Some Examples of Scaling Sets

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THE BASIC DEFINITIONS

An *orthonormal wavelet* is a function $\psi \in \mathrm{L}^2(\mathbb{R})$ such that $\big(2^{j/2}\psi(2^j \cdot - k)\big)_{j,k\in\mathbb{Z}}$ forms an orthonormal basis for $\mathrm{L}^2(\mathbb{R})$.

ψ is an *MSF wavelet* if ψ is an orthonormal wavelet and $|\hat{\psi}| = \chi_K$ a.e. for some $K \in \mathcal{B}(\mathbb{R})$.

ψ is an *MRA wavelet* associated with an MRA $(V_j)_{j\in\mathbb{Z}}$ if $(\psi(\cdot - k))_{k\in\mathbb{Z}}$ forms an orthonormal basis for W_0, where $V_1 = V_0 \oplus W_0$.

A function $\varphi \in \mathrm{L}^2(\mathbb{R})$ is a *scaling function* associated with an MRA $(V_j)_{j\in\mathbb{Z}}$ if $(\varphi(\cdot - k))_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

Theorem. Suppose that $(V_j)_{j\in\mathbb{Z}}$ is an MRA with some MSF wavelet associated with it. There exist $K, S \in \mathcal{B}(\mathbb{R})$ such that:
for each associated wavelet ψ we have $|\hat{\psi}| = \chi_K$ a.e.,
for each associated scaling function φ we have $|\hat{\varphi}| = \chi_S$ a.e.
Moreover, $S = \bigcup_{j=1}^{\infty} 2^{-j}K$ a.e. and $K = 2S \setminus S$ a.e.

`[The WUTAM Consortium (1998)]`
`[M.Papadakis, H.Šikić, G.Weiss (1999)]`

Measurable sets S and K obtained in this way are called *scaling sets* and *MRA wavelet sets*, respectively.

Theorem. If K is an MRA wavelet set then **any** $\psi \in \mathrm{L}^2(\mathbb{R})$ such that $|\hat{\psi}| = \chi_K$ a.e. is an MRA and MSF wavelet.

`[The WUTAM Consortium (1998)]`

The characterization of scaling sets

$S \in \mathcal{B}(\mathbb{R})$ is a scaling set iff

(1) $\{S + 2k\pi;\ k \in \mathbb{Z}\}$ is an a.e.-partition of $\mathbb{R}$

(2) $S \subseteq 2S$ a.e.

(3) $\bigcup_{j\in\mathbb{N}} 2^j S = \mathbb{R}$ a.e.

[M.Papadakis, H.Šikić, G.Weiss (1999)]

The characterization of MRA wavelet sets

$K \in \mathcal{B}(\mathbb{R})$ is an MRA wavelet set iff

(1) $\{K + 2k\pi;\ k \in \mathbb{Z}\}$ is an a.e.-partition of $\mathbb{R}$

(2) $\left\{2^j K;\ j \in \mathbb{Z}\right\}$ is an a.e.-partition of $\mathbb{R}$

(3) $\left\{2^{-j} K + 2k\pi;\ j \in \mathbb{N}, k \in \mathbb{Z}\right\}$ is an a.e.-partition of $\mathbb{R}$

(The first two conditions characterize general *wavelet sets*.)
[E.Hernández, G.Weiss (1996)]

Note that $|S| = 2\pi$ and $|K| = 2\pi$.

There is a 1–1 correspondence between scaling sets and MRA wavelet sets given by

$$\begin{array}{rcl} \text{Scaling sets} & & \text{MRA wavelet sets} \\ S & \longmapsto & K = 2S \setminus S \\ \bigcup_{j=1}^{\infty} 2^{-j} K = S & \longleftarrow\!\!\dashv & K \end{array}$$

Examples?

It is still not clear how to construct some (interesting) examples of scaling sets / MRA wavelet sets.

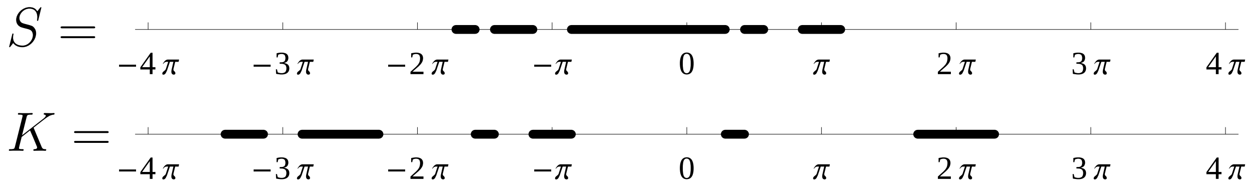
Simple examples (finite unions of intervals)

1. $S = [-\pi, \pi], \quad K = [-2\pi, -\pi] \cup [\pi, 2\pi]$

(the Shannon wavelet)

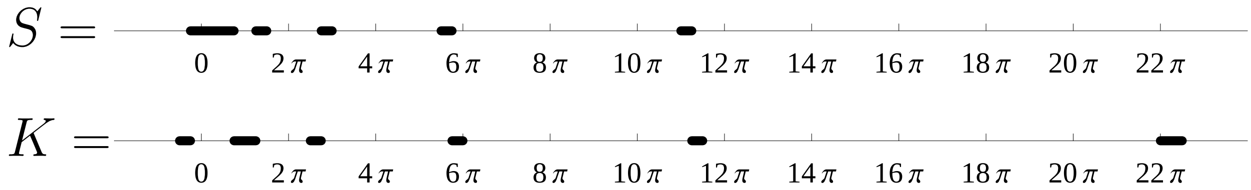
2. $S = [-\frac{12\pi}{7}, -\frac{11\pi}{7}] \cup [-\frac{10\pi}{7}, -\frac{8\pi}{7}] \cup [-\frac{6\pi}{7}, \frac{2\pi}{7}] \cup [\frac{3\pi}{7}, \frac{4\pi}{7}] \cup [\frac{6\pi}{7}, \frac{8\pi}{7}]$

$K = [-\frac{24\pi}{7}, -\frac{22\pi}{7}] \cup [-\frac{20\pi}{7}, -\frac{16\pi}{7}] \cup [-\frac{11\pi}{7}, -\frac{10\pi}{7}] \cup [-\frac{8\pi}{7}, -\frac{6\pi}{7}] \cup [\frac{2\pi}{7}, \frac{3\pi}{7}] \cup [\frac{12\pi}{7}, \frac{16\pi}{7}]$



3. $S = [-\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{3\pi}{2}] \cup [\frac{11\pi}{4}, 3\pi] \cup [\frac{11\pi}{2}, \frac{23\pi}{4}] \cup [11\pi, \frac{45\pi}{4}]$

$K = [-\frac{\pi}{2}, -\frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{5\pi}{2}, \frac{11\pi}{4}] \cup [\frac{23\pi}{4}, 6\pi] \cup [\frac{45\pi}{4}, \frac{23\pi}{2}] \cup [22\pi, \frac{45\pi}{2}]$



Some questions

[1.] Does every wavelet set vanish a.e. outside some bounded interval?

X.Fang and X.Wang (1994) gave a counterexample:

Example. Let

$$l_j = 2^{j+1} - 1, \quad a_j = \frac{2^{-j}\pi}{2^{l_j+1}-1}; \quad j \in \mathbb{N}$$

and

$$\begin{aligned} K_0 &= \left[\tfrac{\pi}{2} + \tfrac{1}{2}\textstyle\sum_{j=1}^{\infty} a_j,\ \pi\right], \\ K_j &= \left[2^{l_j}\pi + 2^{l_j}\textstyle\sum_{i=1}^{j-1} a_i,\ 2^{l_j}\pi + 2^{l_j}\textstyle\sum_{i=1}^{j} a_i\right]; \quad j \in \mathbb{N}. \end{aligned}$$

Then

$$K = -\bigcup_{j=0}^{\infty} K_j \cup \bigcup_{j=0}^{\infty} K_j$$

is a wavelet set.

The above example is a countable union of disjoint intervals with total length 2π. Therefore, we may ask ourselves if there exists a wavelet set which does not vanish a.e. outside a closed set of finite measure.

[1.'] Is there any wavelet set whose topological support has infinite measure?

We shall construct such an example which is also an MRA wavelet set.

2. Does every wavelet set have a "hole" around 0, i.e. does every wavelet set have to be a.e.-disjoint from an interval about 0?

L.Brandolini, G.Garrigós, Z.Rzeszotnik and G.Weiss (1999) gave a counterexample:

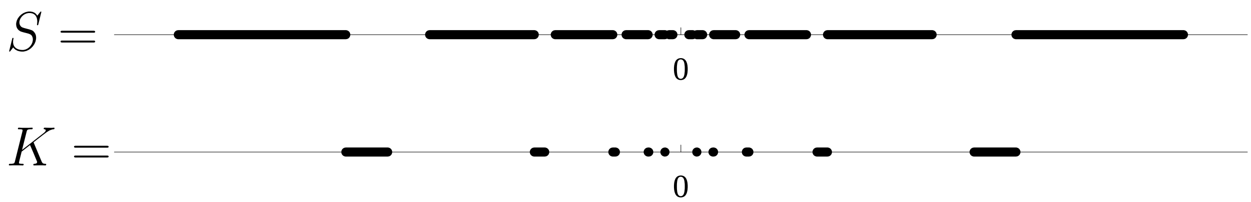
Example. Suppose that $(a_j)_{j\in\mathbb{N}}$ is a sequence of reals such that

$$\tfrac{\pi}{4} < a_1 \le \tfrac{\pi}{3}, \quad \tfrac{\pi}{2^{j+1}} < a_j < \tfrac{1}{2}a_{j-1}, \quad \lim_{j\to\infty} 2^j a_j = \tfrac{\pi}{2}.$$

Then

$$S = \bigcup_{j=1}^{\infty}[-2\pi + \tfrac{\pi}{2^j}, -2\pi + 2a_j] \cup \bigcup_{j=1}^{\infty}[-\tfrac{\pi}{2^{j-1}}, -2a_j] \cup \bigcup_{j=1}^{\infty}[2a_j, \tfrac{\pi}{2^{j-1}}] \cup \bigcup_{j=1}^{\infty}[2\pi - 2a_j, 2\pi - \tfrac{\pi}{2^j}]$$

is a scaling set $S \subseteq [-2\pi, 2\pi]$ and $K = 2S \setminus S \subseteq [-4\pi, 4\pi]$ is the corresponding MRA wavelet set.



This example is again a countable union of disjoint intervals with total length 2π. It has no hole about 0, but it has many "holes" in every neighborhood of 0.

2.' Is there a wavelet set K and an open neighborhood U of 0 such that for every interval $I \subseteq U$ we have $|I \cap K| > 0$.

The answer is positive. Our examples will also have some additional qualities.

THE RADEMACHER FUNCTIONS $(R_n)_{n\in\mathbb{N}}$

$$R_n\colon [0,1] \to \mathbb{R}, \quad R_n(\xi) := (-1)^{\alpha_n},$$

where

$$\xi = 0.\alpha_1\alpha_2\alpha_3\ldots$$

is a binary representation of $\xi \in [0,1]$.

$(R_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ and each of them has a coin-toss distribution, $\sim \begin{pmatrix} -1 & 1 \\ 1/2 & 1/2 \end{pmatrix}$.

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of reals satisfying

$$\textstyle\sum_{n=1}^{\infty} a_n^2 < +\infty, \quad a_1 > a_2 > a_3 > \ldots > 0.$$

Since

$$\textstyle\sum_{n=1}^{\infty} \mathrm{Var}(a_n R_n) = \sum_{n=1}^{\infty} a_n^2 < +\infty,$$

it follows that the series $\sum_{n=1}^{\infty} a_n R_n$ converges a.s., so we define

$$X := \textstyle\sum_{n=1}^{\infty} a_n R_n.$$

Note that

$$\varphi_X(t) = \textstyle\prod_{n=1}^{\infty} \cos(a_n t)$$

(the characteristic function of the distribution).

Theorem. For a.e. $\xi \in [0,1]$

(a) $X(\frac{\xi+1}{2}) \le X(\xi) \le X(\frac{\xi}{2})$,

(b) $\lim_{j\to\infty} X(\frac{\xi}{2^j}) = \sum_{n=1}^{\infty} a_n \quad \in \langle 0, +\infty]$,

(c) $\lim_{j\to\infty} X(\frac{\xi+2^j-1}{2^j}) = -\sum_{n=1}^{\infty} a_n \quad \in [-\infty, 0\rangle$.

Sketch of Proof.

$$X(\tfrac{\xi}{2^j}) \; = \; \sum_{n=1}^{j} a_n + \sum_{n=1}^{\infty} a_{n+j} R_n(\xi)$$
$$X(\tfrac{\xi+2^j-1}{2^j}) \; = \; -\sum_{n=1}^{j} a_n + \sum_{n=1}^{\infty} a_{n+j} R_n(\xi)$$

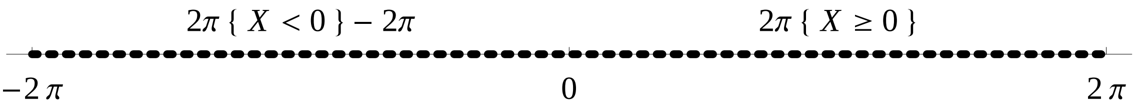
□

Examples of scaling sets $S \subseteq [-2\pi, 2\pi]$

Proposition.

$$S := (2\pi\,\{X < 0\} - 2\pi) \cup 2\pi\,\{X \ge 0\}$$

is a scaling set.

$S =$ 

Sketch of Proof.

For a.e. $\zeta \in S$

1° If $\zeta = 2\pi\xi$, $X(\xi) \ge 0$, then $\frac{\zeta}{2} = 2\pi\cdot\frac{\xi}{2}$, $X(\frac{\xi}{2}) \ge X(\xi) \ge 0$, so $\frac{\zeta}{2} \in S$.

2° If $\zeta = 2\pi\xi - 2\pi$, $X(\xi) < 0$, then $\frac{\zeta}{2} = 2\pi \cdot \frac{\xi+1}{2} - 2\pi$, $X(\frac{\xi+1}{2}) \le X(\xi) < 0$, so $\frac{\zeta}{2} \in S$.

Therefore $S \subseteq 2S$ a.e.

For a.e. $\zeta \in [-2\pi, 2\pi]$

1° If $\zeta = 2\pi\xi$, $\xi \in [0,1]$, then $\frac{\zeta}{2^j} = 2\pi \cdot \frac{\xi}{2^j}$ and (for j large enough) $X(\frac{\xi}{2^j}) \geq 0$, so $\frac{\zeta}{2^j} \in S$.

2° If $\zeta = 2\pi\xi - 2\pi$, $\xi \in [0,1]$, then $\frac{\zeta}{2^j} = 2\pi \cdot \frac{\xi+2^j-1}{2^j} - 2\pi$, and (for j large enough) $X(\frac{\xi+2^j-1}{2^j}) < 0$, so $\frac{\zeta}{2^j} \in S$.

Thus, $\bigcup_{j\in\mathbb{N}} 2^j S = \mathbb{R}$ a.e. □

In the following we additionally suppose that $\sum_{n=1}^{\infty} a_n = \infty$.

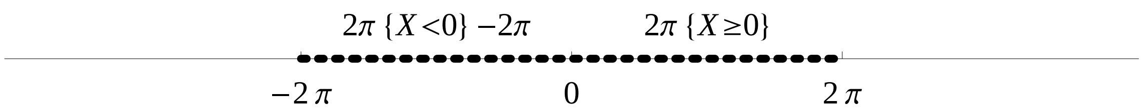
Lemma. Suppose that a positive sequence $(b_n)_{n\in\mathbb{N}}$ satisfies

$$\sum_{n=1}^{\infty} b_n^2 < +\infty, \quad \sum_{n=1}^{\infty} b_n = +\infty$$

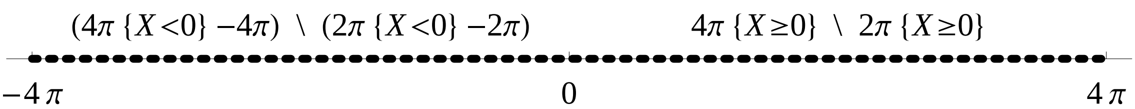
and define $Y := \sum_{n=1}^{\infty} b_n R_n$. Then for any $c, d \in \mathbb{R}$, $c < d$ we have

$$\mathbb{P}\left(Y \in \langle c, d \rangle\right) > 0.$$

Theorem.

$S =$ 

For every interval $I \subseteq [-2\pi, 2\pi]$ we have $0 < |I \cap S| < |I|$. (Neither S nor $[-2\pi, 2\pi] \setminus S$ contains an interval a.e.)

$K =$ 

For every interval $I \subseteq [-4\pi, 4\pi]$ we have $0 < |I \cap K| < |I|$. (Neither K nor $[-4\pi, 4\pi] \setminus K$ contains an interval a.e.)

Sketch of Proof.

Consider a dyadic interval $J \subseteq [0,1]$, i.e.

$$J = \{R_1 = \varepsilon_1, \ldots, R_m = \varepsilon_m\}$$

for some $\varepsilon_1, \ldots, \varepsilon_m \in \{-1, 1\}$.

$$\begin{aligned} |2\pi J \cap S| &= 2\pi \cdot \mathbb{P}\left(R_1 = \varepsilon_1, \ldots, R_m = \varepsilon_m, X \geq 0\right) = \\ &= [\text{independence of } (R_n)_{n \in \mathbb{N}}] = \\ &= 2\pi \cdot \left(\tfrac{1}{2}\right)^m \cdot \mathbb{P}\left(Y \geq c\right) > 0, \end{aligned}$$

where $Y = \sum_{n=m+1}^{\infty} a_n R_n$, $c = -\sum_{n=1}^{m} a_n \varepsilon_n$. Similarly,

$$|(2\pi J - 2\pi) \cap S| = 2\pi \cdot \left(\tfrac{1}{2}\right)^m \cdot \mathbb{P}\left(Y < c\right) > 0.$$

Analogously, $|2\pi J \cap S^{\mathrm{c}}| > 0$ and $|(2\pi J - 2\pi) \cap S^{\mathrm{c}}| > 0$.

The calculations for K are similar. For example,

$$\begin{aligned} &|2\pi\, J \cap K| = \\ &= 2\pi \cdot \mathbb{P}\Big(R_1 = \varepsilon_1, \ldots, R_m = \varepsilon_m, X(\tfrac{\cdot}{2}) \geq 0, X < 0\Big) \geq \\ &\geq 2\pi \cdot \left(\tfrac{1}{2}\right)^{m+1} \cdot \mathbb{P}\left(c < Y < d\right), \end{aligned}$$

where $Y = \sum_{n=m+2}^{\infty} a_n R_n$, $d = -\sum_{n=1}^{m} a_n \varepsilon_n + a_{m+1}$,

$c = d - 2(a_{m+1} - a_{m+2})$.

□

Examples of unbounded scaling sets

Theorem.

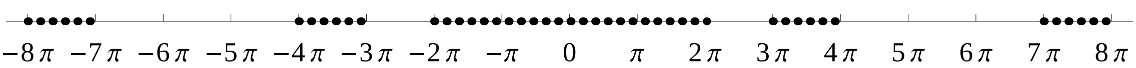
$$S := \bigcup_{k=0}^{\infty}\left(2\pi\left\{R_1 = 1, X(\tfrac{\cdot}{2^k}) < 0, X(\tfrac{\cdot}{2^{k+1}}) \geq 0\right\} - 2^{k+1}\pi\right)$$
$$\cup\ (2\pi\{R_1 = -1, X < 0\} - 2\pi) \cup 2\pi\{R_1 = 1, X \geq 0\}$$
$$\cup \bigcup_{k=0}^{\infty}\left(2\pi\left\{R_1 = -1, X(\tfrac{\cdot + 2^k - 1}{2^k}) \geq 0, X(\tfrac{\cdot + 2^{k+1} - 1}{2^{k+1}}) < 0\right\} + (2^{k+1} - 2)\pi\right)$$

is a scaling set. Let $K = 2S \setminus S$ be the corresponding MRA wavelet set. For every interval I such that

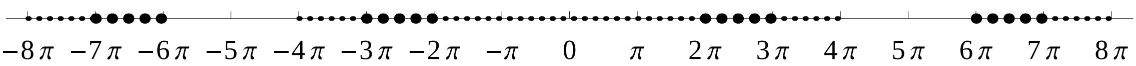
$$I \subseteq \bigcup_{k=2}^{\infty}[-(2^k-1)\pi, -(2^k-2)\pi] \cup \bigcup_{k=2}^{\infty}[(2^k-2)\pi, (2^k-1)\pi]$$

we have $|I \cap K| > 0$.

$S =$



$K =$



Therefore, the topological support of K contains

$$\bigcup_{k=2}^{\infty}[-(2^k-1)\pi, -(2^k-2)\pi] \cup \bigcup_{k=2}^{\infty}[(2^k-2)\pi, (2^k-1)\pi],$$

so it is a set of infinite measure.