Problemi klasične i maksimalne teorije Fourierove restrikcije

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Restrikcija Fourierove transformacije

Fourierova transformacija: $\mathcal{F}\colon \mathrm{L}^1(\mathbb{R}^d) \to \mathrm{C}_0(\mathbb{R}^d)$, $\mathcal{F}\colon f \mapsto \widehat{f}$

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} \, \mathrm{d}x$$

$\mathcal{F}$ se proširuje do unitarnog operatora $\mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$
$\mathcal{F}$ se proširuje do kontrakcije $\mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^{p'}(\mathbb{R}^d)$ za $1 < p < 2$

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$S =$ neka hiperploha u $\mathbb{R}^d$ (npr. paraboloid, stožac, sfera)
Može li se smisleno definirati $\widehat{f}|_S$? (*Stein*, kasne 1960-e)

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Može li se smisleno definirati $\widehat{f}|_S$? (*Stein*, kasne 1960-e)

- $p = 1 \rightsquigarrow$ DA, jer je $\widehat{f}$ neprekidna
- $p = 2 \rightsquigarrow$ NE, jer je $\widehat{f}$ proizvoljna L^2 funkcija
- Što se može za $1 < p < 2$? Pitanje u ovisnosti o S i p

Apriorna ocjena

σ = neka mjera na S, npr. (pogodno normalizirana) plošna mjera
Želimo apriornu ocjenu:

$$\big\|\widehat{f}\big\|_{\mathrm{L}^q(S,\sigma)} \leq C\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

za neki $1 \leq q \leq \infty$ i sve funkcije $f \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$ ili samo $f \in \mathcal{S}(\mathbb{R}^d)$

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Nakon toga će se $\widehat{f}|_S \in \mathrm{L}^q(S,\sigma)$ moći definirati proširenjem od $f \mapsto \widehat{f}|_S$ po neprekidnosti do $\mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^q(S,\sigma)$

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$\chi \in \mathcal{S}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \chi = 1$, $\chi_\varepsilon(x) := \varepsilon^{-d}\chi(\varepsilon^{-1}x)$

$$\lim_{\varepsilon\to 0^+} \big(\widehat{f} * \chi_\varepsilon\big)\big|_S \text{ postoji po normi od } \mathrm{L}^q(S,\sigma)$$

Protuprimjer

Za općenitu hiperplohu odgovor je NE

$S = \mathbb{R}^{d-1} \times \{0\}$ hiperravnina

$f = f_1 \otimes f_2 \implies \widehat{f} = \widehat{f_1} \otimes \widehat{f_2} \implies \|\widehat{f}\|_{\mathrm{L}^q(S)} = \|\widehat{f_1}\|_{\mathrm{L}^q(\mathbb{R}^{d-1})} |\widehat{f_2}(0)|$

Trebali bismo $|\widehat{f_2}(0)| \leq C \|f_2\|_{\mathrm{L}^p(\mathbb{R})} \implies p = 1$

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Promotrimo adjungirani operator $\mathcal{E}$ od $\mathcal{R}\colon f \mapsto \widehat{f}|_S$:

$$\langle \mathcal{R}f, g\rangle_{\mathrm{L}^2(S,\sigma)} = \langle f, \mathcal{E}g\rangle_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$(\mathcal{E}g)(x) = \int_S e^{2\pi i x\cdot\xi} g(\xi)\, \mathrm{d}\sigma(\xi)$$

$\mathcal{R}\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^q(S,\sigma) \iff \mathcal{E}\colon \mathrm{L}^{q'}(S,\sigma) \to \mathrm{L}^{p'}(\mathbb{R}^d)$

T^*T metoda

Za $q = 2$ možemo koristiti T^*T trik: $\|T^*T\| = \|T\|^2$

$$(\mathcal{E}\mathcal{R}f)(x) = \int_{\mathbb{R}^d} f(y)\Big(\int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\xi}\,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}y$$

$$\Longrightarrow \ \mathcal{E}\mathcal{R}f = f * \check{\sigma}$$

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$\mathcal{E}\mathcal{R}\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^{p'}(\mathbb{R}^d)$ (?)

σ = plošna mjera na sferi $\mathbb{S}^{d-1} \implies |\check{\sigma}(x)| \leq C(1+|x|)^{-(d-1)/2}$
Youngova nejednakost za konvoluciju (*Fefferman* / *Stein*, 1970.):

$$S = \mathbb{S}^{d-1}, \quad p < \tfrac{4d}{3d+1}, \quad q \leq 2$$

Primjena po dijadskim vijencima (*Tomas* / *Stein*, 1975.):

$$S = \mathbb{S}^{d-1}, \quad p \leq \tfrac{2(d+1)}{d+3}, \quad q \leq 2$$

Problem Fourierove restrikcije

$d = 2 \rightsquigarrow$ problem je suštinski riješen: $p < \frac{4}{3}$, $q \leq \frac{p'}{3}$

- za $S = \mathbb{S}^1 \rightsquigarrow$ *Zygmund* 1974.
- za kompaktne C^2 krivulje S s nenegativnom zakrivljenosti κ
  $\mathrm{d}\sigma =$ lučna mjera $\mathrm{d}l$ s težinom $\kappa^{1/3}$
  $\rightsquigarrow$ *Carleson i Sjölin*, 1972.; *Sjölin*, 1974.

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$d \geq 3 \rightsquigarrow$ problem je uvelike otvoren

Tri klasične hiperplohe:

- *paraboloid* $\{\xi = (\eta, k|\eta|^2) : \eta \in \mathbb{R}^{d-1}\}$ s mjerom $\mathrm{d}\sigma(\xi) = \mathrm{d}\eta$
  Slutnja: $p < \frac{2d}{d+1}$, $q = \frac{(d-1)p'}{d+1}$ (primijetimo $p < q$)
- *stožac* $\{\xi = (\eta, |\eta|) : \eta \in \mathbb{R}^{d-1}\}$ s mjerom $\mathrm{d}\sigma(\xi) = \mathrm{d}\eta/|\eta|$
  Slutnja: $p < \frac{2(d-1)}{d}$, $q = \frac{(d-2)p'}{d}$ (primijetimo $p < q$)
- *sfera* $\{\xi \in \mathbb{R}^d : |\xi| = r\}$ s plošnom mjerom σ
  Slutnja: $p < \frac{2d}{d+1}$, $q \leq \frac{(d-1)p'}{d+1}$

Već sama apriorna ocjena je zanimljiva (*Strichartz*, 1977)

$$\begin{cases} i\partial_t u + k\Delta u = 0 & \text{u } \mathbb{R}^{d-1} \\ u(\cdot, 0) = u_0 \end{cases}$$

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$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^{d-1}} e^{2\pi i(x\cdot\eta - 2\pi k t|\eta|^2)}\,\widehat{u_0}(\eta)\,\mathrm{d}\eta \\ &= \int_S e^{2\pi i(x,t)\cdot\xi} g(\xi)\,\mathrm{d}\sigma(\xi) = (\mathcal{E}g)(x,t) \end{aligned}$$

$S = \{(\eta, -2\pi k|\eta|^2) : \eta \in \mathbb{R}^{d-1}\}$, $\mathrm{d}\sigma(\xi) = \mathrm{d}\eta$, $g(\xi) = \widehat{u_0}(\eta)$

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Fourierova restrikcija vrijedi npr. za $p' = \frac{2(d+1)}{d-1}$, $q = 2$

$\implies \|u\|_{\mathrm{L}^{2(d+1)/(d-1)}(\mathbb{R}^{d-1}\times\mathbb{R})} \leq C\|\widehat{u_0}\|_{\mathrm{L}^2(\mathbb{R}^{d-1})} = C\|u_0\|_{\mathrm{L}^2(\mathbb{R}^{d-1})}$

Strichartzove ocjene: $\|u(x,t)\|_{\mathrm{L}^a_t(\mathbb{R},\mathrm{L}^b_x(\mathbb{R}^{d-1}))} \leq C\|u_0\|_{\mathrm{L}^2(\mathbb{R}^{d-1})}$

za $2 \leq a, b \leq \infty$, $\frac{2}{a} + \frac{d-1}{b} = \frac{d-1}{2}$, $(a,b,d) \neq (2,\infty,3)$

Valna jednadžba

$$\begin{cases} \partial_t^2 u - k^2 \Delta u = 0 & \text{u } \mathbb{R}^{d-1} \\ u(\cdot, 0) = u_0, \ \partial_t u(\cdot, 0) = u_1 \end{cases}$$

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$$u = (u^- - u^+)/2$$

$$u^\pm(x,t) = \int_{\mathbb{R}^{d-1}} e^{2\pi i (x\cdot\eta \pm kt|\eta|)} \widehat{u_0}(\eta)\,\mathrm{d}\eta + \frac{i}{2\pi k} \int_{\mathbb{R}^{d-1}} e^{2\pi i (x\cdot\eta \pm kt|\eta|)} \widehat{u_1}(\eta) \frac{\mathrm{d}\eta}{|\eta|}$$

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$$\|g_1\|_{\mathrm{L}^2(S,\sigma)} = \big\||\eta|^{-1/2}\widehat{u_1}(\eta)\big\|_{\mathrm{L}^2(\mathbb{R}^{d-1})} = \|u_1\|_{\dot{\mathrm{H}}^{-1/2}(\mathbb{R}^{d-1})}$$

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$(-4\pi^2|\xi|^2 + k^2)\widehat{u}(\xi) = 0$
Tražimo rješenja oblika $\widehat{u}(\xi) = \delta\big(|\xi|^2 - (k/2\pi)^2\big) m(\xi/|\xi|)$

$$u(x) = \int_S e^{2\pi i x\cdot\xi} m(\xi/|\xi|)\, \mathrm{d}\sigma(\xi)$$

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$\implies \|u\|_{\mathrm{L}^{2(d+1)/(d-1)}(\mathbb{R}^d)} \le C\|m\|_{\mathrm{L}^2(S,\sigma)}$

Maksimalna Fourierova restrikcija

Teorem (*Müller, Ricci i Wright*, 2016.)

$d = 2$, $S = C^2$ *krivulja sa zakrivljenosti* $\kappa \geq 0$, $\mathrm{d}\sigma = \kappa^{1/3}\,\mathrm{d}l$,
$\chi \in \mathcal{S}(\mathbb{R}^d)$, $p < \frac{4}{3}$, $q \leq \frac{p'}{3}$

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Teorem (*Vitturi*, 2017.)

$d \geq 3$, $S = \mathbb{S}^{d-1}$, $\sigma =$ *plošna mjera*, $\chi \in \mathcal{S}(\mathbb{R}^d)$, $p \leq \frac{4}{3}$, $q \leq \frac{(d-1)p'}{d+1}$ *(strogi podskup Tomas–Steinovog raspona za* $d \geq 4$*)*

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Za $f \in \mathrm{L}^p(\mathbb{R}^d)$ restrikcija $\widehat{f}|_S$ ima smisla po točkama, tj.

$$\lim_{\varepsilon \to 0^+} \big(\widehat{f} * \chi_\varepsilon\big)(\xi)$$

postoji za σ-g.s. $\xi \in S$ (konv. na $\mathcal{S}(\mathbb{R}^d)$ + maksimalna ocjena)

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Varijacijska Fourierova restrikcija

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Teorem (*K. i Oliveira e Silva*, 2018.)

σ = *plošna mjera na* $\mathbb{S}^2 \subset \mathbb{R}^3$, $\chi \in \mathcal{S}(\mathbb{R}^3)$ *ili* $\chi = \mathbb{1}_{\mathrm{B}(0,1)}$, $\varrho > 2$

$$\Big\| \sup_{0<t_0<t_1<\cdots<t_m} \Big(\sum_{j=1}^{m} \big| \widehat{f} \ast \chi_{t_j} - \widehat{f} \ast \chi_{t_{j-1}} \big|^{\varrho} \Big)^{1/\varrho} \Big\|_{\mathrm{L}^2(\mathbb{S}^2,\sigma)} \leq C \|f\|_{\mathrm{L}^{4/3}(\mathbb{R}^3)}$$

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Ako je $f \in \mathrm{L}^{4/3}(\mathbb{R}^3)$, tada je za σ-g.s. $\xi \in \mathbb{S}^2$:

$$\sup_{0<t_0<t_1<\cdots<t_m} \Big(\sum_{j=1}^{m} \big| (\widehat{f} * \chi_{t_j})(\xi) - (\widehat{f} * \chi_{t_{j-1}})(\xi) \big|^{\varrho} \Big)^{1/\varrho} < \infty$$

$\Longrightarrow$ $\big((\widehat{f} * \chi_{\varepsilon})(\xi) \big)_{\varepsilon>0}$ pravi $O(\delta^{-\varrho})$ skokova veličine $\geq \delta$

Apstraktni princip

Teorem (*K.*, 2018.)

$S \subseteq \mathbb{R}^d$ izmjeriv, σ = mjera na S, μ = kompleksna mjera na $\mathbb{R}^d$, $\mu_t(E) := \mu(t^{-1}E)$, $\widehat{\mu} \in \mathrm{C}^\infty$, $\eta > 0$

$$|\nabla \widehat{\mu}(x)| \leq D(1 + |x|)^{-1-\eta}$$

Pretpostavimo da za neke $1 \leq p \leq 2$, $\boxed{p < q} < \infty$ vrijedi apriorna ocjena F. r. Tada vrijedi maksimalna ocjena F. r.:

$$\Big\| \sup_{t \in (0,\infty)} \big| \widehat{f} * \mu_t \big| \Big\|_{\mathrm{L}^q(S,\sigma)} \leq C \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

i za svaki $p < \varrho < \infty$ vrijedi varijacijska ocjena F. r.:

$$\Big\| \sup_{0<t_0<t_1<\cdots<t_m} \Big(\sum_{j=1}^{m} \big| \widehat{f} * \mu_{t_j} - \widehat{f} * \mu_{t_{j-1}} \big|^{\varrho} \Big)^{1/\varrho} \Big\|_{\mathrm{L}^q(S,\sigma)} \leq C \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

Posljedice i nadogradnje

- Teorem ponovno dokazuje prethodno navedene rezultate
- Pokriva cijeli Tomas–Steinov raspon za sferu $\mathbb{S}^{d-1}$
- Pokriva svaki dokazani raspon za paraboloid ili stožac
- Možemo uzeti $\mathrm{d}\mu(x) = \chi(x)\,\mathrm{d}x$, $\chi \in \mathcal{S}(\mathbb{R}^d)$ ili $\chi = \mathbb{1}_{\mathrm{B}(0,1)}$
- Možemo za μ uzeti plošnu mjeru na $\mathbb{S}^{d-1}$ u dimenzijama $d \geq 4 \rightsquigarrow$ sferična usrednjenja $\frac{1}{\mu(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} \widehat{f}(\xi + \varepsilon\zeta)\,\mathrm{d}\mu(\zeta)$

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Teorem (*Ramos*, 2019.)

μ = mjera na $\mathbb{R}^2$, $p < \frac{4}{3}$, $q \leq \frac{p'}{3}$. Maksimalna F. r. za $\mathbb{S}^1$ vrijedi čim je $\mathrm{M}_\mu g := \sup_{t>0} |g * \mu_t|$ ograničena na $\mathrm{L}^r(\mathbb{R}^2)$ za $r > 2$.
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$\Longrightarrow$ sferična usrednjenja u $d = 2, 3$ (*Bourgain* 1986., *Stein* 1976.)

Ideja dokaza

Ideje u pozadini apstraktnog principa za maksimalnu F. r.:

- Pretp. da se supremum uzima po $t \in \{t_1 > t_2 > \cdots > t_N\}$
- Nakon linearizacije:

$$\big\| \big(\widehat{f} * \mu_{t(\xi)}\big)(\xi) \big\|_{\mathrm{L}^q_\xi(S,\sigma)} \leq C \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

- Nakon dualizacije:

$$\begin{aligned} \Lambda(f,g) &:= \int_S \big(\widehat{f} * \mu_{t(y)}\big)(y) g(y) \,\mathrm{d}\sigma(y) \\ &= \int_S \int_{\mathbb{R}^d} f(x) g(y) e^{-2\pi i x \cdot y} \check{\mu}(t(y)x) \,\mathrm{d}x \,\mathrm{d}\sigma(y) \end{aligned}$$

- Označimo $S_n := \{y \in S : t(y) = t_n\}$
- Zamislimo da je $\check{\mu}(t_n \cdot) = \mathbb{1}_{E_1 \cup E_2 \cup \cdots \cup E_n}$ za neke disjunktne $E_1, E_2, \ldots, E_N \subseteq \mathbb{R}^d$

  Ovo je mjesto gdje malo varamo

Dobivamo bilinearnu formu:

$$\Lambda(f,g) = \int_S \int_{\mathbb{R}^d} f(x)g(y)e^{-2\pi i x\cdot y} \sum_{\substack{m,n\\ 1\le m\le n\le N}} \mathbb{1}_{E_m}(x)\mathbb{1}_{S_n}(y)\,\mathrm{d}x\,\mathrm{d}\sigma(y)$$

Ovo je blok-trokutasto okrnjenje jezgre $K(x,y) = e^{-2\pi i x\cdot y}$

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Trebamo ocjenu:

$$|\Lambda(f,g)| \le C \|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \|g\|_{\mathrm{L}^{q'}(S,\sigma)}$$

s konstantom neovisnom o broju N
Ona se izvodi iz ne-maksimalne ocjene indukcijom(!) po broju N
Ključno je imati pretpostavku $p < q$

Hvala na pažnji!