

Norm-variation of bilinear averages

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joint work with

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§1. Motivation: ergodic averages

Single averages

$$M_n f(x) := \frac{1}{n} \sum_{i=0}^{n-1} f(S^i x)$$

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The setting:

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- $S: X \rightarrow X$ measure-preserving: $\mu(S^{-1}E) = \mu(E)$
- $f \in L^2(X)$

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Convergence as $n \rightarrow \infty$:

- in $L^2(X) \rightsquigarrow$ von Neumann (1932)
- pointwise a.e. \rightsquigarrow Birkhoff (1931)
- Can we quantify the L^2 convergence by controlling the number of jumps in the norm?

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Norm-variation estimate [Jones, Ostrovskii, and Rosenblatt (1996)]

$$\sup_{n_0 < n_1 < \dots < n_m} \sum_{j=1}^m \|M_{n_j} f - M_{n_{j-1}} f\|_{L^2}^2 \leq C \|f\|_{L^2}^2$$

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Consequence

If $\|f\|_{L^2} = 1$, then $(M_n f)_{n=1}^\infty$ has $O(\varepsilon^{-2})$ jumps of size $\geq \varepsilon$ in the L^2 norm

$m_1 < n_1 \leq m_2 < n_2 \leq \dots \leq m_j < n_j$ s.t. $\|M_{n_j} f - M_{m_j} f\|_{L^2} \geq \varepsilon$

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$$M_n(f, g)(x) := \frac{1}{n} \sum_{i=0}^{n-1} f(S^i x) g(T^i x)$$

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- motivated by Furstenberg's work in additive combinatorics

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Convergence as $n \rightarrow \infty$:

- in $L^2(X)$ (and in $L^p(X)$, $p < \infty$) \rightsquigarrow Conze and Lesigne (1984)
- pointwise a.e. \rightsquigarrow an old open problem! (Calderón?)
- quantitative L^2 convergence? metric entropy bounds?

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Previous results:

- $T = S^m$, $m \in \mathbb{Z}$, pointwise a.e. convergence
 \rightsquigarrow Bourgain (1990), Demeter (2007)
- $T = S^m$, $m \in \mathbb{Z}$, pointwise variation estimate
 \rightsquigarrow Do, Oberlin, and Palsson (2015)

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General commuting and measure-preserving S, T :
any explicit quantitative results for norm convergence
 \rightsquigarrow posed by Avigad and Rute (2012) and others

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Norm-variation estimate [Durcik, K., Škreb, and Thiele (2016)]

$$\sup_{n_0 < n_1 < \dots < n_m} \sum_{j=1}^m \|M_{n_j}(f, g) - M_{n_{j-1}}(f, g)\|_{L^2}^2 \leq C \|f\|_{L^4}^2 \|g\|_{L^4}^2$$

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If $\|f\|_{L^4} = \|g\|_{L^4} = 1$, then $(M_n(f, g))_{n=1}^{\infty}$ has $O(\varepsilon^{-2})$ jumps of size $\geq \varepsilon$ in the L^2 norm

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Multiple averages

$$M_n(f_1, f_2, \dots, f_r)(x) := \frac{1}{n} \sum_{i=0}^{n-1} f_1(T_1^i x) f_2(T_2^i x) \cdots f_r(T_r^i x)$$

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The setting:

- $r \geq 3$
- (X, \mathcal{F}, μ) a probability space
- $T_1, T_2, \dots, T_r: X \rightarrow X$ commuting and measure-preserving
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Convergence as $n \rightarrow \infty$:

- in $L^2(X)$ (and in $L^p(X)$, $p < \infty$) \rightsquigarrow Tao (2007) and also by Austin (2008), Walsh (2011), Zorin-Kranich (2011)
- no quantitative norm-convergence results known

§2. (Smooth) averages on \mathbb{R}^2

Bilinear averages

$$A_t^\varphi(F, G)(x, y) := \int_{\mathbb{R}} F(x + s, y)G(x, y + s) t^{-1}\varphi(t^{-1}s) ds$$

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$$\sup_{t_0 < \dots < t_m} \sum_{j=1}^m \|A_{t_j}^\varphi(F, G) - A_{t_{j-1}}^\varphi(F, G)\|_{L^2(\mathbb{R}^2)}^2 \leq C_\varphi \|F\|_{L^4(\mathbb{R}^2)}^2 \|G\|_{L^4(\mathbb{R}^2)}^2$$

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- φ is a Schwartz function

- $\varphi = \mathbb{1}_{[0,1)} \implies A_t(F, G)(x, y) = \frac{1}{t} \int_0^t F(x+s, y)G(x, y+s) ds$

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$t_j = 2^j$, φ Schwartz

$$\psi(s) := \varphi(s) - 2\varphi(2s) \implies \psi_{2^j} = \varphi_{2^j} - \varphi_{2^{j-1}}$$

$$S(F, G)(x, y) := \left(\sum_{j \in \mathbb{Z}} \underbrace{\left| \int_{\mathbb{R}} F(x + s, y) G(x, y + s) \psi_{2^j}(s) ds \right|^2}_{A_{2^j}^\varphi(F, G)(x, y) - A_{2^{j-1}}^\varphi(F, G)(x, y)} \right)^{1/2}$$

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Consequence

$$\|S(F, G)\|_{L^2(\mathbb{R}^2)} \leq C_\psi \|F\|_{L^4(\mathbb{R}^2)} \|G\|_{L^4(\mathbb{R}^2)}$$

Bilinear averages

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$$\tilde{S}(f, g)(x) := \left(\sum_{j \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(x + s) g(x - s) 2^{-j} \psi(2^{-j} s) ds \right|^2 \right)^{1/2}$$

$$F(x, y) := f(x - y) R^{-1/4} \vartheta(R^{-1} y)$$

$$G(x, y) := g(x - y) R^{-1/4} \vartheta(R^{-1} x)$$

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Consequence [Lacey, Thiele (1997), bilinear Hilbert transform]

$$\|\tilde{S}(f, g)\|_{L^2(\mathbb{R})} \leq C_\psi \|f\|_{L^4(\mathbb{R})} \|g\|_{L^4(\mathbb{R})}$$

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Bilinear averages

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Compare with the “triangular Hilbert transform”:

$$T(F, G)(x, y) := \text{p.v.} \int_{\mathbb{R}} F(x + s, y) G(x, y + s) \frac{ds}{s}$$

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- one function specialized \rightsquigarrow K., Thiele, Zorin-Kranich (2015)
- nontrivial cancellation \rightsquigarrow Zorin-Kranich (2015)
- no-estimates known!
- $\text{p.v.} \frac{1}{s} = \sum_j \psi_{2^j}(s) \rightsquigarrow$ the square function vs. the singular integral

§2. (Smooth) averages on \mathbb{R}^2

The “triangular Hilbert transform”:

$$T(F, G)(x, y) := \text{p.v.} \int_{\mathbb{R}} F(x + s, y) G(x, y + s) \frac{ds}{s}$$

It has to be difficult! Dualize:

$$\Lambda(F, G, H) = \int_{\mathbb{R}^2} \text{p.v.} \int_{\mathbb{R}} F(x + s, y) G(x, y + s) H(x, y) \frac{ds}{s} dx dy$$

By taking

$$F(x, y) = f(x), \quad G(x, y) = g(x)e^{iN(x)y}, \quad H(x, y) = h(x)e^{-iN(x)y}$$

we obtain bounds for the linearized Carleson operator

$$(Cf)(x) = \int_{\mathbb{R}} f(x + s) e^{iN(x)s} \frac{ds}{s}$$

$$\Lambda(F, G, H) = \langle Cf, gh \rangle$$

↪ Carleson (1966)

§2. (Smooth) averages on \mathbb{R}^2

Bilinear averages on \mathbb{Z}^2

$$\tilde{A}_n(\tilde{F}, \tilde{G})(k, l) := \frac{1}{n} \sum_{i=0}^{n-1} \tilde{F}(k+i, l) \tilde{G}(k, l+i)$$

§2. (Smooth) averages on \mathbb{R}^2

Bilinear averages on \mathbb{Z}^2

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Norm-variation estimate [DKŠT (2016)]

$$\sup_{n_0 < \dots < n_m} \sum_{j=1}^m \|A_{n_j}(\tilde{F}, \tilde{G}) - A_{n_{j-1}}(\tilde{F}, \tilde{G})\|_{\ell^2(\mathbb{Z}^2)}^2 \leq C \|\tilde{F}\|_{\ell^4(\mathbb{Z}^2)}^2 \|\tilde{G}\|_{\ell^4(\mathbb{Z}^2)}^2$$

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Calderón's transference: $\mathbb{R}^2 \dashrightarrow \mathbb{Z}^2 \dashrightarrow (X, \mathcal{F}, \mu, S, T)$

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Calderón's transference:

$$\begin{array}{ccc} \mathbb{R}^2 & \dashrightarrow & \mathbb{Z}^2 \\ F, G & & \tilde{F}, \tilde{G} \end{array} \quad \dashrightarrow \quad \begin{array}{c} (X, \mathcal{F}, \mu, S, T) \\ f, g \end{array}$$

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$$F(x, y) := \sum_{i, l \in \mathbb{Z}} \tilde{F}(i - l, l) \mathbb{1}_{[i, i+1)}(x + y) \mathbb{1}_{[l, l+1)}(y)$$

$$G(x, y) := \sum_{i, k \in \mathbb{Z}} \tilde{G}(k, i - k) \mathbb{1}_{[k, k+1)}(x) \mathbb{1}_{[i, i+1)}(x + y)$$

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$$\tilde{F}_x(k, l) := f(S^k T^l x)$$

$$\tilde{G}_x(k, l) := g(S^k T^l x)$$

§3. Cheap tricks

- just the ideas, technical oversimplifications
- sometimes work in discrete, sometimes in continuous scales
- sometimes apply before, sometimes after decompositions

§3. Cheap tricks, #1 Long and short variations

Split into “long” and “short” variations

[Jones, Seeger, Wright (2004)]:

$$t_0 < t_1 < \cdots < t_{j-1} < t_j < \cdots < t_m$$

Insert the closest powers 2^k , $k \in \mathbb{Z}$

- Long variation / long jumps:
corresponding to the scales $t_j \in \{2^k : k \in \mathbb{Z}\}$
- Short variation / short jumps:
corresponding to t_j from a fixed interval $[2^k, 2^{k+1}]$

§3. Cheap tricks, #2 Telescoping

Expand out the L^2 norm for the long variation:

$$\begin{aligned} & \sum_{j=1}^m \|A_{2^{k_j}}^\varphi(F, G) - A_{2^{k_{j-1}}}^\varphi(F, G)\|_{L^2(\mathbb{R}^2)}^2 \\ &= \sum_{j=1}^m \int_{\mathbb{R}^4} F(x+u, y)G(x, y+u)F(x+v, y)G(x, y+v) \\ & \quad (\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v) \, dx dy du dv \end{aligned}$$

A telescoping identity:

$$\begin{aligned} & (\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v) \\ &= \underbrace{\varphi_{2^{k_{j-1}}}(u)\varphi_{2^{k_{j-1}}}(v) - \varphi_{2^{k_j}}(u)\varphi_{2^{k_j}}(v)}_{\text{telescopes into a single-scale quantity}} \\ & \quad + \varphi_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v) \\ & \quad + (\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(u)\varphi_{2^{k_j}}(v) \end{aligned}$$

§3. Cheap tricks, #2 Telescoping

Kernel of the 4-linear form:

$$K(u, v) := \sum_{j=1}^m \varphi_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)$$

Denote $\psi := \varphi - \varphi_2$

$$\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}} = \sum_{l=k_{j-1}}^{k_j-1} \psi_{2^l}$$

Leads to a single-index sum (one parameter),
not a double-index one (two parameters)

§3. Cheap tricks, #3 Moving the cancellation

$$F(x+u, y)G(x, y+u)F(x+v, y)G(x, y+v) \varphi(u)\psi(v)$$

Change of variables:

$$z = x + y + u, \quad w = x + y + v$$

leads to:

$$\tilde{F}(y, z)\tilde{G}(x, z)\tilde{F}(y, w)\tilde{G}(x, w) \varphi(z - x - y)\psi(w - x - y)$$

The Fourier inversion formula:

$$\begin{aligned} & \varphi(z - x - y)\psi(w - x - y) \\ &= \int_{\mathbb{R}^2} \widehat{\varphi}(\xi)\widehat{\psi}(\eta)e^{2\pi i\xi(z-x-y)}e^{2\pi i\eta(w-x-y)}d\xi d\eta \end{aligned}$$

§3. Cheap tricks, #3 Moving the cancellation

$$\begin{aligned} & \varphi(z - x - y)\psi(w - x - y) \\ &= \int_{\mathbb{R}^2} \widehat{\varphi}(\xi)\widehat{\psi}(\eta)e^{-2\pi i(\xi+\eta)x}e^{-2\pi i(\xi+\eta)y}e^{2\pi i\xi z}e^{2\pi i\eta w}d\xi d\eta \end{aligned}$$

If φ has only “low” frequencies and ψ has only “high frequencies”, then (ξ, η) is far from the antidiagonal $\eta = -\xi$

$\implies e^{-2\pi i(\xi+\eta)x}$ and $e^{-2\pi i(\xi+\eta)y}$ bring useful cancellation

Approach $\eta = -\xi$ using an appropriate L-P decomposition

§3. Cheap tricks, #4 Positivity

Continuous scales are easier [Durcik (2014)]

Take a Gaussian cutoff function: $g(s) := e^{-\pi s^2}$

$$\Theta(F) := \sum_{j=1}^m \int_{\mathbb{R}^4} F(y, z) F(x, z) F(y, w) F(x, w) \\ g_{2^{k_j}}(z - w) (g_{2^{k_{j-1}}} - g_{2^{k_j}})(x - y) dx dy dz dw$$

$$\tilde{\Theta}(F) := \sum_{j=1}^m \int_{\mathbb{R}^4} F(y, z) F(x, z) F(y, w) F(x, w) \\ (g_{2^{k_{j-1}}} - g_{2^{k_j}})(z - w) g_{2^{k_{j-1}}}(x - y) dx dy dz dw$$

Integration by parts and/or summation by parts:

$$\underbrace{\Theta(F)}_{\geq 0} + \underbrace{\tilde{\Theta}(F)}_{\geq 0} = \underbrace{\text{a single scale quantity}}_{\text{easy to bound}}$$

§3. Cheap tricks, #5 Dominating by the Gaussians

Can we dominate a Schwartz function by the Gaussian function? Tail decaying too rapidly!

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Can we dominate a Schwartz function by the Gaussian function? Tail decaying too rapidly!

We can dominate by a superposition of the Gaussians
[Durcik (2014)]:

$$g(s) := e^{-\pi s^2}, \quad \sigma(s) := \int_1^\infty g_\alpha(s) \alpha^{-\lambda} d\alpha, \quad \lambda > 1$$

$$\lim_{|s| \rightarrow \infty} |s|^\lambda \sigma(s) \in (0, \infty) \implies (1 + |s|)^{-\lambda} \lesssim_\lambda \sigma(s)$$

§4. The actual auxiliary results

Proposition

Let $\lambda > 1$ and let $\vartheta, \varphi \in \mathcal{S}(\mathbb{R})$ be such that

$$|\vartheta(s)| \leq (1 + |s|)^{-\lambda}, \quad |\varphi(s)| \leq (1 + |s|)^{-\lambda}.$$

Assume that $\widehat{\vartheta}$ is supported in $[-2^{-4}, 2^{-4}]$, while $\widehat{\varphi}$ is supported in $[-1, 1]$ and constant on $[-2^{-2}, 2^{-2}]$.

Then for any $m \in \mathbb{N}$, $k_0, \dots, k_m \in \mathbb{Z}$, and any $F, G \in \mathcal{S}(\mathbb{R}^2)$ we have

$$\left| \sum_{j=1}^m \int_{\mathbb{R}^4} F(x+u, y)G(x, y+u)F(x+v, y)G(x, y+v) \vartheta_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v) dx dy du dv \right| \lesssim_{\lambda} \|F\|_{L^4(\mathbb{R}^2)}^2 \|G\|_{L^4(\mathbb{R}^2)}^2.$$

§4. The actual auxiliary results

Proposition

Let $\lambda > 1$ and let $\Phi \in \mathcal{S}(\mathbb{R}^2)$ be such that

$$|\Phi(u, v)| \leq (1 + |u + v|)^{-\lambda} (1 + |u - v|)^{-2\lambda}.$$

Assume that $\widehat{\Phi}$ is supported in $([-2, -2^{-5}] \cup [2^{-5}, 2])^2$.

Then for any $F, G \in \mathcal{S}(\mathbb{R}^2)$ and any $N \in \mathbb{N}$ we have

$$\left| \sum_{j=-N}^N \int_{\mathbb{R}^4} F(x + u, y) G(x, y + u) F(x + v, y) G(x, y + v) \Phi_{2^j}(u, v) dx dy du dv \right| \lesssim_{\lambda} \|F\|_{L^4(\mathbb{R}^2)}^2 \|G\|_{L^4(\mathbb{R}^2)}^2.$$

§4. The actual auxiliary results

Treating the long variation:

$$V(F, G) := \sum_{j=1}^m \left\| A_{2^{k_j}}^\varphi(F, G) - A_{2^{k_{j-1}}}^\varphi(F, G) \right\|_{L^2(\mathbb{R}^2)}^2$$

$$\varphi = \varphi * \chi + \varphi * \omega$$

$$\text{supp } \widehat{\chi} \subseteq [-2^{-4}, 2^{-4}], \quad \text{supp } \widehat{\omega} \subseteq [-2, -2^{-5}] \cup [2^{-5}, 2]$$

$$\begin{aligned} \Lambda(F, G) := \sum_{j=1}^m \int_{\mathbb{R}^4} & F(x+u, y)G(x, y+u)F(x+v, y)G(x, y+v) \\ & (\varphi * \chi)_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v) \, dx dy du dv \end{aligned}$$

$$\widetilde{\Lambda}(F, G) := \sum_{j=-N}^N \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} F(x+u, y)G(x, y+u)(\varphi * \omega)_{2^j}(u) \, du \right)^2 dx dy$$

A bootstrapping estimate:

$$V(F, G) \lesssim \|F\|_{L^4(\mathbb{R}^2)}^2 \|G\|_{L^4(\mathbb{R}^2)}^2 + |\Lambda(F, G)| + \widetilde{\Lambda}(F, G)^{1/2} V(F, G)^{1/2}$$

Thank you for your attention!