TALES ON TWO COMMUTING TRANSFORMATIONS OR FLOWS

Vjekoslav Kovač (University of Zagreb)

Supported by HRZZ IP-2018-01-7491 (DEPOMOS)

Based on joint work with
M. Christ, *P. Durcik*, *J. Roos*, *M. Stipčić*, *K. A. Škreb*, and *C. Thiele*

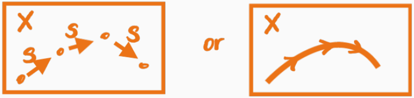
June 23, 2021

8th European Congress of Mathematics, Portorož, Slovenia

§1. Single linear ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x)\,\mathrm{d}t$$

- $(X, \mathcal{F}, \mu)$ a probability space or a σ-finite measure space
- $S\colon X \to X$ a measure-preserving transf. (i.e., $\mu(S^{-1}E) = \mu(E)$), or $S^t\colon X \to X$ is a measure-preserving flow (i.e., an $\mathbb{R}$-action)
- $f \in \mathrm{L}^2(X)$ or $f \in \mathrm{L}^p(X)$, $1 \leq p < \infty$
- Motivated by statistical mechanics (Boltzmann) about 100 yrs ago
- Pioneering work by von Neumann, Birkhoff, and Koopman in the 1930s



$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x)\,\mathrm{d}t$$

L^2 convergence as $N \to \infty$:

- Proved by von Neumann (1932)
- Can we quantify the L^2 convergence by controlling the number of jumps in the norm?

Norm-variation estimate [Jones, Ostrovskii, and Rosenblatt (1996)]

$$\sup_{N_0<N_1<\cdots<N_m} \Big(\sum_{j=1}^{m} \big\|A_{N_j}f - A_{N_{j-1}}f\big\|_{\mathrm{L}^2}^2\Big)^{1/2} \le C\,\|f\|_{\mathrm{L}^2}$$

- They work on $\mathbb{T} \equiv \mathbb{S}^1$ and then use the spectral theorem for the unitary operator $f \mapsto f \circ S$ (the Koopman operator)

Consequence

$A_N f$ make $O(\varepsilon^{-2}\|f\|_{\mathrm{L}^2}^2)$ jumps of size $\ge \varepsilon$ in the L^2 norm

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x)\,\mathrm{d}t$$

Pointwise a.e. convergence as $N \to \infty$:

- Proved by Birkhoff (1931)
- Can we quantify the a.e. convergence by controlling the number of jumps along almost all trajectories?

Pointwise variational estimate [Bourgain (1988), Jones, Kaufman, Rosenblatt, and Wierdl (1998)]

$$\Big\| \sup_{N_0<N_1<\cdots<N_m} \Big(\sum_{j=1}^{m} |A_{N_j} f - A_{N_{j-1}} f|^{\varrho} \Big)^{1/\varrho} \Big\|_{\mathrm{L}^p} \le C_{p,\varrho} \, \|f\|_{\mathrm{L}^p}$$

for $2 < \varrho < \infty$, $1 < p < \infty$

- Bourgain actually studied more general, discrete single polynomial averages

§1. Single linear ergodic averages — Calderón's transference principle

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x)\,\mathrm{d}t$$

- These correspond to the following averages on $\mathbb{R}$:

$$\frac{1}{N}\int_0^N F(x+t)\,\mathrm{d}t$$

  or, in turn, to

$$\int_{\mathbb{R}} F(x+t)\varphi_N(t)\,\mathrm{d}t$$

  for a fixed Schwartz function φ, where $\varphi_N(t) := N^{-1}\varphi(N^{-1}t)$
- The latter averages (for $N = 2^k$) can be further compared to a martingale (e.g., dyadic) model

§2. Multiple ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(S^n x) f_2(S^{2n} x) \cdots f_d(S^{dn} x)$$

- $f_1, f_2, \ldots, f_d \in \mathrm{L}^\infty(X)$
- Introduced by Furstenberg (1977) to reprove Szemerédi's theorem
- Initially interested only in $\limsup_{n\to\infty} > 0$ for $f_j = \mathbb{1}_A$

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(S_1^n x) f_2(S_2^n x) \cdots f_d(S_d^n x)$$

- $S_i S_j = S_j S_i$ for $1 \leq i < j \leq d$
- Introduced by Furstenberg and Katznelson (1977) to prove higher-dimensional Szemerédi's theorem

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(S_1^{P_1(n)} x) f_2(S_2^{P_2(n)} x) \cdots f_d(S_d^{P_d(n)} x)$$

- Introduced by Bergelson and Leibman (1996) to prove polynomial Szemerédi's theorem

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) g(T^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x) g(T^t x)\, \mathrm{d}t$$

The setting:

- $ST = TS$ or $S^s T^t = T^t S^s$
- $f, g \in \mathrm{L}^\infty(X)$

Convergence in $\mathrm{L}^2(X)$ (and in $\mathrm{L}^p(X)$, $p < \infty$) as $N \to \infty$:

- Proved by Conze and Lesigne (1984)

Convergence a.e. as $N \to \infty$:

- It is an old open problem! (Calderón? Furstenberg?)
- Particular case $T = S^{-1}$ shown by Bourgain (1990), with a pointwise variational estimate by Do, Oberlin, and Palsson (2015)
- “Additionally averaged” aver. handled by Donoso and Sun (2016)

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) g(T^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x) g(T^t x)\, \mathrm{d}t$$

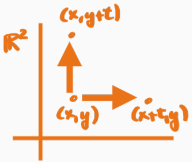
- Can we quantify L^2 convergence in the sense of controlling the number of jumps?
- This can also be considered as partial progress towards a.e. convergence (Bourgain's metric entropy bounds)
- A question by Avigad and Rute (2012); also by Bourgain?

Norm-variation estimate [Durcik, K., Škreb, and Thiele (2016)]

$$\sup_{N_0<N_1<\cdots<N_m} \Big(\sum_{j=1}^{m} \big\| A_{N_j}(f,g) - A_{N_{j-1}}(f,g) \big\|_{\mathrm{L}^2}^2 \Big)^{1/2} \leq C \, \|f\|_{\mathrm{L}^4} \|g\|_{\mathrm{L}^4}$$

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x) g(T^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x) g(T^t x)\,\mathrm{d}t$$

correspond to the following averages on $\mathbb{R}^2$:



$$\frac{1}{N}\int_0^N F(x+t,y)G(x,y+t)\,\mathrm{d}t$$

or, in turn, to

$$\int_{\mathbb{R}} F(x+t,y)G(x,y+t)\varphi_N(t)\,\mathrm{d}t$$

for a fixed Schwartz function φ

$$A_N(F,G)(x,y) := \int_{\mathbb{R}} F(x+t,y)G(x,y+t)\varphi_N(t)\,\mathrm{d}t$$

Beginning of the proof of a special case (a bilinear square function)

$$\sum_{k\in\mathbb{Z}} \big\| A_{2^{k+1}}(f,g) - A_{2^k}(F,G) \big\|_{\mathrm{L}^2}^2 \;\leq\; C \, \|F\|_{\mathrm{L}^4}^2 \|G\|_{\mathrm{L}^4}^2$$

- Expand the LHS as

$$\int_{\mathbb{R}^4} F(x+s,y)F(x+t,y)G(x,y+s)G(x,y+t)K(s,t)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$

- Substitute $u = x+y+s$, $v = x+y+t$ to obtain

$$\int_{\mathbb{R}^4} F_1(u,y)F_2(v,y)F_3(x,u)F_4(x,v)K(u-x-y, v-x-y)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}y$$

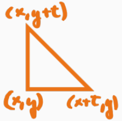
- This quadrilinear singular integral form has a certain “entangled” structure and relates to the previous work by K. (2010, 2011), Durcik (2014, 2015), etc.

$$A_N(F,G)(x,y) := \int_{\mathbb{R}} F(x+t,y)G(x,y+t)\varphi_N(t)\,\mathrm{d}t$$

$$\sum_{j=1}^{m} \left\| A_{2^{k_j}}(f,g) - A_{2^{k_{j-1}}}(F,G) \right\|_{\mathrm{L}^2}^2 \leq C \, \|F\|_{\mathrm{L}^4}^2 \|G\|_{\mathrm{L}^4}^2$$

- Do not dualize the L^2-variation expression as it would lead to the "triangular Hilbert transform":

$$T(F,G)(x,y) := \mathrm{p.v.} \int_{\mathbb{R}} F(x+t,y)\,G(x,y+t)\,\frac{\mathrm{d}t}{t},$$



  boundedness of which is still an open problem
- Partial results by K., Thiele, and Zorin-Kranich (2015); Zorin-Kranich (2016); Durcik, K., and Thiele (2016)

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x)g(T^{n^2} x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x)g(T^{t^2} x)\,dt$$

The setting:

- $ST = TS$ or $S^s T^t = T^t S^s$
- $f, g \in L^\infty(X)$

Convergence in $L^2(X)$ (and in $L^p(X)$, $p < \infty$) as $N \to \infty$:

- Proved by Bergelson and Leibman (1996), in much higher generality
- Continuous-time result is strictly speaking a consequence of discrete-to-continuous transference by Bergelson, Leibman, and Moreira (2011)

$$\frac{1}{N}\sum_{n=0}^{N-1} f(S^n x)g(T^{n^2} x) \quad \text{or} \quad \frac{1}{N}\int_0^N f(S^t x)g(T^{t^2} x)\,\mathrm{d}t$$

Convergence a.e. as $N \to \infty$:

Continuous-time averages

- Proved by Christ, Durcik, K., and Roos (2020)
- Hypotheses can be relaxed to $f \in \mathrm{L}^p(X)$, $g \in \mathrm{L}^q(X)$, $1 < p, q \leq \infty$, $1/p + 1/q \leq 1$

Discrete-time averages

- Still an open problem ??
- Particular case $T = S$ (which was already a big open problem) shown by Krause, Mirek, and Tao (2020); they also establish pointwise variational estimates

Pointwise convergence result [Christ, Durcik, K., and Roos (2020)]

$$\lim_{N\to\infty} \frac{1}{N}\int_0^N f(S^t x) g(T^{t^2} x)\, \mathrm{d}t \quad \text{exists a.e.}$$

A few ingredients of the proof:

- We transfer the estimate

$$\Big\| \frac{1}{N}\int_0^N \big(F(x+t+\delta, y) - F(x+t, y)\big) G(x, y+t^2)\, \mathrm{d}t \Big\|_{\mathrm{L}^1_{(x,y)}} \leq C_{\gamma,\delta} N^{-\gamma} \|F\|_{\mathrm{L}^2} \|G\|_{\mathrm{L}^2}$$

  for $\delta > 0$ and $\gamma = \gamma(\delta) > 0$ from $\mathbb{R}^2$ to the meas.-preserving system
- This estimate is, in turn, shown as a consequence of a powerful "trilinear smoothing estimate" (a "local estimate") by Christ, Durcik, and Roos (2020)
- We are partly quantitative, partly relying on density arguments

$$\sum_{k=0}^{N-1}\Big(\frac{1}{\lfloor a^k\rfloor}\sum_{n=0}^{\lfloor a^k\rfloor-1} f(S^n x)\Big)\big(\mathbb{E}(g|\mathcal{F}_{k+1})-\mathbb{E}(g|\mathcal{F}_k)\big)(x)$$

- A hybrid object that combines ergodic averages and martingales
- Motivated by an open-ended question of Kakutani (1950)
- $S\colon X\to X$ is measure-preserving
- $\mathcal{F}_0\supseteq\mathcal{F}_1\supseteq\mathcal{F}_2\supseteq\cdots$ so that $\mathbb{E}(g|\mathcal{F}_n)$ is a backward (i.e., reversed) martingale
- Commutativity assumption: $\mathbb{E}(h\circ S|\mathcal{F}_n)=\mathbb{E}(h|\mathcal{F}_n)\circ S$ ←
- $1<a<\infty$, $f\in \mathrm{L}^p(X)$, $g\in \mathrm{L}^q(X)$

$$\sum_{k=0}^{N-1} \Big(\frac{1}{\lfloor a^k \rfloor} \sum_{n=0}^{\lfloor a^k \rfloor - 1} f(S^n x) \Big) \big(\mathbb{E}(g|\mathcal{F}_{k+1}) - \mathbb{E}(g|\mathcal{F}_k) \big)(x)$$

Convergence in norm as $N \to \infty$:

- If $1/p + 1/q = 1/r$, $p, q \in [4/3, 4]$, $r \in [1, 4/3]$, then convergence in the L^r norm was shown by K. and Stipčić (2020)
- The main part is mere $\mathrm{L}^p \times \mathrm{L}^q \to \mathrm{L}^r$ boundedness (nontrivial here)

Convergence pointwise a.e. as $N \to \infty$:

- Still an open problem
- Could be a nice toy-problem for the famous open problem on double linear averages w.r.t. two commuting transformations

$$\sum_{k=0}^{N-1}\Big(\frac{1}{\lfloor a^k\rfloor}\sum_{n=0}^{\lfloor a^k\rfloor-1} f(S^n x)\Big)\big(\mathbb{E}(g|\mathcal{F}_{k+1})-\mathbb{E}(g|\mathcal{F}_k)\big)(x)$$

Boundedness of the erg.–mart. paraprod. [K. and Stipčić (2020)]

$$\|\Pi_N(f,g)\|_{\mathrm{L}^r} = \Big\|\sum_{k=0}^{N-1}(A_k f)(E_{k+1}g - E_k g)\Big\|_{\mathrm{L}^r} \leq C_{a,p,q,r}\,\|f\|_{\mathrm{L}^p}\|g\|_{\mathrm{L}^q}$$

for $1/p + 1/q = 1/r,\ \ p,q\in[4/3,4],\ \ r\in[1,4/3]$

$$\Longrightarrow\ \|\Pi_N(f,g)-\Pi_M(f,g)\|_{\mathrm{L}^r} \leq C_{a,p,q,r}\,\|f\|_{\mathrm{L}^p}\|E_N g - E_M g\|_{\mathrm{L}^q}$$

so convergence in $\mathrm{L}^r(X)$ follows

$$\sum_{k=0}^{N-1}\Big(\frac{1}{\lfloor a^k\rfloor}\sum_{n=0}^{\lfloor a^k\rfloor-1} f(S^n x)\Big)\big(\mathbb{E}(g|\mathcal{F}_{k+1})-\mathbb{E}(g|\mathcal{F}_k)\big)(x)$$

- For $a=2$ it corresponds to an object obtained by lifting to $\mathbb{R}\times X$:

$$\sum_{k=0}^{N-1}\Big(\frac{1}{2^k}\int_0^{2^k} F(x+t,y)\,\mathrm{d}t\Big)\big(\mathbb{E}_2(G(x,y)|\mathcal{F}_{k+1})-\mathbb{E}_2(G(x,y)|\mathcal{F}_k)\big)$$

  and, in turn, to

$$\sum_{k=0}^{N-1}\mathbb{E}_1(F|\mathcal{D}_k)\big(\mathbb{E}_2(G|\mathcal{F}_{k+1})-\mathbb{E}_2(G|\mathcal{F}_k)\big),$$

  where $\mathcal{D}_k$ is the dyadic filtration
- Reduces (up to several technical details) to estimates for two "commuting" martingales by K. and Škreb (2013)

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(S_1^n x) f_2(S_2^n x)\cdots f_d(S_d^n x) \quad \text{or} \quad \frac{1}{N}\int_0^N f_1(S_1^t x) f_2(S_2^t x)\cdots f_d(S_d^t x)\,\mathrm{d}t$$

Convergence in $\mathrm{L}^2(X)$ (and in $\mathrm{L}^p(X)$, $p < \infty$) as $N \to \infty$:

- Proved by Tao (2007) and also by Austin (2008), Walsh (2011), Zorin-Kranich (2011)
- No quantitative norm-convergence results known for $d \geq 3$

Convergence a.e. as $N \to \infty$ is open already for $d \geq 2$

$$\frac{1}{N}\int_0^N f_1(S_1^{P_1(t)}x)f_2(S_2^{P_2(t)}x)\cdots f_d(S_d^{P_d(t)}x)\,\mathrm{d}t$$

Convergence in $\mathrm{L}^2(X)$ (and in $\mathrm{L}^p(X)$, $p < \infty$) as $N \to \infty$:

- Proved by Austin (2011)

Additionally assume:

$$\deg P_1 < \deg P_2 < \cdots < \deg P_d$$

Convergence a.e. as $N \to \infty$:

- Proved by Frantzikinakis (2021)
- He uses beautifully elegant density argument
- Emphasizes simplifications coming from continuous time and polynomials (or general Hardy-field functions) with different orders of growth

Thank you for your attention!