lim_{k\to\infty} a_k^{1/\beta^k} = \infty \right\}$$

has a non-empty interior in $\mathbb{R}^d$.

In particular, there is a sequence with $\lim_{k\to\infty} a_k^{1/\beta^k} = \infty$ and $\sum_{k=1}^{\infty} \frac{1}{a_k+j} \in \mathbb{Q}$ for $j = 0, \dots, d-1$ (**double exponential growth**).

It is well-known that $\lim_{k\to\infty} a_k^{1/2^k} = \infty$ is impossible.

Already the non-empty interior with no growth requirement was posed as an open problem by Erdős, Graham, and Straus. (EP #268)

Corollary

For every $d \in \mathbb{N}$

$$\left\{ \left(\sum_{n\in A} \frac{1}{n}, \sum_{n\in A} \frac{1}{n+1}, \ldots, \sum_{n\in A} \frac{1}{n+d-1} \right) \,:\, A \subset \mathbb{N} \text{ infinite}, \sum_{n\in A} \frac{1}{n} < \infty \right\}$$

has a non-empty interior in $\mathbb{R}^d$.

Special cases:

- $d = 2$ claimed by Erdős and Straus;
- $d = 3$ posed by Erdős and Graham in 1980, proved by K. in 2024 (producing a concrete ball of radius 10^{-24} inside the set).

Kenneth B. Stolarsky (1942)

Photo: University of Illinois Urbana-Champaign

Erdős and Graham, 1980: *The following pretty conjecture is due to Stolarsky:*

$$\sum_{n=1}^{\infty} \frac{1}{a_n + t}$$

cannot be rational for every positive integer t. (EP #266)

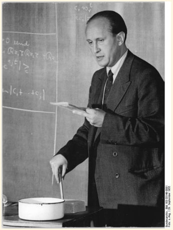
Theorem (K. and Tao, 2024)

There exists $(a_n)_{n=1}^{\infty}$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n + t} \in \mathbb{Q}$ for every $t \in \mathbb{Q} \setminus \{-a_n : n \in \mathbb{N}\}$.

The proof is constructive but far from “explicit.”

Aftermath:

- There is a lot of recent interest on irrationality sequences.
- Similar tricks work for at least one other irrationality problem (Crmarić and K., 2025).



Pál Turán (1910–1976)

Endre Szemerédi (1940)

Erdős and Turan, 1936, $ 1000:
Let $r_k(N)$ be the size of the largest subset of $\{1, \ldots, N\}$ which does not contain a non-trivial k-term arithmetic progression. Prove that $r_k(N) = o(N)$. (EP #139)

Szemerédi, 1975

$r_k(N) = o(N)$.

There is still a lot of work on bounds for $r_k(N)$:

- $k = 3$ — Kelley and Meka, 2023;
- $k = 4$ — Green and Tao, 2017;
- $k \geqslant 5$ — Leng, Sah, and Sawhney, 2024.

Szemerédi’s paper has 509 citations on *Mathematical Reviews*.

The closest I ever got to the problem was the following.

We say that y is a *gap* of a progression

$$x,\ x+y,\ x+2y,\ \dots, x+(k-1)y\ \in \mathbb{R}^d$$

of length k.

> ***(Durcik and K., 2020)***
>
> Take $k \geqslant 3,\ p \neq 1, 2, \dots, k-1, \infty,\ d \geqslant D(k,p),\ A \subseteq [0,1]^d$ measurable, $|A| \geqslant \delta > 0$. Then the set of ℓ^p-norms of the gaps of k-term arithmetic progressions in the set A contains an interval of length at least $c(k,p,d,\delta) > 0$.

We were using Szemerédi's theorem + techniques from multilinear harmonic analysis developed in our dissertations and in several later papers.

Thank you for your attention!