A sharp nonlinear Hausdorff-Young inequality for small potentials

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Real Analysis, Harmonic Analysis, and Applications

Oberwolfach, July 28, 2017

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(the SU(1,1)-scattering transform, the Dirac scattering transform)

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$f : \mathbb{R} \to \mathbb{C}$ measurable, bounded, compactly supported, $\quad \xi \in \mathbb{R}$

$$\frac{d}{dx}\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} = \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} \begin{bmatrix} 0 & f(x)e^{-2\pi i x\xi} \\ \overline{f(x)}e^{2\pi i x\xi} & 0 \end{bmatrix}$$

$$\begin{bmatrix} a(-\infty,\xi) & b(-\infty,\xi) \\ \overline{b(-\infty,\xi)} & \overline{a(-\infty,\xi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a(\cdot,\xi)$ and $b(\cdot,\xi)$ exist as absolutely continuous solutions

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We define $a, b \colon \mathbb{R} \to \mathbb{C}$ by

$$a(\xi) := a(+\infty,\xi), \quad b(\xi) := b(+\infty,\xi)$$

and study mapping properties of the "forward transform"

$$f \quad \mapsto \quad a, b \quad \text{or better} \quad \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix}$$

The nonlinear Fourier transform

The matrix group SU(1, 1) and its Lie algebra $\mathfrak{su}(1,1)$:

$$\mathrm{SU}(1,1) := \left\{ \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} : A, B \in \mathbb{C},\ |A|^2 - |B|^2 = 1 \right\}$$

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$$\frac{d}{dx} \underbrace{\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix}}_{\mathrm{SU(1,1)}} = \underbrace{\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix}}_{\mathrm{SU(1,1)}} \underbrace{\begin{bmatrix} 0 & f(x)e^{-2\pi i x\xi} \\ \overline{f(x)}e^{2\pi i x\xi} & 0 \end{bmatrix}}_{\mathfrak{su}(1,1)}$$

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Alternatively:

$$|a(-\infty,\xi)|^2 - |b(-\infty,\xi)|^2 = 1, \quad \frac{d}{dx}\left(|a(x,\xi)|^2 - |b(x,\xi)|^2\right) = 0$$

The nonlinear Fourier transform

In the scalar form:

$$\partial_x a(x,\xi) = \overline{f(x)} e^{2\pi i x\xi} b(x,\xi), \quad \partial_x b(x,\xi) = f(x) e^{-2\pi i x\xi} a(x,\xi)$$
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Integral equations:

$$a(x,\xi) = 1 + \int_{-\infty}^{x} \overline{f(y)} e^{2\pi i y\xi} b(y,\xi) dy$$
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- Picard's iteration gives a multilinear series expansion: $a = 1 + \cap + \cap\cap + \cap\cap\cap \cdots$
- Born's approximation: $b(x,\xi) \approx (f \mathbb{1}_{(-\infty,x]})\widehat{\ }(\xi)$ when $\|f\|_{L^1(\mathbb{R})} \ll 1$

$$|a|^2 - |b|^2 = 1 \iff \left|\frac{1}{a}\right|^2 + \left|\frac{b}{a}\right|^2 = 1$$

$t = 1/a =$ the transmission coefficient
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Riccati's differential equation:

$$\partial_x r(x,\xi) = f(x)e^{-2\pi i x\xi} - \overline{f(x)}e^{2\pi i x\xi} r(x,\xi)^2, \qquad r(-\infty,\xi) = 0$$

and the corresponding integral equation:

$$r(x,\xi) = \int_{-\infty}^{x} f(y)e^{-2\pi i y\xi}dt - \int_{-\infty}^{x} \overline{f(y)}e^{2\pi i y\xi} r(y,\xi)^2 dt$$

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This is not the linear Fourier transform on SU(1,1)!
$f \mapsto a, b, r$ are "very" nonlinear transformations
Still, they share many symmetries with the linear Fourier transform (w.r.t. L^1-dilations, translations, modulations, etc.)

Motivation #1: the Dirac operator

Eigenproblem for the Dirac operator:

$$L := \begin{bmatrix} \frac{d}{dx} & -\overline{f} \\ f & -\frac{d}{dx} \end{bmatrix}, \quad \text{i.e.} \quad L \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \varphi' - \overline{f}\psi \\ f\varphi - \psi' \end{bmatrix}$$

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For $\xi \in \mathbb{R}$ we consider the eigenproblem:

$$L \begin{bmatrix} \varphi(\cdot, \xi) \\ \psi(\cdot, \xi) \end{bmatrix} = -\pi i \xi \begin{bmatrix} \varphi(\cdot, \xi) \\ \psi(\cdot, \xi) \end{bmatrix}$$

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i.e.

$$\partial_x \varphi(x,\xi) + \pi i \xi \varphi(x,\xi) = \overline{f(x)}\psi(x,\xi)$$
$$\partial_x \psi(x,\xi) - \pi i \xi \psi(x,\xi) = f(x)\varphi(x,\xi)$$

i.e.

$$\partial_x \big(\underbrace{\varphi(x,\xi)e^{\pi i x \xi}}_{a(x,\xi)} \big) = \overline{f(x)} e^{2\pi i x \xi} \underbrace{\psi(x,\xi)e^{-\pi i x \xi}}_{b(x,\xi)}$$
$$\partial_x \big(\underbrace{\psi(x,\xi)e^{-\pi i x \xi}}_{b(x,\xi)} \big) = f(x) e^{-2\pi i x \xi} \underbrace{\varphi(x,\xi)e^{\pi i x \xi}}_{a(x,\xi)}$$

Motivation #2: AKNS-ZS systems

Two bodies in the plane with interactions:
$u_1(t), u_2(t) \in \mathbb{C} =$ positions at time t, $\quad \omega_1 \neq \omega_2, \quad \lambda \in \mathbb{R}$

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$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = i\lambda \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \overline{f(t)} \\ f(t) & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Does the system remain bounded for a.e. $\lambda \in \mathbb{R}$?

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$$\frac{d}{dt}\Big(\overbrace{u_1(t)e^{-i\omega_1\lambda t}}^{a(t,\lambda)}\Big) = \overline{f(t)}e^{i(\omega_2-\omega_1)\lambda t}\,\overbrace{u_2(t)e^{-i\omega_2\lambda t}}^{b(t,\lambda)}$$

$$\frac{d}{dt}\Big(\underbrace{u_2(t)e^{-i\omega_2\lambda t}}_{b(t,\lambda)}\Big) = f(t)e^{-i(\omega_2-\omega_1)\lambda t}\,\underbrace{u_1(t)e^{-i\omega_1\lambda t}}_{a(t,\lambda)}$$

Introduce: $\xi = (\omega_2 - \omega_1)\lambda/2\pi$

Open questions — nonlinear analogue of Carleson

Carleson (1966)

$$f \in \mathrm{L}^2(\mathbb{R}) \implies \lim_{x \to +\infty} \int_{-x}^{x} f(y) e^{-2\pi i y \xi} dy \text{ exists for a.e. } \xi \in \mathbb{R}$$

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The nonlinear Carleson problem — open question

$$f \in \mathrm{L}^2(\mathbb{R}), \ \operatorname{supp}(f) \subseteq [0, +\infty)$$

$$\overset{?}{\implies} \lim_{x \to +\infty} \begin{bmatrix} \frac{a(x,\xi)}{b(x,\xi)} & \frac{b(x,\xi)}{a(x,\xi)} \end{bmatrix} \text{ exists for a.e. } \xi \in \mathbb{R}$$

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- Even finiteness of $\sup_x |a(x,\xi)|$ for a.e. $\xi \in \mathbb{R}$ is open
- Christ and Kiselev (2001): for $f \in \mathrm{L}^p(\mathbb{R})$, $p < 2$
- Muscalu, Tao, Thiele (2002): the Cantor group "toy-model", the exponentials replaced with characters of a different group

Open questions — nonlinear analogue of Hausdorff-Young

Young (1913), Hausdorff (1923), Babenko (1961), Beckner (1975)

$$1 \le p \le 2,\ 1/p + 1/p' = 1 \implies \|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R})} \le \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

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Nonlinear H-Y inequalities — Christ and Kiselev (2001)

$$1 \le p \le 2 \implies \|(\log |a|^2)^{1/2}\|_{L^{p'}(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}$$

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- $p = 1$ trivial (with $C_1 = 1$) by Grönwall's lemma
- $p = 2$ an identity (with $C_2 = 1$) by the contour integration
- Open question: Does C_p remain bounded as $p \uparrow 2$?
  K. (2010): confirmed in the Cantor group "toy-model"

The main result

Fix $1 < p < 2$, $H > 0$ (the "height"), and $W > 0$ (the "width")

Recall Babenko-Beckner's constant: $\mathbf{B}_p = p^{1/2p} p'^{-1/2p'}$

We only consider potentials f s.t. $|f| \leq H\mathbb{1}_I$ for intervals $|I| \leq W$

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K., Oliveira e Silva, Rupčić (2017)

There exist $\delta > 0$ and $\varepsilon > 0$ (depending on p, H, W) s.t.:

$$\|f\|_{\mathrm{L}^1(\mathbb{R})} \leq \delta \implies \left\|(\log|a|^2)^{\frac{1}{2}}\right\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \left(\mathbf{B}_p - \varepsilon \|f\|^2_{\mathrm{L}^1(\mathbb{R})}\right) \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

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One gets $C_p \geq \mathbf{B}_p$ by considering

$$f(x) = e^{-(Nx)^2} \mathbb{1}_{[-1,1]}(x), \quad N \to +\infty$$

Thus, it is tempting to conjecture $C_p = \mathbf{B}_p$

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- Nonlinear H-Y ratio beats the linear one as $\|f\|_{\mathrm{L}^1(\mathbb{R})} \to 0$

Proof: the dichotomy

$\mathfrak{G}$ = (modulated) Gaussians $G(x) = Ce^{-Ax^2+Bx}$, $A > 0$, $B, C \in \mathbb{C}$

$\mathrm{dist}_p(f, \mathfrak{G}) := \inf_{G \in \mathfrak{G}} \|f - G\|_{\mathrm{L}^p(\mathbb{R})}$

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Case #1:
far from the Gaussians

$$\frac{\mathrm{dist}_p(f, \mathfrak{G})}{\|f\|_{\mathrm{L}^p(\mathbb{R})}} \gtrsim \|f\|_{\mathrm{L}^1(\mathbb{R})}^{1/2}$$

Use Christ's sharpened linear Hausdorff-Young inequality + Grönwall's lemma

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Case #2:
close to the Gaussians

$$\frac{\mathrm{dist}_p(f, \mathfrak{G})}{\|f\|_{L^p(\mathbb{R})}} \lesssim \|f\|_{L^1(\mathbb{R})}^{1/2}$$

Use a few terms of the multilinear expansion + approximate by a Gaussian

Case #1: functions far from the Gaussians

Christ (2014)

$$\|\widehat{f}\|_{\mathsf{L}^{p'}(\mathbb{R})} \leq \left(\mathbf{B}_p - c_p \frac{\mathrm{dist}_p^2(f,\mathfrak{G})}{\|f\|^2_{\mathsf{L}^p(\mathbb{R})}}\right)\|f\|_{\mathsf{L}^p(\mathbb{R})}$$

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From the integral equations:

$$\Big\| |b(x,\xi)| + |a(x,\xi) - 1| \Big\|_{\mathrm{L}^{p'}_\xi(\mathbb{R})}$$
$$\leq \big\|\widehat{f\mathbb{1}_{(-\infty,x]}}\big\|_{\mathrm{L}^{p'}(\mathbb{R})} + \int_{-\infty}^{x} |f(y)| \Big\| |b(y,\xi)| + |a(y,\xi) - 1| \Big\|_{\mathrm{L}^{p'}_\xi(\mathbb{R})} dy$$

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Christ's inequality & Grönwall's lemma in this case give:

$$\big\| (\log |a|^2)^{\frac{1}{2}} \big\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \mathbf{B}_p \underbrace{\left(1 - 2\|f\|_{\mathrm{L}^1(\mathbb{R})}\right) \exp\left(\|f\|_{\mathrm{L}^1(\mathbb{R})}\right)}_{1 - \|f\|_{\mathrm{L}^1(\mathbb{R})} + O(\|f\|_{\mathrm{L}^1(\mathbb{R})}^2)} \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

Case #2: functions close to the Gaussians

For some $G \in \mathfrak{G}$ we have:

$$\|f - G\|_{\mathsf{L}^p(\mathbb{R})} \ll \|f\|_{\mathsf{L}^p(\mathbb{R})}, \quad \|f\|_{\mathsf{L}^p(\mathbb{R})} \lesssim \|G\|_{\mathsf{L}^p(\mathbb{R})}, \quad \|f\|_{\mathsf{L}^1(\mathbb{R})} \lesssim \|G\|_{\mathsf{L}^1(\mathbb{R})}$$

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$$\log(|a(\xi)|^2) = 2\,\mathrm{Re} \int_{\mathbb{R}} f(x_1) e^{-2\pi i x_1 \xi}\, \overline{r(x_1, \xi)}\, dx_1$$

Case #2: functions close to the Gaussians

For some $G \in \mathfrak{G}$ we have:

$$\|f - G\|_{\mathrm{L}^p(\mathbb{R})} \ll \|f\|_{\mathrm{L}^p(\mathbb{R})}, \quad \|f\|_{\mathrm{L}^p(\mathbb{R})} \lesssim \|G\|_{\mathrm{L}^p(\mathbb{R})}, \quad \|f\|_{\mathrm{L}^1(\mathbb{R})} \lesssim \|G\|_{\mathrm{L}^1(\mathbb{R})}$$

$$\log(|a(\xi)|^2) = 2\,\mathrm{Re} \int_{\mathbb{R}} f(x_1) e^{-2\pi i x_1 \xi}\, \overline{r(x_1, \xi)}\, dx_1$$

$$= \overbrace{2\,\mathrm{Re} \int_{\mathbb{R}} \Big(\int_{-\infty}^{x_1} f(x_1)\overline{f(x_2)} e^{-2\pi i (x_1 - x_2)\xi}\, dx_2 \Big)\, dx_1}^{|\widehat{f}(\xi)|^2}$$
$$- 2\,\mathrm{Re} \int_{\mathbb{R}} \Big(\int_{-\infty}^{x_1} f(x_1) f(x_2) e^{-2\pi i (x_1 + x_2)\xi}\, \overline{r(x_2, \xi)^2}\, dx_2 \Big)\, dx_1$$

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$$(\mathcal{Q}f)(\xi) := \mathrm{Re} \int_{\{(x_1 \wedge x_2) > (x_3 \vee x_4)\}} f(x_1) f(x_2) \overline{f(x_3) f(x_4)} \, e^{2\pi i(-x_1 - x_2 + x_3 + x_4)\xi} \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

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Case #2: functions close to the Gaussians

$$\left\|(\log|a|^2)^{\frac{1}{2}}\right\|^{p'}_{\mathsf{L}^{p'}(\mathbb{R})} \leq \underbrace{\|\widehat{f}\|^{p'}_{\mathsf{L}^{p'}(\mathbb{R})}}_{\leq \mathbf{B}_p^{p'}\|f\|^{p'}_{\mathsf{L}^p}} - \frac{p'}{2}\mathcal{H}(f) + \mathcal{R}(f)$$

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$$\mathcal{H}(f) := \int_{\mathbb{R}} (\mathcal{Q}f)(\xi)|\widehat{f}(\xi)|^{p'-2} d\xi$$

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The quotient is invariant under scalar multiplications, modulations, translations, and L^1-normalized dilations

$\implies$ Essentially enough to take $G(x) = e^{-\pi x^2}$ and verify $\mathcal{H}(G) > 0$

An example

It would be brave to conjecture:

$$\left\|(\log |a|^2)^{\frac{1}{2}}\right\|_{\mathsf{L}^{p'}(\mathbb{R})} \leq \left\|\widehat{f}\right\|_{\mathsf{L}^{p'}(\mathbb{R})}$$

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but this is simply wrong!

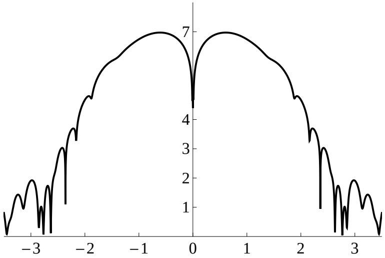
Take $p' = 4$, $\alpha = \frac{1}{100}$, and $f = \alpha F$, where

$$F = -\mathbb{1}_{[0,1)} + 8\mathbb{1}_{[1,2)} + 7\mathbb{1}_{[2,3)} - 6\mathbb{1}_{[3,4)} + 5\mathbb{1}_{[4,5)} - 3\mathbb{1}_{[5,6)}$$

$$\left\|(\log|a|^2)^{\frac{1}{2}}\right\|_{\mathrm{L}^{4}(\mathbb{R})} = 0.12075839\ldots$$

$$\left\|\widehat{f}\right\|_{\mathrm{L}^{4}(\mathbb{R})} = 0.12075670\ldots$$

$(\log|a|^2)^{\frac{1}{2}}$ for $f = \alpha F$, $\alpha = 1, \frac{1}{2}, \frac{1}{10}, \frac{1}{100}$



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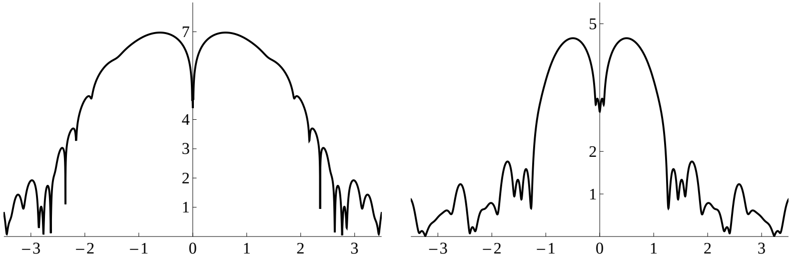

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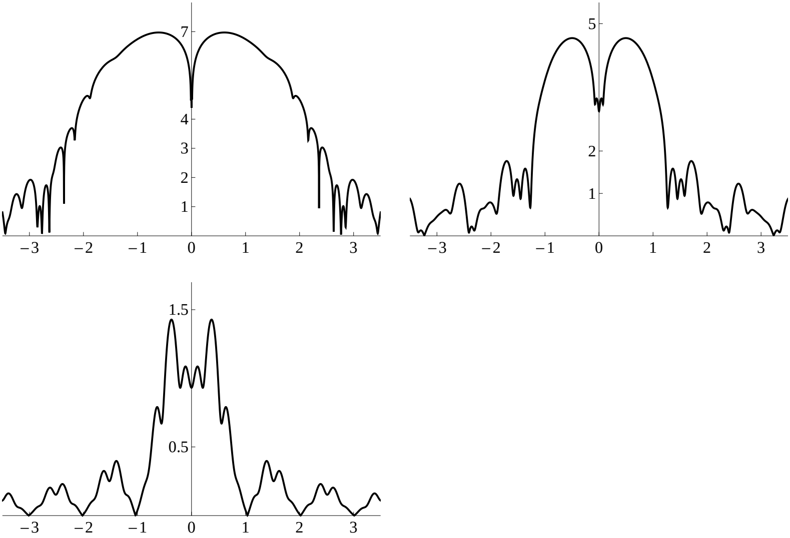

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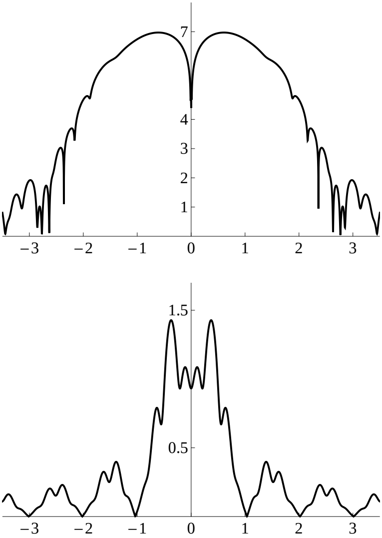

Thank you!

Thank you for your attention!