Scattering theory analogues of several classical estimates in Fourier analysis

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Classical Fourier analysis

The trigonometric/Fourier series

$$a(t) = \sum_{n=-\infty}^{+\infty} A_n e^{2\pi i n t} \qquad \text{(at least formally)}$$

Typically we take $A_n = \int_{\mathbb{T}} f(u) e^{-2\pi i n u} du, \quad \mathbb{T} \equiv \mathbb{R}/\mathbb{Z} \equiv [0, 1)$

Convergence? In which sense? Under which conditions?

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The Fourier transform

$$\widehat{f}(\xi) := \int_{-\infty}^{+\infty} f(x) e^{-2\pi ix\xi} dx \qquad \text{(at least formally)}$$

Suppose that f is locally integrable and supported in $[0, +\infty)$

$$\tfrac{d}{dx} a(x,\xi) = f(x) e^{-2\pi ix\xi}, \quad a(0,\xi) = 0$$

$$\Longrightarrow \; a(x,\xi) = \int_0^x f(y) e^{-2\pi iy\xi} dy = \widehat{f \mathbb{1}_{[0,x]}}(\xi), \;\; \text{``}a(+\infty,\xi) = \widehat{f}(\xi)\text{''}$$

Nonlinear Fourier analysis

The SU(1,1) trigonometric product

$$\begin{bmatrix} a_N(t) & b_N(t) \\ \overline{b_N(t)} & \overline{a_N(t)} \end{bmatrix} = \prod_{n=0}^{N} \begin{bmatrix} A_n & B_n e^{2\pi i n t} \\ \overline{B_n} e^{-2\pi i n t} & A_n \end{bmatrix}$$

$A_n > 0, \;\; B_n \in \mathbb{C}, \;\; A_n^2 - |B_n|^2 = 1$

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The SU(1,1) Fourier transform / the Dirac scattering transform

$$\frac{d}{dx} \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} = \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} \begin{bmatrix} 0 & f(x) e^{-2\pi i x \xi} \\ \overline{f(x)} e^{2\pi i x \xi} & 0 \end{bmatrix}$$

$$\begin{bmatrix} a(0,\xi) & b(0,\xi) \\ \overline{b(0,\xi)} & \overline{a(0,\xi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{“}\widehat{f}(\xi) = \begin{bmatrix} a(+\infty,\xi) & b(+\infty,\xi) \\ \overline{b(+\infty,\xi)} & \overline{a(+\infty,\xi)} \end{bmatrix}\text{”}$$

Suppose that f is locally integrable and supported in $[0,+\infty)$
$\Longrightarrow$ $a(\cdot,\xi)$ and $b(\cdot,\xi)$ exist as absolutely continuous solutions

Nonlinear Fourier analysis

$$\mathrm{SU}(1,1) := \left\{ \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} \;:\; A, B \in \mathbb{C},\;\; |A|^2 - |B|^2 = 1 \right\}$$

$$\underbrace{\begin{bmatrix} a_N(t) & b_N(t) \\ \overline{b_N(t)} & \overline{a_N(t)} \end{bmatrix}}_{\mathrm{SU(1,1)}} = \prod_{n=0}^{N} \underbrace{\begin{bmatrix} A_n & B_n e^{2\pi i n t} \\ \overline{B_n} e^{-2\pi i n t} & A_n \end{bmatrix}}_{\mathrm{SU(1,1)}}$$

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$$\mathfrak{su}(1,1) = \left\{ \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} : A, B \in \mathbb{C},\ A \in i\mathbb{R} \right\}$$

$$\frac{d}{dx} \underbrace{\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix}}_{\mathrm{SU(1,1)}} = \underbrace{\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix}}_{\mathrm{SU(1,1)}} \underbrace{\begin{bmatrix} 0 & f(x) e^{-2\pi i x \xi} \\ \overline{f(x)} e^{2\pi i x \xi} & 0 \end{bmatrix}}_{\mathfrak{su}(1,1)}$$

This is not the linear Fourier transform on SU(1,1)!

Nonlinear Fourier analysis

In the scalar form:

$$\partial_x a(x,\xi) = \overline{f(x)} e^{2\pi i x\xi} b(x,\xi), \quad \partial_x b(x,\xi) = f(x) e^{-2\pi i x\xi} a(x,\xi)$$
$$a(0,\xi) = 1, \quad b(0,\xi) = 0$$

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Integral equations:

$$a(x,\xi) = 1 + \int_0^x \overline{f(y)} e^{2\pi i y\xi} b(y,\xi) dy$$
$$b(x,\xi) = \int_0^x f(y) e^{-2\pi i y\xi} a(y,\xi) dy$$

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- Picard's iteration gives certain multilinear series expansions
- Born's approximation: $b(x,\xi) \approx \widehat{f \mathbb{1}_{[0,x]}}(\xi)$ when $\|f\|_{L^1(\mathbb{R})} \ll 1$
- We care about the long term behavior and cannot linearize!

Motivation

Eigenproblem for the Dirac operator:

$$L := \begin{bmatrix} \frac{d}{dx} & -\bar{f} \\ f & -\frac{d}{dx} \end{bmatrix}, \quad \text{i.e.} \quad L \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \varphi' - \bar{f}\psi \\ f\varphi - \psi' \end{bmatrix}$$

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L is skew-adjoint, so for $\xi \in \mathbb{R}$ we consider the eigenproblem:

$$L \begin{bmatrix} \varphi(\cdot, \xi) \\ \psi(\cdot, \xi) \end{bmatrix} = -\pi i \xi \begin{bmatrix} \varphi(\cdot, \xi) \\ \psi(\cdot, \xi) \end{bmatrix}$$

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i.e.

$$\begin{aligned} \partial_x \varphi(x,\xi) + \pi i \xi \varphi(x,\xi) &= \overline{f(x)}\psi(x,\xi) \\ \partial_x \psi(x,\xi) - \pi i \xi \psi(x,\xi) &= f(x)\varphi(x,\xi) \end{aligned}$$

i.e.

$$\begin{aligned} \partial_x \big(\underbrace{\varphi(x,\xi)e^{\pi i x \xi}}_{a(x,\xi)} \big) &= \overline{f(x)} e^{2\pi i x \xi} \underbrace{\psi(x,\xi)e^{-\pi i x \xi}}_{b(x,\xi)} \\ \partial_x \big(\underbrace{\psi(x,\xi)e^{-\pi i x \xi}}_{b(x,\xi)} \big) &= f(x) e^{-2\pi i x \xi} \underbrace{\varphi(x,\xi)e^{\pi i x \xi}}_{a(x,\xi)} \end{aligned}$$

Carleson's theorem

Classical/linear — Carleson (1966)

$$(A_n) \in \ell^2(\mathbb{Z}) \implies \lim_{N\to+\infty} \sum_{n=-N}^{N} A_n e^{2\pi i n t} \text{ exists for a.e. } t \in \mathbb{T}$$

$$f \in \mathrm{L}^2(\mathbb{R}) \implies \lim_{R\to+\infty} \int_{-R}^{+R} f(x) e^{-2\pi i x\xi} dx \text{ exists for a.e. } \xi \in \mathbb{R}$$

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Nonlinear analogues — Open question

$$\sum_{n=0}^{+\infty} |B_n|^2 < \infty \overset{?}{\implies} \lim_{N\to+\infty} \begin{bmatrix} \frac{a_N(t)}{b_N(t)} & \frac{b_N(t)}{a_N(t)} \end{bmatrix} \text{ exists for a.e. } t \in \mathbb{T}$$

$$f \in \mathrm{L}^2(\mathbb{R}) \overset{?}{\implies} \lim_{x\to+\infty} \begin{bmatrix} \frac{a(x,\xi)}{b(x,\xi)} & \frac{b(x,\xi)}{a(x,\xi)} \end{bmatrix} \text{ exists for a.e. } \xi \in \mathbb{R}$$

- Even finiteness of $\sup_{x\in[0,+\infty)} |a(x,\xi)|$ for a.e. $\xi \in \mathbb{R}$ is open
- Muscalu, Tao, Thiele (2002): the Cantor group "toy-model"

Hausdorff-Young inequalities

Classical/linear — Young (1913), Hausdorff (1923)

$$1 \leq p \leq 2,\ 1/p + 1/p' = 1 \implies \|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

Babenko (1961), Beckner (1975)

$$1 < p < 2 \implies \|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \underbrace{p^{1/2p} {p'}^{1/2p'}}_{<1} \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

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Nonlinear analogues — Christ and Kiselev (2001)

$$1 \le p \le 2 \implies \|(\log|a(+\infty,\cdot)|^2)^{1/2}\|_{\mathrm{L}^{p'}(\mathbb{R})} \le C_p \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

- $p = 1$ trivial by Gronwall's lemma
- $p = 2$ an identity (with $C_2 = 1$) by the contour integration
- Open question: Does C_p stay bounded as $p \uparrow 2$?
- K. (2010): confirmed in the Cantor group "toy-model"

Lacunary trigonometric series

$$1 \le m_1 < m_2 < m_3 < \cdots, \quad q > 1, \quad m_{j+1} \ge q m_j$$

Norm convergence — Zygmund (1920s)

$$\Big\| \sum_{j=1}^{N} A_j e^{2\pi i m_j t} \Big\|_{\mathrm{L}^p_t(\mathbb{T})} \sim_{p,q} \Big(\sum_{j=1}^{N} |A_j|^2 \Big)^{1/2}, \quad 0 < p < \infty$$

$$\sum_{j=1}^{\infty} A_j e^{2\pi i m_j t} \text{ converges in } \mathrm{L}^p \iff (A_j) \in \ell^2(\mathbb{N})$$

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$$\sum_{j=1}^{\infty} A_j e^{2\pi i m_j t} \text{ converges in } L^p \iff (A_j) \in \ell^2(\mathbb{N})$$

Convergence a.e. — Kolmogorov (1924)

$$(A_j) \in \ell^2(\mathbb{N}) \implies \sum_{j=1}^{\infty} A_j e^{2\pi i m_j t} \text{ converges for a.e. } t \in \mathbb{T}$$

Converse of convergence a.e. — Zygmund (1930)

$$\sum_{j=1}^{\infty} A_j e^{2\pi i m_j t} \text{ conv. on a set of measure} > 0 \implies (A_j) \in \ell^2(\mathbb{N})$$

Lacunary SU(1,1) trigonometric products

$$\rho\colon \mathsf{SU}(1,1) \times \mathsf{SU}(1,1) \to [0,+\infty)$$
$$\rho(G_1, G_2) := \log\left(1 + \|G_1^{-1}G_2 - I\|_{\mathrm{op}}\right)$$

ρ is a complete metric on $\mathsf{SU}(1,1)$

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ρ is a complete metric on $\mathrm{SU}(1,1)$

$$d_p(g_1, g_2) := \begin{cases} \|\rho(g_1(t), g_2(t))\|_{\mathrm{L}^p_t(\mathbb{T})} & \text{for } 1 \le p < \infty \\ \|\rho(g_1(t), g_2(t))\|^p_{\mathrm{L}^p_t(\mathbb{T})} & \text{for } 0 < p < 1 \end{cases}$$
$$\mathrm{L}^p(\mathbb{T}, \mathrm{SU}(1,1)) := \left\{ g\colon \mathbb{T} \to \mathrm{SU}(1,1) : d_p(I, g) < +\infty \right\}$$

d_p is a complete metric on $\mathrm{L}^p(\mathbb{T}, \mathrm{SU}(1,1))$

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$$1 \le m_1 < m_2 < m_3 < \cdots, \quad m_{j+1} \ge q m_j$$

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K. and Rupčić (2016)

Assume $q \ge 2$ and take $0 < p < \infty$

$$\lim_{N\to+\infty} \begin{bmatrix} a_N & b_N \\ \overline{b_N} & \overline{a_N} \end{bmatrix} \text{ exists in } d_p \iff \sum_{j=1}^{\infty} |B_j|^2 < +\infty$$

Recall $\sum_{j=1}^{\infty} |B_j|^2 < +\infty \iff \sum_{j=1}^{\infty} \log A_j < +\infty$

$\iff \prod_{j=1}^{\infty} (A_j^2 + |B_j|^2) < +\infty$

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K. and Rupčić (2016)

Assume $q \geq 3$

$$\lim_{N\to+\infty} \begin{bmatrix} \frac{a_N(t)}{b_N(t)} & \frac{b_N(t)}{a_N(t)} \end{bmatrix} \text{ exists on a set of measure } > 0$$

$$\implies \sum_{j=1}^{\infty} |B_j|^2 < +\infty$$

References

Introductory literature:

- Tao, Thiele, *Nonlinear Fourier Analysis*, IAS/Park City Graduate Summer School, unpublished lecture notes, 2003, available at arXiv:1201.5129 [math.CA]
- Tao, *An introduction to the nonlinear Fourier transform*, unpublished note, 2002
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Thank you for your attention!