PARAPRODUCTS WITH GENERAL DILATIONS

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What is a paraproduct?

S. Janson, J. Peetre (1988)

The name "paraproduct" denotes an idea rather than a unique definition.

Á. Bényi, D. Maldonado, V. Naibo (2010)

Anything called a "paraproduct" and denoted $(f,g) \mapsto \Pi(f,g)$ should:

- reconstruct the product, $fg = \Pi(f,g) + \Pi(g,f) + \text{error}$,
- satisfy estimates of the same type as $(f,g) \mapsto fg$,
- linearize a nonlinearity, $F(f) = F(0) + \Pi(F'(f), f) + \text{error}$,
- satisfy some Leibniz-type rule, $\partial^\alpha \Pi(f,g) = \tilde{\Pi}(f, \partial^\alpha g)$.

Classical examples

A. P. Calderón (1965)

$$\Pi(F,G)(s) := -i\int_0^\infty F(s+it)G'(s+it)dt,$$

$$\|\Pi(F,G)\|_{\mathrm{L}^r(\mathbb{R})} \lesssim_{p,q} \|F\|_{\mathrm{H}^p(U)}\|G\|_{\mathrm{H}^q(U)},$$

$U =$ the upper half-plane, $1 < p, q < \infty$, $1/r = 1/p + 1/q$.

C. Pommerenke (1977)

$$\Pi(F,G)(\zeta) := \int_0^\zeta F(\xi)G'(\xi)d\xi,$$

$$\|\Pi(F,G)\|_{\mathrm{H}^2(\mathbb{D})} \lesssim \|F\|_{\mathrm{H}^2(\mathbb{D})}\|G\|_{\mathrm{BMOA}(\mathbb{D})},$$

$D =$ the unit disk, $\mathrm{BMOA}(\mathbb{D}) =$ the dual od $\mathrm{H}^1(\mathbb{D})$.

J.-M. Bony (1981), “opérateurs de paramultiplication”

$$\Pi(f,g) := \int_0^\infty \underbrace{(f * \varphi_t)}_{\mathrm{P}_t f} \underbrace{(g * \psi_t)}_{\mathrm{Q}_t g} \frac{dt}{t},$$

$\varphi_t(x) := t^{-d}\varphi(t^{-1}x)$, φ, ψ Schwartz, $\hat{\psi}(\xi) = 0$ around $\xi = \mathbf{0}$,

$$\|\Pi(f,g)\|_{\mathrm{L}^r(\mathbb{R}^d)} \lesssim_{p,q} \|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}.$$

R. R. Coifman and Y. Meyer (1990)

$$\Pi(f,g) := \sum_{Q,\eta} |Q|^{-1/2} \langle f, \varphi_Q \rangle \langle g, \psi_Q^\eta \rangle \psi_Q^\eta,$$

$Q \in$ dyadic cubes, $\varphi =$ father wavelet, $\psi^\eta =$ mother wavelets,

$$\|\Pi(f,g)\|_{\mathrm{L}^2(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{L}^2(\mathbb{R}^d)} \|g\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

Classical examples

D. L. Burkholder (1966), "martingale transform"

$$\Pi(\mathsf{X},\mathsf{Y}) := \sum_{k=0}^{\infty} X_k\big(Y_{k+1}-Y_k\big),$$

$\mathsf{Y}=(Y_k)_{k=0}^{\infty}$ martingale w.r.t. $(\mathcal{F}_k)$, $\mathsf{X}=(X_k)_{k=0}^{\infty}$ adapted to $(\mathcal{F}_k)$,

$$\|\Pi(\mathsf{X},\mathsf{Y})\|_{\mathrm{L}^q} \lesssim_q \|\mathsf{X}\|_{\mathrm{L}^\infty}\|\mathsf{Y}\|_{\mathrm{L}^q}; \quad 1<q<\infty.$$

R. Bañuelos and A. G. Bennett (1988), "paraproduct of martingales"

$$\Pi(\mathsf{X},\mathsf{Y}) := \int_0^\infty X_t dY_t,$$

$\mathsf{X}=(X_t)_{t\geq 0}$, $\mathsf{Y}=(Y_t)_{t\geq 0}$ martingales w.r.t. Brownian filtration,

$$\|\Pi(\mathsf{X},\mathsf{Y})\|_{\mathrm{H}^r} \lesssim_{p,q} \|\mathsf{X}\|_{\mathrm{H}^p}\|\mathsf{Y}\|_{\mathrm{H}^q}; \quad 0<p,q<\infty,\ 1/r=1/p+1/q.$$

Non-classical example

C. Demeter and C. Thiele (2008)

$$\Pi(f,g) := \sum_{k\in\mathbb{Z}} (\mathrm{P}^{(1)}_{2^k} f)(\mathrm{Q}^{(2)}_{2^k} g) \ \text{ or } \ \int_0^\infty (\mathrm{P}^{(1)}_t f)(\mathrm{Q}^{(2)}_t g)\frac{dt}{t},$$

$\varphi_t(x) := t^{-1}\varphi(t^{-1}x)$, φ, ψ Schwartz, $\operatorname{supp}\hat{\psi} \subseteq \{\xi : \frac{1}{2} \le |\xi| \le 2\}$,
$\mathrm{P}^{(1)}_t$, $\mathrm{Q}^{(2)}_t$ are L-P projections in the 1st and the 2nd variable:

$$\begin{aligned} (\mathrm{P}^{(1)}_t f)(x,y) &:= \textstyle\int_{\mathbb{R}} f(x-s,y)\varphi_t(s)ds, \\ (\mathrm{Q}^{(2)}_t g)(x,y) &:= \textstyle\int_{\mathbb{R}} g(x,y-s)\psi_t(s)ds. \end{aligned}$$

Non-classical example

F. Bernicot (2010), V. K. (2010), F. Bernicot and V. K. (2013)

$$\|\Pi(f,g)\|_{\mathrm{L}^r} \lesssim_{p,q} \|f\|_{\mathrm{L}^p}\|g\|_{\mathrm{L}^q},$$

$1 < p,q < \infty,\ \ r < 2,\ \ 1/r = 1/p + 1/q$
(fails for $p = \infty$ or $q = \infty$),

$$\|\Pi(f,g)\|_{\mathrm{L}^r_y(\mathrm{W}^{s,r}_x)} \lesssim_{p,q,s} \|f\|_{\mathrm{L}^p}\|g\|_{\mathrm{W}^{s,q}}$$

if additionally $r > 1,\ \ s \geq 0$.

P. Durcik (2014, 2015)

L^p estimates for a trilinear operator with a similar structure.

Non-classical example

Idea of the proof:

(1) Compare with the dyadic version:

$$\Pi_{\text{dyadic}}(f,g) := \sum_{k\in\mathbb{Z}} (\mathbb{E}_k^{(1)} f)(\Delta_k^{(2)} g),$$

$\mathbb{E}_k^{(1)}$ dyadic martingale averages in the 1st variable,
$\Delta_k^{(2)}$ dyadic martingale differences in the 2nd variable:

$$\begin{aligned} (\mathbb{E}_k^{(1)} f)(x,y) &:= \textstyle\sum_{|I|=2^{-k}} \Big(\frac{1}{|I|} \int_I f(u,y)du \Big) \mathbf{1}_I(x), \\ (\Delta_k^{(2)} g)(x,y) &:= (\mathbb{E}_{k+1}^{(2)} g)(x,y) - (\mathbb{E}_k^{(2)} g)(x,y). \end{aligned}$$

(2) Use the Bellman function technique i.e. the induction on scales.

General dilations

$\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,

$(\delta_t^{(1)})_{t>0}$, $(\delta_t^{(2)})_{t>0}$ groups of dilations:

- $\delta_t^{(j)} \colon \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$ is a linear transformation,
- $\delta_1^{(j)}$ is the identity and $\delta_{st}^{(j)} = \delta_s^{(j)} \delta_t^{(j)}$ for $s, t > 0$,
- $(0, +\infty) \times \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$, $(t, x) \mapsto \delta_t^{(j)} x$ is continuous,
- $\lim_{t \to 0} \delta_t^{(j)} x = \mathbf{0}$.

Equivalently, for some $A_j \in \mathrm{M}_{d_j}(\mathbb{R})$, $\mathrm{Re}(\sigma(A_j)) > 0$:

$$\delta_t^{(j)} = t^{A_j} = e^{(\log t) A_j}, \quad t > 0, \ j = 1, 2.$$

Example: Non-isotropic dilations on $\mathbb{R}^d$:

$$\delta_t(x_1, \ldots, x_d) := \left(t^{a_1} x_1, \ldots, t^{a_d} x_d\right).$$

General dilations

$\varphi^{(j)}: \mathbb{R}^{d_j} \to \mathbb{C}$, $j = 1, 2$ Schwartz functions, $\int_{\mathbb{R}^{d_j}} \varphi^{(j)}(x)dx = 1$.

Denote the dilates:

$$\varphi_t^{(j)}(x) := (\det \delta_t^{(j)})\, \varphi^{(j)}\big(\delta_t^{(j)} x\big), \quad x \in \mathbb{R}^{d_j},\ t > 0,\ j = 1, 2.$$

Denote the L-P projections:

$$\begin{aligned} (\mathrm{P}_t^{(1)} f)(x, y) &:= \int_{\mathbb{R}^{d_1}} f(x - s, y)\varphi_t^{(1)}(s)ds, \\ (\mathrm{P}_t^{(2)} g)(x, y) &:= \int_{\mathbb{R}^{d_2}} g(x, y - s)\varphi_t^{(2)}(s)ds. \end{aligned}$$

V. K. and K. A. Škreb (2014)

$$\Pi_{\alpha,\beta}(f,g) := \sum_{k\in\mathbb{Z}} \big(\mathrm{P}^{(1)}_{\alpha^k} f\big)\, \big(\mathrm{P}^{(2)}_{\beta^{k+1}} g - \mathrm{P}^{(2)}_{\beta^k} g\big).$$

There exist parameters $0 < \alpha, \beta < 1$, depending only on the dilation structure, such that the estimate

$$\big\|\Pi_{\alpha,\beta}(f,g)\big\|_{\mathrm{L}^r(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}$$

holds whenever $1 < r < 2 < p, q < \infty$ and $1/r = 1/p + 1/q$. The constant depends on $p, q, r, \alpha, \beta, \varphi^{(1)}, \varphi^{(2)}$, and the dilation groups.

Sketch of the proof:

(1) Establish estimates for martingale paraproducts w.r.t. a Cartesian product structure.
$(\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}, \mathbb{P}_1 \times \mathbb{P}_2)$, $\mathcal{F}_k := \mathcal{A}_k \otimes \mathcal{B}$, $\mathcal{G}_k := \mathcal{A} \otimes \mathcal{B}_k$,
$(X_k)_{k=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_k)_{k=0}^{\infty}$, $(Y_k)_{k=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{G}_k)_{k=0}^{\infty}$. If

$$\left\| \mathbb{E}(U | \mathcal{F}_{k+1} \cap \mathcal{G}_{k+1}) \right\|_{\mathrm{L}^\infty} \leq C \left\| \mathbb{E}(U | \mathcal{F}_k \cap \mathcal{G}_k) \right\|_{\mathrm{L}^\infty},$$

then

$$\Big\| \sum_{k=1}^{n} X_{k-1}(Y_k - Y_{k-1}) \Big\|_{\mathrm{L}^r} \lesssim_{p,q} C^{3/2} \|X_n\|_{\mathrm{L}^p} \|Y_n\|_{\mathrm{L}^q}.$$

Sketch of the proof:

(2) Apply the estimates to martingales on abstract dyadic cubes constructed by M. Christ (1990).

$$\{Q_{k,i}^{(j)} \, : \, k \in \mathbb{Z}, \; i \in I_k^{(j)}\}, \quad j = 1, 2,$$

$$\mathcal{A}_k := \sigma\big(\{Q_{k,i}^{(1)} \, : \, i \in I_k^{(1)}\}\big),$$
$$\mathcal{B}_k := \sigma\big(\{Q_{k,i}^{(2)} \, : \, i \in I_k^{(2)}\}\big).$$

Sketch of the proof:

(3) Use the square function of R. L. Jones, A. Seeger, and J. Wright (2008).

$$\mathcal{S}^{(1)}f := \Big(\sum_{k\in\mathbb{Z}} \big|\mathrm{P}^{(1)}_{\alpha^k} f - \mathbb{E}(f|\mathcal{F}_k)\big|^2\Big)^{1/2},$$

$$\mathcal{S}^{(2)}g := \Big(\sum_{k\in\mathbb{Z}} \big|\mathrm{P}^{(2)}_{\beta^k} g - \mathbb{E}(g|\mathcal{G}_k)\big|^2\Big)^{1/2},$$

$$\|\mathcal{S}^{(1)}f\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim_p \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}; \quad 1 < p < \infty,$$

$$\|\mathcal{S}^{(2)}g\|_{\mathrm{L}^q(\mathbb{R}^d)} \lesssim_q \|g\|_{\mathrm{L}^q(\mathbb{R}^d)}; \quad 1 < q < \infty.$$

Further generalizations?

Forget about the product structure.

Let $(\mathrm{P}^{(1)}_{e^s})_{s>0}$ and $(\mathrm{P}^{(2)}_{e^s})_{s>0}$ be two semigroups of operators. ($t = e^s$ switches from multiplicative to additive writing.)

Take

$$\Pi(f,g) := \sum_{k=1}^{\infty} \big(\mathrm{P}^{(1)}_{2^k} f\big)\big(\mathrm{P}^{(2)}_{2^{k+1}} g - \mathrm{P}^{(2)}_{2^k} g\big)$$

or

$$\Pi(f,g) := \int_0^\infty (\mathrm{P}^{(1)}_{e^s} f)\Big(\frac{d}{ds}\mathrm{P}^{(2)}_{e^s} g\Big) ds = \int_1^\infty (\mathrm{P}^{(1)}_t f)\Big(t\frac{d}{dt}\mathrm{P}^{(2)}_t g\Big)\frac{dt}{t}.$$

How much structure has to be imposed on $\mathrm{P}^{(1)}$ and $\mathrm{P}^{(2)}$ in order to have L^p bounds?

Further generalizations

Let $(\mathrm{P}^{(1)}_{e^s})_{s>0}$, $(\mathrm{P}^{(2)}_{e^s})_{s>0}$ be two commuting diffusion semigroups on a measure space $(M, \mathcal{M}, \mu)$:

- $\mathrm{P}^{(j)}_{e^{s_1+s_2}} = \mathrm{P}^{(j)}_{e^{s_1}} \mathrm{P}^{(j)}_{e^{s_2}}$, $\mathrm{P}^{(1)}_{e^{s_1}} \mathrm{P}^{(2)}_{e^{s_2}} = \mathrm{P}^{(2)}_{e^{s_2}} \mathrm{P}^{(1)}_{e^{s_1}}$,
- $\mathrm{P}^{(j)}_{e^s}$ is self-adjoint on $\mathrm{L}^2(M)$,
- $f \in \mathrm{L}^2(M) \implies \lim_{s\to 0} \mathrm{P}^{(j)}_{e^s} f = f$ in L^2 (strong continuity),
- $\|\mathrm{P}^{(j)}_{e^s} f\|_{\mathrm{L}^p} \leq \|f\|_{\mathrm{L}^p}$; $1 \leq p \leq \infty$ (contractivity),
- $f \geq 0 \implies \mathrm{P}^{(j)}_{e^s} f \geq 0$ (positivity),
- $f \in \mathrm{L}^1(M) \implies \int_M \mathrm{P}^{(j)}_{e^s} f \, d\mu = \int_M f d\mu$ (conservativeness).

V. K. and K. A. Škreb (2015)

$$\big\|\Pi(f,g)\big\|_{\mathrm{L}^r} \lesssim_{p,q} \|f\|_{\mathrm{L}^p} \|g\|_{\mathrm{L}^q}$$

in a certain range of exponents such that $1/r = 1/p + 1/q$.

Further generalizations

Idea of the proof:

(1) Extract the product structure using conditional probabilities, the "lifting" of A. Ionescu Tulcea, and "diagonal measures".
(2) Establish estimates for martingale paraproducts of arbitrary martingales w.r.t. general "commuting filtrations".
(3) Reduce semigroups to martingales using the construction of G.-C. Rota (1962).

Thank you!