Several irrationality problems for Ahmes series

May 20, 2025

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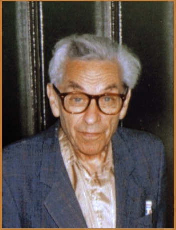
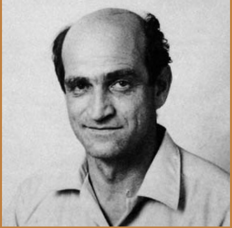
Combinatorial and Additive Number Theory (CANT 2025), CUNY + online

This work was supported by the Croatian Science Foundation under the project number HRZZ-IP-2022-10-5116 (FANAP)

Ahmes series



Paul Erdős (1913–1996)



Ernst G. Straus (1922–1983)

Photo: AMS Mathematical Reviews

In the 1960s Erdős and Straus used the term *Ahmes series* for

$$\sum_{k=1}^{\infty} \frac{1}{a_k}, \qquad a_1 < a_2 < a_3 < \cdots \text{ positive integers.}$$

The main question: Is the sum $\in \mathbb{Q}$?

The Seated Scribe (2613–2494 B.C.), Louvre

Ahmes was an Egyptian scribe who (re)wrote the Rhind Mathematical Papyrus around 1550 B.C.

Alternative terms *sums of unit fractions* and *Egyptian fractions* have finitary connotations.

Which of these series have irrational sums?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \tfrac{\pi^2}{6} \notin \mathbb{Q}$$

Lindemann 1882 (π is transcendental)

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) \notin \mathbb{Q}$$

Apéry 1978 (much more difficult!)

$$\sum_{n=1}^{\infty} \frac{1}{n^5} = \zeta(5)\ \textbf{?}$$

open problem

Which of these series have irrational sums?

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

$= e - 1 \notin \mathbb{Q}$ Euler 1737 (*e* has infinite continued fraction)

$$\sum_{n=2}^{\infty} \frac{1}{n! - 1}$$

? open problem, posed by Erdős in the 1960s, EP #68

$$\sum_{n=1}^{\infty} \frac{1}{n!(n+2)}$$

$= \frac{1}{2} \in \mathbb{Q}$ easy

Irrationality vs. growth — A folklore example

> How fast can $(a_k)_{k=1}^{\infty}$ grow if $\sum_{k=1}^{\infty} \frac{1}{a_k} \in \mathbb{Q}$?

For example, numbers given by a very rapidly convergent series, e.g.,

$$\sum_{k=1}^{\infty} \frac{1}{2^{k!}}$$

are Liouville numbers and thus $\notin \mathbb{Q}$.

How fast can $(a_k)_{k=1}^{\infty}$ grow if $\displaystyle\sum_{k=1}^{\infty} \frac{1}{a_k} \in \mathbb{Q}$?

Consider

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = \frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\cdots = \sum_{k=1}^{\infty} \left(\frac{1}{s_k-1} - \frac{1}{s_{k+1}-1}\right) = 1$$

for *Sylvester's sequence* $s_1 = 2, s_{k+1} = s_k^2 - s_k + 1$, so that $s_k \sim c_0^{2^k}$ for $c_0 = 1.2640847\ldots$.
By shifting, $a_k = s_{k+t}$, we get $a_k \sim C^{2^k}$ for $C = c_0^{2^t}$, so

$$\lim_{k\to\infty} a_k^{1/2^k} \in \mathbb{R}$$

can be as large as we wish.

How fast can $(a_k)_{k=1}^{\infty}$ grow if $\displaystyle\sum_{k=1}^{\infty} \frac{1}{a_k} \in \mathbb{Q}$?

Conversely,

$$\lim_{k\to\infty} a_k^{1/2^k} = \infty$$

is sufficient for $\sum_{k=1}^{\infty} \frac{1}{a_k} \notin \mathbb{Q}$.

Erdős (in 1975) proved that $\limsup_{k\to\infty} a_k^{1/2^k} = \infty$ and $a_k \geqslant k^{1+\varepsilon}$ for large k are already sufficient.

New results

- **One-dimensional results**
  E.g., (ir)rationality of certain "perturbations" of $\sum_k \frac{1}{a_k}$.
- **Higher-dimensional results**
  E.g., simultaneous rationality of
  $$\left(\sum_k \frac{1}{a_k}, \sum_k \frac{1}{a_k + 1}, \ldots, \sum_k \frac{1}{a_k + d - 1} \right) \in \mathbb{Q}^d.$$
- **Infinite-dimensional results**
  E.g., simultaneous rationality of
  $$\left(\sum_k \frac{1}{a_k + t} : t \in \mathbb{N} \right) \in \mathbb{Q}^{\mathbb{N}}.$$

Heuristics that often actually works!

Rationals are countable $\implies$ easy to avoid.

Rationals are dense $\implies$ difficult to miss.

Ronald L. Graham (1935–2020)

Erdős and Graham, 1980, gave a possible definition of an *irrationality sequence* $a_1 < a_2 < a_3 < \cdots \in \mathbb{N}$.

(This was the third one appearing in the literature!)

Definition

We require that

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n} \notin \mathbb{Q}$$

for every bounded $(b_n)_{n=1}^{\infty}$ such that $b_n \in \mathbb{Z} \setminus \{0\}$, $a_n + b_n \neq 0$.

> 2^{2^n} *is an irrationality sequence although we do not know about* 2^n *or* $n!$*.* (Erdős and Graham, 1980, EP #264)

> *Is there an irrationality sequence* a_n *of this type which increases exponentially? It is not hard to show that it cannot increase slower than exponentially.* (Erdős, 1986)

Theorem: a negative result — K. and Tao, 2024

$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty \implies (a_n)_{n=1}^{\infty}$ is not an irrationality sequence.

In particular,

$$a_n = 2^n$$

and sequences

$$a_n \sim \theta^n, \quad \theta > 1$$

are **not** irrationality sequences

The condition can be weakened to

$$\liminf_{n\to\infty} \left(a_n^2 \sum_{k=n+1}^{\infty} \frac{1}{a_k^2} \right) > 0$$

Theorem: a negative result — K. and Tao, 2024

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty \implies (a_n)_{n=1}^{\infty} \text{ is not an irrationality sequence.}$$

Proof

$$\frac{1}{a_n^2} \leqslant C \sum_{k=n+1}^{\infty} \frac{1}{a_k^2}, \qquad a_n \geqslant 4C+1$$

$$J_n := \{a_n+1, a_n+2, a_n+3, \ldots, a_n+4C+1\}$$

$$I_n := \left[\sum_{k=n}^{\infty} \frac{1}{\max J_k}, \sum_{k=n}^{\infty} \frac{1}{\min J_k}\right] \subset (0,\infty)$$

We first claim that $I_n = \left\{\frac{1}{j} : j \in J_n\right\} + I_{n+1}$.

The gaps $\frac{1}{j} - \frac{1}{j+1} < \frac{1}{a_n^2}$ are smaller than the length of I_{n+1},

$$\sum_{k=n+1}^{\infty} \left(\frac{1}{a_n+1} - \frac{1}{a_n+4C+1}\right) \geqslant \sum_{k=n+1}^{\infty} \frac{C}{a_k^2}.$$

Theorem: a negative result — K. and Tao, 2024

$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty \implies (a_n)_{n=1}^{\infty}$ is not an irrationality sequence.

Thus, we have $I_n = \left\{ \frac{1}{j} : j \in J_n \right\} + I_{n+1}$ for all $n \geqslant m$.
Now we claim that the tail sums

$$\left\{ \sum_{k=m}^{\infty} \frac{1}{x_k} : x_k \in J_k \text{ for every } k \geqslant m \right\}$$

fill in the whole segment I_m.
Fix some $x \in I_m$ and inductively construct $x_k \in J_k$ for $k \geqslant m$ s.t.

$$x \in \sum_{k=m}^{n-1} \frac{1}{x_k} + I_n \quad \implies \quad x = \sum_{k=m}^{\infty} \frac{1}{x_k}.$$

Theorem: a negative result — K. and Tao, 2024

$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \infty \implies (a_n)_{n=1}^{\infty}$ is not an irrationality sequence.

Taking

$$x \in I_m \cap \mathbb{Q}$$

and writing

$$x_n = a_n + b_n$$

we conclude:
there exists $(b_n)_{n=1}^{\infty}$ in $[1, 4C+1]^{\mathbb{N}}$ s.t.

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n} \in \mathbb{Q}.$$

∎

Corollary

$a_n = 2^n$ is not an irrationality sequence.

Note that the representations are not explicit, e.g.,

$$\frac{3}{4} = \sum_{n=1}^{\infty} \frac{1}{2^n + b_n}$$

with $1 \leqslant b_n \leqslant 5$ for every n.

1D results — Irrationality sequences

Theorem: a positive result — K. and Tao, 2024

For $F\colon \mathbb{N} \to (0,\infty)$, $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \infty$ there exists an irrationality sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \sim F(n)$.

In particular, there exists an irrationality sequence with, say,

$$a_n \sim 2^{n \log_2 \log_2 \log_2 n}.$$

The proof gives more: there exists an irrationality sequence with, say,

$$a_n = n! + O(\log_2 \log_2 n).$$

Remember that we do not know if $\sum_{n=2}^{\infty} \frac{1}{n!-1} \notin \mathbb{Q}$.

Theorem: a positive result — K. and Tao, 2024

For $F\colon \mathbb{N} \to (0,\infty)$, $\lim_{n\to\infty} \dfrac{F(n+1)}{F(n)} = \infty$ there exists an irrationality sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \sim F(n)$.

Proof

Choose $(c_n)_{n=1}^{\infty}$ with "much slower" growth than F.
Consider a sequence $(a_n)_{n=1}^{\infty}$ constructed randomly with

$$a_n \in \lfloor F(n) \rfloor + \{1, 2, 3, \ldots, c_n\}$$

uniformly and independently for each $n \geqslant n_0$.
Then $a_n \sim F(n)$ and even $a_n = F(n) + O(c_n)$ and we claim that

$$\mathbb{P}\Big((a_n)_{n=1}^{\infty} \text{ is an irrationality sequence} \Big) = 1.$$

Theorem: a positive result — K. and Tao, 2024

For $F\colon \mathbb{N} \to (0, \infty)$, $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \infty$ there exists an irrationality sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \sim F(n)$.

Proof

The assumptions on the growth of F and $(c_n)_{n=1}^{\infty}$ guarantee that, for sufficiently large m, every $q \in \mathbb{Q}$ has at most one representation

$$q = \sum_{n=m}^{\infty} \frac{1}{\lfloor F(n) \rfloor + d_n}$$

with $-c_n < d_n \leqslant 2c_n$.
The gaps between $\frac{1}{\lfloor F(k) \rfloor + d_k}$ and $\frac{1}{\lfloor F(k) \rfloor + d'_k}$ decay very rapidly.

Theorem: a positive result — K. and Tao, 2024

For $F\colon \mathbb{N} \to (0,\infty)$, $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \infty$ there exists an irrationality sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \sim F(n)$

Proof $E_{C,q} := \Big\{ (a_n)_{n=1}^{\infty} : \exists\, (b_n)_{n=m_C}^{\infty}$ s.t. $-C \leqslant b_n \leqslant C$ for $n \geqslant m_C$ and $\sum_{n=m_C}^{\infty} \frac{1}{a_n+b_n} = q \Big\}$

If $a_n + b_n = \lfloor F(n) \rfloor + d_n$ is "the only possibility" for q then

$$E_{C,q} = \bigcap_{n=m_C}^{\infty} \left\{ (a_n)_{n=1}^{\infty} : \lfloor F(n)\rfloor + d_n - C \leqslant a_n \leqslant \lfloor F(n)\rfloor + d_n + C \right\}.$$

$$\mathbb{P}(E_{C,q}) \leqslant \lim_{N\to\infty} \prod_{n=m_C}^{N} \frac{2C+1}{c_n} = 0$$

Take a union over $q \in \mathbb{Q}$ and $C \in \mathbb{N}$. ∎

Once I asked: Assume that $\sum \frac{1}{n_k}$ *and* $\sum \frac{1}{n_k - 1}$ *are both rational. How fast can* n_k *tend to infinity? I was (and am) sure that* $n_k^{1/k} \to \infty$ *is possible but* $n_k^{1/2^k}$ *must tend to 1. Unfortunately almost nothing is known. David Cantor observed that*

$$\sum_{k=3}^{\infty} \frac{1}{\binom{k}{2}} \quad \text{and} \quad \sum_{k=3}^{\infty} \frac{1}{\binom{k}{2} - 1}$$

are both rational and we do not know any sequence with this property which tends to infinity faster than polynomially. (Erdős, 1986, EP #265)

(...) and we could never decide if n_k *can increase exponentially or even faster.* (Erdős, 1983)

Theorem — K. and Tao, 2024

For every $d \in \mathbb{N}$ there exists $\beta > 1$ such that

$$\Big\{ \Big(\sum_{k=1}^{\infty} \frac{1}{a_k}, \sum_{k=1}^{\infty} \frac{1}{a_k + 1}, \ldots, \sum_{k=1}^{\infty} \frac{1}{a_k + d - 1} \Big) : (a_k)_{k=1}^{\infty} \text{ is a strictly increasing sequence in } \mathbb{N} \text{ such that } \lim_{k \to \infty} a_k^{1/\beta^k} = \infty \Big\}$$

has a non-empty interior in $\mathbb{R}^d$.

In particular, there is a sequence with $\lim_{k \to \infty} a_k^{1/\beta^k} = \infty$ and $\sum_{k=1}^{\infty} \frac{1}{a_k + j} \in \mathbb{Q}$ for $j = 0, \ldots, d-1$. (**double exponential growth**)

$\mathbb{R}^d$ results — Simultaneous rationality

Theorem — K. and Tao, 2024

$$\mathrm{Int}\left\{\left(\sum_{k=1}^{\infty}\frac{1}{a_k},\dots,\sum_{k=1}^{\infty}\frac{1}{a_k+d-1}\right):(a_k)_{k=1}^{\infty}\text{ s.t. }\lim_{k\to\infty}a_k^{1/\beta^k}=\infty\right\}\neq\emptyset$$

Proof

Linear change of variables:

$$U\left(\frac{1}{x},\frac{1}{x+1},\frac{1}{x+2},\dots,\frac{1}{x+d-1}\right)=\left(f_1(x),f_2(x),f_3(x),\dots,f_d(x)\right)$$

$$f_i(x):=\frac{1}{x(x+1)\cdots(x+i-1)}$$

It “decouples” the dynamics so that we can imitate the 1D proof.

$\mathbb{R}^d$ results — Simultaneous rationality

Theorem — K. and Tao, 2024

$$\operatorname{Int}\left\{\left(\sum_{k=1}^{\infty}\frac{1}{a_k},\dots,\sum_{k=1}^{\infty}\frac{1}{a_k+d-1}\right) : (a_k)_{k=1}^{\infty} \text{ s.t. } \lim_{k\to\infty} a_k^{1/\beta^k}=\infty\right\}\neq\emptyset$$

Proof

$$1<\beta<\left(\frac{2d+2}{2d+1}\right)^{1/d}, \qquad \beta^d<\alpha<\frac{2d+2}{2d+1}$$

$$N_k := \begin{cases}(2d+1)^k & \text{for } 1\leqslant k\leqslant k_0,\\ \lfloor 2^{\alpha^k}\rfloor & \text{for } k>k_0,\end{cases} \qquad M_k\sim N_k^{1/2}$$

Consider the collection of sequences

$$\mathcal{A} := \left\{(a_n)_{n=1}^{\infty}\in\mathbb{Z}^{\mathbb{N}} : jN_k - M_k \leqslant a_{(k-1)d+j} \leqslant jN_k + M_k \text{ for } 1\leqslant j\leqslant d\right\}.$$

We want to prove $\operatorname{Int}\left\{\left(\sum_{n=1}^{\infty} f_i(a_n)\right)_{i=1}^{d} : (a_n)_{n=1}^{\infty}\in\mathcal{A}\right\}\neq\emptyset.$

Theorem — K. and Tao, 2024

$$\operatorname{Int}\left\{\left(\sum_{k=1}^{\infty}\frac{1}{a_k},\dots,\sum_{k=1}^{\infty}\frac{1}{a_k+d-1}\right):(a_k)_{k=1}^{\infty}\text{ s.t. }\lim_{k\to\infty}a_k^{1/\beta^k}=\infty\right\}\neq\emptyset$$

Proof

We first show

$$s_k+R_k\subseteq S_k+R_{k+1},$$

$$s_k:=\left(\sum_{j=1}^{d}f_i(jN_k)\right)_{i=1}^{d}\in\mathbb{R}^d,$$

$$S_k:=\left\{\left(\sum_{j=1}^{d}f_i(jN_k+n_j)\right)_{i=1}^{d}\;:\;n_1,\dots,n_d\in[-M_k,M_k]\cap\mathbb{Z}\right\}\subset\mathbb{R}^d,$$

$$R_k:=\prod_{i=1}^{d}\left[-\frac{\varepsilon_d M_k}{N_k^{i+1}},\frac{\varepsilon_d M_k}{N_k^{i+1}}\right].$$

Theorem — K. and Tao, 2024

$$\mathrm{Int}\left\{\left(\sum_{k=1}^{\infty}\frac{1}{a_k},\dots,\sum_{k=1}^{\infty}\frac{1}{a_k+d-1}\right):(a_k)_{k=1}^{\infty}\text{ s.t. }\lim_{k\to\infty}a_k^{1/\beta^k}=\infty\right\}\neq\emptyset$$

Proof

Using

$$f_i(N+n)=f_i(N)-\frac{in}{N^{i+1}}+O_i\left(\frac{n^2}{N^{i+2}}\right)$$

we can rewrite

$$S_k=\left\{s_k-p_k(n_1,\dots,n_d)-\Delta_k(n_1,\dots,n_d)\ :\ n_1,\dots,n_d\in[-M_k,M_k]\cap\mathbb{Z}\right\},$$

$$p_k(n_1,\dots,n_d):=\left(\frac{i}{N_k^{i+1}}\sum_{j=1}^{d}\frac{n_j}{j^{i+1}}\right)_{i=1}^{d},$$

$$\Delta_k(n_1,\dots,n_d)\in\prod_{i=1}^{d}\left[-\frac{C_iM_k^2}{N_k^{i+2}},\frac{C_iM_k^2}{N_k^{i+2}}\right].$$

Theorem — K. and Tao, 2024

$$\mathrm{Int}\left\{\left(\sum_{k=1}^{\infty}\frac{1}{a_k},\ldots,\sum_{k=1}^{\infty}\frac{1}{a_k+d-1}\right) : (a_k)_{k=1}^{\infty} \text{ s.t. } \lim_{k\to\infty} a_k^{1/\beta^k}=\infty\right\}\neq\emptyset$$

Proof

Vandermonde $\implies$

$$\left\{\left(\sum_{j=1}^{d}\frac{n_j}{j^{i+1}}\right)_{i=1}^{d} : n_1,\ldots,n_d\in\mathbb{Z}\right\}$$

contains an integer sub-lattice $v_d\mathbb{Z}^d$.

Now the argument becomes essentially 1D.

The error-term Δ_k is small if

$$\frac{1}{N_k^{i+1}}+\frac{M_k^2}{N_k^{i+2}}\ll\frac{M_{k+1}}{N_{k+1}^{i+1}}.$$

Theorem — K. and Tao, 2024

$$\operatorname{Int}\left\{\left(\sum_{k=1}^{\infty}\frac{1}{a_k},\ldots,\sum_{k=1}^{\infty}\frac{1}{a_k+d-1}\right) : (a_k)_{k=1}^{\infty} \text{ s.t. } \lim_{k\to\infty} a_k^{1/\beta^k}=\infty\right\}\neq\emptyset$$

Proof

Now that we have $s_k + R_k \subseteq S_k + R_{k+1}$ for $k \geqslant m$, we take

$$x \in \sum_{k=m}^{\infty} s_k + R_m$$

and inductively construct $y_k \in S_k$ such that

$$x \in \sum_{k=m}^{n-1} y_k + \sum_{k=n}^{\infty} s_k + R_n \quad \Longrightarrow \quad x = \sum_{k=m}^{\infty} y_k.$$

∎

Already the non-empty interior with no growth requirement was posed as an open problem by Erdős, Graham, and Straus. (EP #268)

Corollary

For every $d \in \mathbb{N}$

$$\left\{ \left(\sum_{n\in A} \frac{1}{n}, \sum_{n\in A} \frac{1}{n+1}, \dots, \sum_{n\in A} \frac{1}{n+d-1} \right) : A \subset \mathbb{N} \text{ infinite, } \sum_{n\in A} \frac{1}{n} < \infty \right\}$$

has a non-empty interior in $\mathbb{R}^d$.

Special cases:

- $d = 2$ claimed by Erdős and Straus.
- $d = 3$ posed by Erdős and Graham in 1980, proved by K., 2024, producing a concrete ball of radius 10^{-24} inside the set.

Kenneth B. Stolarsky (1942)

Photo: University of Illinois Urbana-Champaign

The following pretty conjecture is due to Stolarsky:

$$\sum_{n=1}^{\infty} \frac{1}{a_n + t}$$

cannot be rational for every positive integer t.

(Erdős and Graham, 1980, EP #266)

∞D results — Stolarsky's problem

Theorem — K. and Tao, 2024

No. There exists $(a_n)_{n=1}^{\infty}$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n + t} \in \mathbb{Q}$ for every $t \in \mathbb{Q} \setminus \{-a_n : n \in \mathbb{N}\}$.

Proof

Order the rationals as $(t_i)_{i=1}^{\infty}$.
This time use the linear change of variables $U \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$,

$$U\Big(\frac{1}{x+t_1}, \frac{1}{x+t_2}, \frac{1}{x+t_3}, \dots\Big) = \big(f_1(x), f_2(x), f_3(x), \dots\big),$$

$$f_i(x) := \begin{cases} \dfrac{1}{\prod_{j=1}^{i}(x+t_j)} & \text{for } x \in \mathbb{R} \setminus \{-t_1, \dots, -t_i\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$U\mathbb{Q}^{\mathbb{N}} = \mathbb{Q}^{\mathbb{N}}.$$

Theorem — K. and Tao, 2024

No. There exists $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \frac{1}{a_n + t} \in \mathbb{Q}$ for every $t \in \mathbb{Q} \setminus \{-a_n : n \in \mathbb{N}\}$.

Proof

We find an algorithm that generates $x_i \in \mathbb{Q}$ and $a_n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} f_i(a_n) = x_i \quad \text{for } i = 1, 2, 3, \ldots.$$

The idea is to preserve the old relations

$$x_i \in \sum_{l \leqslant k-1} \sum_{j} f_i(a_{n(l)+j}) + \sum_{l \geqslant k} \sum_{j} f_i(jN_l) + [-\delta_{i,k}, \delta_{i,k}]$$

and gradually introduce the new x's, i.e., increase the dimension. ∎

Open problem #1

Open problem

Is $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^{n_k}-1} \notin \mathbb{Q}$ for any positive integers $n_1 < n_2 < n_3 < \cdots$?

(Erdős and Graham, 1980, EP #257)

Is $\displaystyle\sum_{k=1}^{\infty} \frac{1}{t^{n_k}-1} \notin \mathbb{Q}$ for every integer $t \geqslant 2$ and positive integers $n_1 < n_2 < n_3 < \cdots$?

(Erdős, 1968)

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n}-1} \notin \mathbb{Q}$ conjectured by Chowla, 1947, proved by Erdős, 1948.
- $\displaystyle\sum_{p \text{ prime}} \frac{1}{2^{p}-1} \notin \mathbb{Q}$ posed by Erdős in the 1940s, proved by Pratt, 2024, assuming a certain uniform version of the Hardy–Littlewood prime tuples conjecture.

Open problem #1

- $2 \leqslant t_1 < \cdots < t_m,\ \sum_{j=1}^{m} \frac{1}{t_j - 1} > 1 \implies \sum_{j=1}^{m} \sum_{n \in S_j} \frac{1}{t_j^n - 1} \in \mathbb{Q}$ for some $S_1, \ldots, S_m \subseteq \mathbb{N}$ with $S_1 \cup \cdots \cup S_m$ infinite.
  K. and Tao, 2024

- $\left\{ \sum_{n \in S} \frac{1}{2^n - 1} : S \subseteq \mathbb{N} \right\}$ has empty interior, but positive Lebesgue measure (a fat Cantor set).

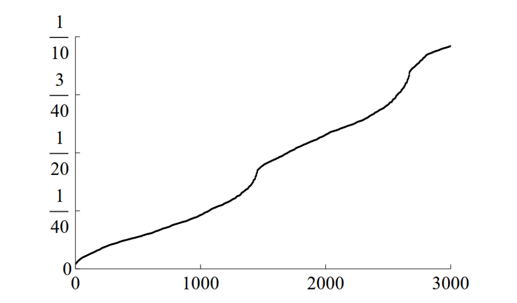
- Boes, Darst, and Erdős, 1981, showed that there exist fat, symmetric, irrational Cantor sets.

Open problem #2

Open problem

Is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+f(n))} \notin \mathbb{Q}$ whenever $f(1) \leqslant f(2) \leqslant f(3) \leqslant \cdots \to \infty$? (Erdős and Graham, 1980)

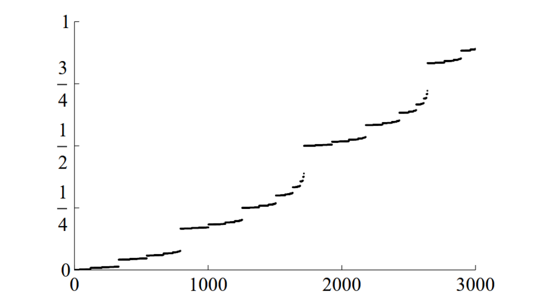
- **No** if only $f(n) \to \infty$. Crmarić and K., 2025, EP #270



Partial series with $1 \leqslant f(1) \leqslant 5,\ 2 \leqslant f(2), f(3) \leqslant 5,\ 3 \leqslant f(4), f(5), f(6), f(7) \leqslant 5$; the smallest 3000 sums sorted.

Open problem #2

- The sums from the open problem form a set of Lebesgue measure 0. Crmarić and K., 2025



Partial series with $1 \leqslant f(1) \leqslant f(2) \leqslant f(3) \leqslant f(4) \leqslant f(5) \leqslant f(6) \leqslant f(7) \leqslant 8$; the smallest 3000 sums sorted.

Thank you for your attention!