Neke primjene multilinearnih singularnih integrala

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PMF, Sveučilište u Splitu

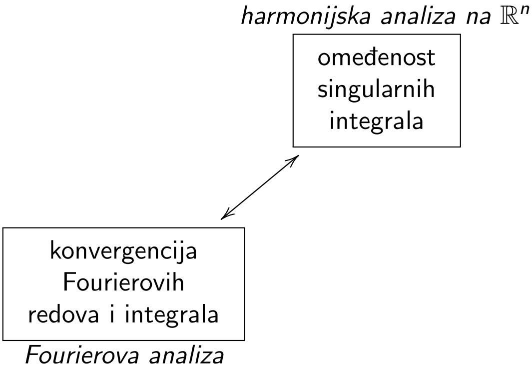
18. 1. 2019.

Veze s drugim područjima

harmonijska analiza na $\mathbb{R}^n$

omeđenost
singularnih
integrala

Veze s drugim područjima



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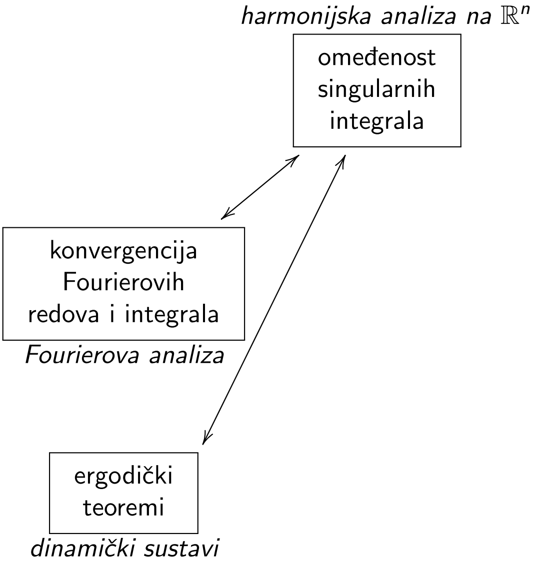

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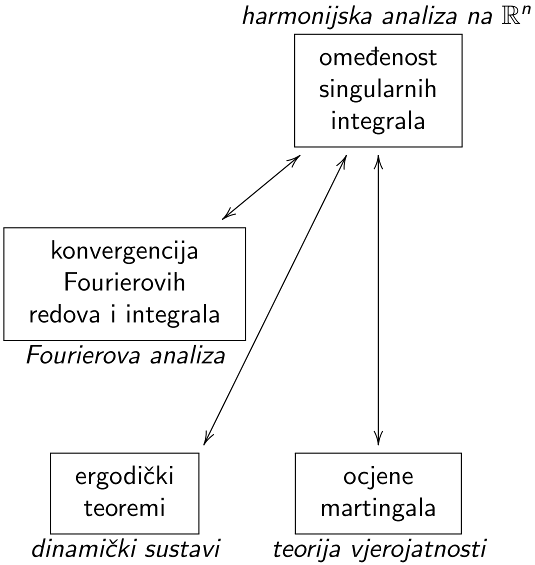

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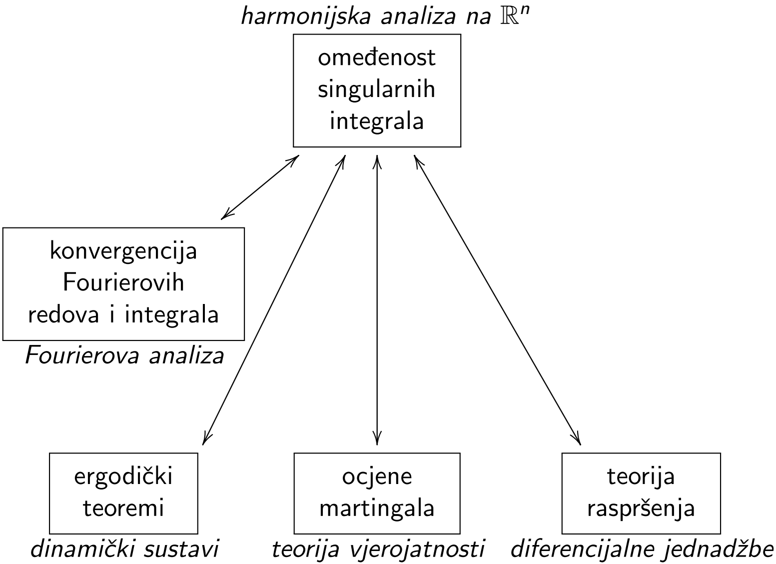

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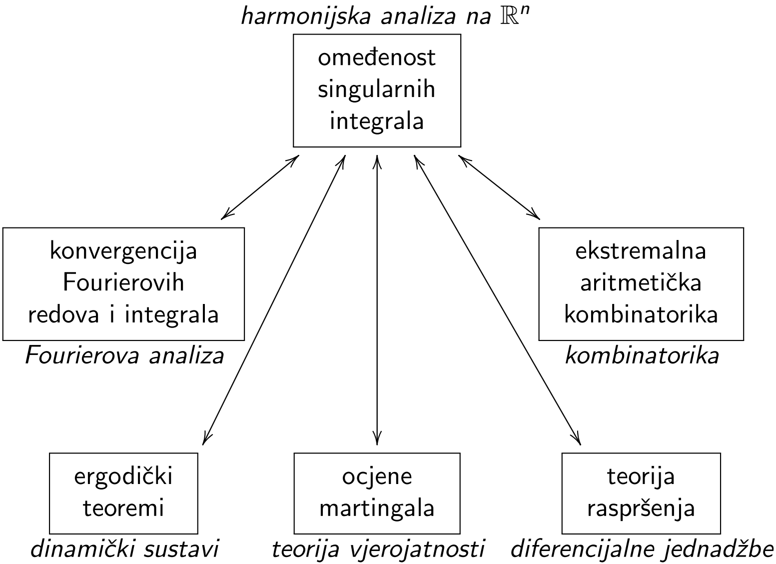

Singularni integral

Integralni operator sa singularnom jezgrom

Linearan

$$(Tf)(x) \ := \ p.v. \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

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Jezgra $K(x,y)$ je "kontrolirano singularna" blizu dijagonale $x = y$
Npr. $K(x,y) = \frac{1}{x-y}$ (za $n = 1$) ili $K(x,y) = \frac{x_1 - y_1}{|x-y|^{n+1}}$ (za $n \geq 2$)

Glavna vrijednost integrala: $p.v. \int_{\mathbb{R}^n} \ = \ \lim_{\varepsilon \searrow 0} \int_{\{y \in \mathbb{R}^n : |x-y| > \varepsilon\}}$

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$$(Tf)(x) \ := \ p.v. \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

Zanimaju nas L^p ocjene:

$$\|Tf\|_{L^p(\mathbb{R}^n)} \ \leq \ C_p \, \|f\|_{L^p(\mathbb{R}^n)}$$

$\|f\|_{L^p(\mathbb{R}^n)} := \big(\int_{\mathbb{R}^n} |f(x)|^p dx \big)^{1/p}, \quad 0 < p < \infty$
$\|f\|_{L^\infty(\mathbb{R}^n)} := ess\, sup_{x \in \mathbb{R}^n} |f(x)|$

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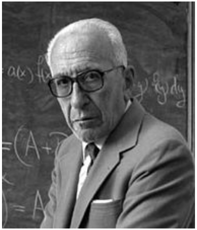
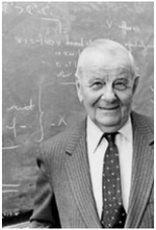


Alberto P. Calderón
(1920.–1998.)



Antoni Zygmund
(1900.–1992.)

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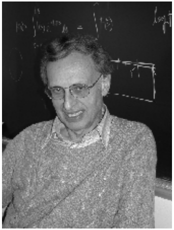
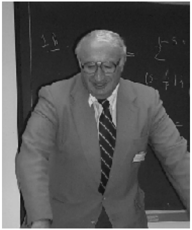


Elias M. Stein
(1931.–2018.)



Guido L. Weiss

Singularni integral

Integralni operator sa singularnom jezgrom

Maksimalan

$$(T^{max} f)(x) \ := \ \sup_{\varepsilon>0} \left| \int_{\{y\in\mathbb{R}^n \,:\, |x-y|>\varepsilon\}} K_\varepsilon(x,y) f(y) dy \right|$$

$K_\varepsilon(x,y) = K(x,y)$ ili $K_\varepsilon(x,y) = \frac{1}{\varepsilon^n} K(\frac{1}{\varepsilon}x, \frac{1}{\varepsilon}y)$

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Varijacijski

$$(T^{r-var}f)(x) \ := \ \Big\| \int_{\{y\in\mathbb{R}^n : |x-y|>\varepsilon\}} K_\varepsilon(x,y)f(y)dy \Big\|_{V^r_\varepsilon}$$

$$\|\theta(\varepsilon)\|_{V^r_\varepsilon} := \sup_{0<\varepsilon_0<\varepsilon_1<\ldots<\varepsilon_m} \big(\textstyle\sum_{j=1}^m |\theta(\varepsilon_j) - \theta(\varepsilon_{j-1})|^r \big)^{1/r} + \sup_{\varepsilon>0} |\theta(\varepsilon)|$$

$\|\theta\|_{V^r_\varepsilon}$ daje kvantitativnu ocjenu broja skokova funkcije θ

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Multilinearan

$$(f_1, f_2, \ldots, f_k) \mapsto T(f_1, f_2, \ldots, f_k)$$

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$$(f_1, f_2, \ldots, f_k) \mapsto T(f_1, f_2, \ldots, f_k)$$

$$\begin{aligned} T(f_1, f_2, \ldots, f_k)(x) \; := p.v. \int_\Omega & K(x, y_1, y_2, \ldots, y_k) \\ & f_1(y_1) f_2(y_2) \ldots f_k(y_k)\, d\sigma(y_1, y_2, \ldots, y_k) \end{aligned}$$

Ω = neka (hiper-)ravnina, σ = Hausdorffova mjera na Ω

Npr. $T(f,g)(x) := p.v. \int_{\mathbb{R}} \frac{1}{t} f(x-t) g(x+t) dt$

$T(f,g)(x,y) := p.v. \int_{\mathbb{R}} \frac{1}{t} f(x+t,y) g(x,y+t) dt$

$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} \frac{s}{(s^2+t^2)^{3/2}} f(x+s,y) g(x,y+t)\, dsdt$

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$$(f_1, f_2, \ldots, f_k) \mapsto T(f_1, f_2, \ldots, f_k)$$

Zanimaju nas L^p ocjene:

$$\| T(f_1, f_2, \ldots, f_k) \|_{L^p(\mathbb{R}^n)} \;\leq\; C_{p,p_1,\ldots,p_k} \prod_{j=1}^{k} \| f_j \|_{L^{p_j}(\mathbb{R}^n)}$$

za eksponente koji zadovoljavaju npr. $\frac{1}{p} = \sum_{j=1}^{k} \frac{1}{p_j}$
(relacija ovisi o "skaliranju", tj. strukturi od T)

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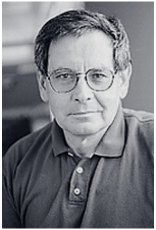
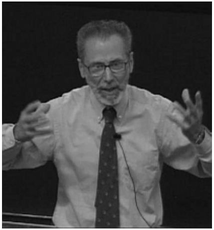


Ronald R. Coifman



Yves F. Meyer

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Michael T. Lacey

Christoph M. Thiele

1. Hilbertova transformacija

$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{\pi t}$$

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$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t) \frac{dt}{\pi t}$$

Rezultati

$\|Tf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}$ za $1 < p < \infty$ (Riesz, 1928.)

$\|T^{r-var} f\|_{L^p(\mathbb{R})} \leq C_{p,r} \|f\|_{L^p(\mathbb{R})}$ za $1 < p < \infty$, $r > 2$
(Campbell, Jones, Reinhold i Wierdl, 2000.)

1. Hilbertova transformacija

$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{\pi t}$$

Tehnike

Fourierova transformacija:

$$\widehat{(Tf)}(\xi) = (-i\, sgn \xi)\, \hat{f}(\xi)$$

Dokaz.

$$(Tf)(x) = \left(p.v. \frac{1}{\pi x}\right) * f(x)$$

$$\widehat{(Tf)}(\xi) = \widehat{\left(p.v. \frac{1}{\pi x}\right)}(\xi)\, \hat{f}(\xi) = (-i\, sgn \xi)\, \hat{f}(\xi)$$

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Fourierova transformacija:

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Izometričnost na $L^2(\mathbb{R})$:

$$\|Tf\|_{L^2(\mathbb{R})} = \|\widehat{Tf}\|_{L^2(\mathbb{R})} = \|(-i\, sgn\xi)\, \hat{f}(\xi)\|_{L^2_\xi(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

po Plancherelovoj jednakosti

1. Hilbertova transformacija

$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{\pi t}$$

Tehnike

Valićne baze:

$\psi_{j,k}(x) := 2^{-j/2}\psi(2^{-j}x - k)$ i.e. $\psi_{j,k} = Dil_{2^j} Tr_k \psi$
$(Dil_a f)(x) := a^{-1/2} f(a^{-1}x)$, $(Tr_k f)(x) := f(x-k)$
$\left(\psi_{j,k}\right)_{j,k\in\mathbb{Z}}$ je ortonormirana baza od $L^2(\mathbb{R})$

T prevodi valićnu bazu u valićnu bazu

1. Hilbertova transformacija

$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{\pi t}$$

Tehnike

T prevodi valićnu bazu u valićnu bazu

Dokaz. T je unitaran $\Rightarrow$ prevodi ONB u ONB
T komutira s translacijama i dilatacijama:

$$T(Tr_k f)(x) = p.v. \int_{\mathbb{R}} f(x-t-k)\frac{dt}{\pi t} = (Tf)(x-k) = Tr_k(Tf)(x)$$

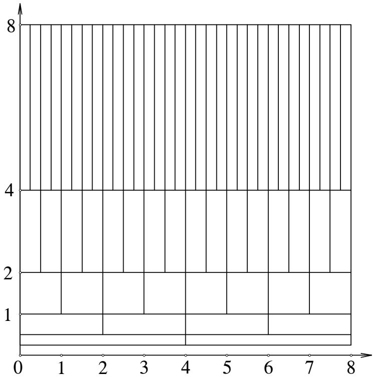
$$T(Dil_a f)(x) = p.v. \int_{\mathbb{R}} \frac{1}{\sqrt{a}} f\left(\frac{x-t}{a}\right)\frac{dt}{\pi t} = \frac{1}{\sqrt{a}}(Tf)\left(\frac{x}{a}\right) = Dil_a(Tf)(x)$$

1. Hilbertova transformacija

$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{\pi t}$$

Tehnike

Vremensko-frekvencijski portret:



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Primjena: Veza s Fourierovom analizom

Konvergencija Fourierovog integrala/reda u L^p normi

(Riesz, 1928.)

Za $f \in L^p(\mathbb{R})$, $1 < p \leq 2$ vrijedi

$$\lim_{N\to\infty} \int_{-N}^{N} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = f(x) \qquad \text{u } L^p(\mathbb{R})$$

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Dokaz.

$\widehat{(Tf)}(\xi) = (-i\, sgn\xi)\, \hat{f}(\xi)$

$\Rightarrow (Tf)(x) = \int_{\mathbb{R}} (-i\, sgn\xi)\, \hat{f}(\xi) e^{2\pi i x\xi} d\xi$

$\Rightarrow (\frac{1}{2} I + \frac{i}{2} T) f(x) = \int_0^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi$

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Dokaz.

Modulacije: $(Mod_\eta f)(x) := e^{2\pi i \eta x} f(x)$

$\int_{-N}^{N} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \frac{i}{2}\big(Mod_{-N}T\,Mod_N - Mod_N T\,Mod_{-N}\big)$

Slijedi uniformna omeđenost parcijalnih integrala u $L^p(\mathbb{R})$

1. Hilbertova transformacija

$$(Tf)(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{\pi t}$$

Poopćenje

Calderón-Zygmundovi operatori

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

"Skoro dijagonalni" u C^1 valićnoj bazi

T(1) teorem (David i Journé, 1984.)

Karakterizacija omeđenosti na $L^2(\mathbb{R}^n)$

2. Hardy-Littlewoodova maksimalna funkcija

$$(T^{max} f)(x) := \sup_{\varepsilon>0} \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x-t) dt \right|$$

2. Hardy-Littlewoodova maksimalna funkcija

$$(T^{max}f)(x) := \sup_{\varepsilon>0}\left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x-t)dt\right|$$

Rezultati

$\|T^{max}f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}$ za $1 < p \leq \infty$

(Hardy i Littlewood, 1910-e)

$\|T^{r-var}f\|_{L^p(\mathbb{R})} \leq C_{p,r} \|f\|_{L^p(\mathbb{R})}$ za $1 < p < \infty$, $r > 2$

(Bourgain 1989.; Jones, Rosenblatt i Wierdl, 2000.)

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Tehnika

Geometrijska teorija pokrivača

Za $\alpha > 0$ postoji familija disjunktnih intervala $\mathcal{I}$ t.d.

- $\frac{1}{|I|} \int_I |f| > \alpha$ za svaki $I \in \mathcal{I}$
- $(3I)_{I \in \mathcal{I}}$ prekrivaju $\{|T^{max} f| > \alpha\}$

$$\alpha \left|\{|T^{max} f| > \alpha\}\right| \leq 3\alpha \sum_{I \in \mathcal{I}} |I| \leq 3 \int_I |f| \leq 3\|f\|_{L^1(\mathbb{R})}$$

Interpolacija između

$\|T^{max} f\|_{L^{1,slabi}(\mathbb{R})} \leq 3\|f\|_{L^1(\mathbb{R})}$ i

$\|T^{max} f\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}$

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Primjena: Veza s ergodičkom teorijom
Ergodička teorija proučava sisteme koji čuvaju mjeru:
$(X, \mathcal{F}, \mu)$ vjerojatnosni prostor,
$S\colon X \to X$, $\mu(S^{-1}B) = \mu(B)$, $f \in L^\infty(X)$

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Točkovni ergodički teorem (Birkhoff, 1931.)

$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(S^n x)$ postoji za μ-g.s. $x \in X$

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Princip prijenosa (Calderón, 1968.)

Ocjene za $\frac{1}{N} \sum_{n=0}^{N-1} f(S^n x)$ $\Leftrightarrow$ ocjene za $\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x-t)dt$

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Dokaz. ($\neq$ od originalnog)
$\Big\| \big\| \frac{1}{N} \sum_{n=0}^{N-1} f(S^n x) \big\|_{V_N^r} \Big\|_{L^p(X)} \leq C_{p,r} \|f\|_{L^p(X)} < \infty$
$\Rightarrow$ $\big\| \frac{1}{N} \sum_{n=0}^{N-1} f(S^n x) \big\|_{V_N^r} < \infty$ za g.s. $x \in X$
$\Rightarrow$ $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(S^n x)$ postoji za g.s. $x \in X$

3. Carlesonov operator

$$(T^{max}f)(x) := \sup_{\eta\in\mathbb{R}} \left| p.v. \int_{\mathbb{R}} f(x-t)\frac{e^{2\pi i\eta t}}{t}dt \right|$$

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Rezultati

$\|T^{max}f\|_{L^p(\mathbb{R})} \leq C\,\|f\|_{L^p(\mathbb{R})}$ za $1 < p < \infty$
($p = 2$, Carleson, 1966.; $p \neq 2$, Hunt, 1967.)

$\|T^{r-var}f\|_{L^p(\mathbb{R})} \leq C_r\,\|f\|_{L^p(\mathbb{R})}$ za $r > 2$, $r' < p < \infty$
(Oberlin, Seeger, Tao, Thiele i Wright, 2010.)

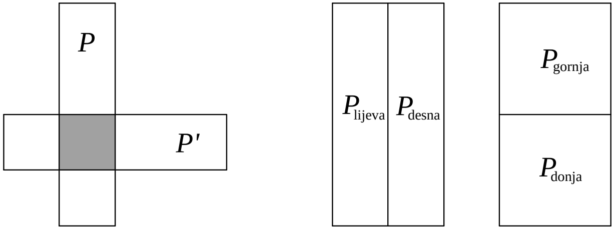
3. Carlesonov operator

$$(T^{max}f)(x) := \sup_{\eta\in\mathbb{R}} \left| p.v. \int_{\mathbb{R}} f(x-t)\frac{e^{2\pi i\eta t}}{t}dt \right|$$

Tehnika (Carleson, 1966.; Fefferman, 1973.; Lacey i Thiele, 2000.)

Valni paketi:
$\psi_{j,k,\ell} := Dil_{2^j} Tr_k Mod_\ell \psi, \quad j,k,\ell \in \mathbb{Z}$

Vremensko-frekvencijski portret:



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Primjena: Veza s Fourierovom analizom

Točkovna konvergencija Fourierovog integrala/reda L^p funkcije
($p = 2$, Carleson, 1966.; $p \neq 2$, Hunt, 1967.)

Za $f \in L^p(\mathbb{R})$, $1 < p \leq 2$ vrijedi

$$\lim_{N \to \infty} \int_{-N}^{N} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = f(x) \qquad \text{za g.s. } x \in \mathbb{R}$$

4. Bilinearna Hilbertova transformacija

$$T(f, g)(x) := p.v. \int_{\mathbb{R}} f(x - t)g(x - \alpha t)\frac{dt}{t}$$

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Rezultat (Lacey i Thiele, 1999.)

$\|T(f,g)\|_{L^p(\mathbb{R})} \leq C\,\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}$

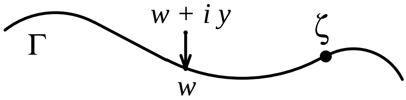
za $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < p_1, p_2 \leq \infty$, $\frac{2}{3} < p < \infty$

4. Bilinearna Hilbertova transformacija

$$T(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)g(x-\alpha t)\frac{dt}{t}$$

Motivacija: Iz kompleksne analize
Cauchyjev integral duž Lipschitzove krivulje Γ:

$$(C_\Gamma f)(w) := \lim_{y \searrow 0} \int_\Gamma \frac{f(\zeta)}{\zeta - (w+iy)} d\zeta$$



4. Bilinearna Hilbertova transformacija

$$T(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)g(x-\alpha t)\frac{dt}{t}$$

Motivacija: Iz kompleksne analize
Cauchyjev integral duž Lipschitzove krivulje Γ:

$$(C_\Gamma f)(w) := \lim_{y\searrow 0} \int_\Gamma \frac{f(\zeta)}{\zeta-(w+iy)}d\zeta$$

Želimo $\|C_\Gamma f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}$

Parametrizacija:
$\zeta = t + i\,\gamma(t), \quad d\zeta = (1+i\gamma'(t))dt, \quad w = x + i\,\gamma(x)$

Razvoj u red i gledanje prvog netrivijalnog člana vode na problem
$\|T(f,\gamma')\|_{L^2(\mathbb{R})} \leq C\,\|f\|_{L^2(\mathbb{R})} \underbrace{\|\gamma'\|_{L^\infty(\mathbb{R})}}_{<\infty}$

4. Bilinearna Hilbertova transformacija

$$T(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)g(x-\alpha t)\frac{dt}{t}$$

Tehnika: Proizlazi iz dokaza omeđenosti Carlesonovog operatora

Trilinearna forma:

$$\Lambda(f,g,h) := \int_{\mathbb{R}} p.v. \int_{\mathbb{R}} f(x-t)g(x-\alpha t)\frac{dt}{t}\, h(x)dx$$

Valni paket:
$\psi_{j,k,\ell} := Dil_{2^j} Tr_k Mod_\ell \psi, \quad j,k,\ell \in \mathbb{Z}$

Dekomponiramo f, g, h u pogodno odabrane elemente sistema valnih paketa

4. Bilinearna Hilbertova transformacija

$$T(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)g(x-\alpha t)\frac{dt}{t}$$

Poopćenje

Multilinearni operatori s modulacijskom simetrijom

Npr. za BHT:

$$\begin{aligned}
&T(Mod_{-\alpha\beta}f, Mod_{\beta}g)(x)\\
&= p.v. \int_{\mathbb{R}} f(x-t)e^{-2\pi i\alpha\beta(x-t)}g(x-\alpha t)e^{2\pi i\beta(x-\alpha t)}\frac{dt}{t}\\
&= e^{2\pi i(1-\alpha)\beta x}\ p.v. \int_{\mathbb{R}} f(x-t)g(x-\alpha t)\frac{dt}{t}\\
&= Mod_{(1-\alpha)\beta}T(f,g)(x)
\end{aligned}$$

5. Trokutasta Hilbertova transformacija

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}} f(x+t,y)g(x,y+t)\frac{dt}{t}$$

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$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}} f(x+t,y)g(x,y+t)\frac{dt}{t}$$

Slutnja (Demeter i Thiele, 2010.)

$\|T(f,g)\|_{L^p(\mathbb{R}^2)} \leq C\,\|f\|_{L^{p_1}(\mathbb{R}^2)}\|g\|_{L^{p_2}(\mathbb{R}^2)}$ za neke p, p_1, p_2

Jača slutnja

$\|T^{r-var}(f,g)\|_{L^p(\mathbb{R}^2)} \leq C\,\|f\|_{L^{p_1}(\mathbb{R}^2)}\|g\|_{L^{p_2}(\mathbb{R}^2)}$ za neke p, p_1, p_2, r

5. Trokutasta Hilbertova transformacija

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}} f(x+t,y)g(x,y+t)\frac{dt}{t}$$

Potencijalna primjena: U ergodičkoj teoriji

$(X, \mathcal{F}, \mu)$ vjerojatnosni prostor,
$S_1, S_2 \colon X \to X$ čuvaju mjeru μ, $S_1 S_2 = S_2 S_1$, $f, g \in L^\infty(X)$

Slutnja

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(S_1^n x) g(S_2^n x) \quad \text{postoji za g.s. } x \in X$$

Slijedila bi iz ocjene za T^{r-var} za neki $r < \infty$

Konvergencija u L^2 normi: Conze i Lesigne, 1984.
Za više komutirajućih transformacija u L^2 normi: Tao, 2007.

5. Trokutasta Hilbertova transformacija

Modifikacija: za $\widehat{\varphi}_k \approx \mathbf{1}_{[2^k,2^{k+1}]}$ definiramo "kvadratnu funkciju"

$$S(f,g)(x,y) := \Big(\sum_{k\in\mathbb{Z}} \Big| \int_{\mathbb{R}} f(x+t,y)g(x,y+t)\varphi_k(t)dt \Big|^2 \Big)^{1/2}$$

5. Trokutasta Hilbertova transformacija

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Primjena #1: U ergodičkoj teoriji

$M_N(f,g)(x) := \frac{1}{N} \sum_{n=0}^{N-1} f(S_1^n x) g(S_2^n x)$

Rezultat (Durcik, K., Škreb i Thiele, 2016.)

Bilinearna ergodička usrednjenja $M_N(f,g)$ čine $O(\varepsilon^{-2})$ skokova veličine $\geq \varepsilon$ u L^2 normi

$A_N(f,g)(x,y) := \frac{1}{N} \int_0^N f(x+t,y) g(x,y+t) dt$

$$\|S(f,g)\|_{L^2}^2 \approx \sum_{k\in\mathbb{Z}} \|A_{2^{k+1}}(f,g) - A_{2^k}(f,g)\|_{L^2}^2$$

5. Trokutasta Hilbertova transformacija

Modifikacija: za $\widehat{\varphi}_k \approx \mathbf{1}_{[2^k, 2^{k+1}]}$ definiramo "kvadratnu funkciju"

$$S(f,g)(x,y) := \Big(\sum_{k\in\mathbb{Z}} \Big| \int_{\mathbb{R}} f(x+t,y) g(x,y+t) \varphi_k(t) dt \Big|^2 \Big)^{1/2}$$

Primjena #2: U aritmetičkoj kombinatorici od $\mathbb{R}^n$

U gornjem izrazu prepoznajemo "ugao" (x,y), $(x+t,y)$, $(x,y+t)$

Rezultat (Durcik, K. i Rimanić, 2016.)

Neka su $1 < p < \infty$, $p \neq 2$ i n "dovoljno velika" dimenzija
Skup $A \subseteq \mathbb{R}^n \times \mathbb{R}^n$ pozitivne gustoće sadrži uglove za sve t takve da je $\|t\|_{\ell_p} \geq c(A)$.

Tvrdnja ne vrijedi za $p = 2$ (euklidska norma): Bourgain, 1986.
Tvrdnja za x, $x+t$, $x+2t$: Cook, Magyar i Pramanik, 2015.
Neki drugi uzorci: Durcik i K., 2018.

6. Zapetljani paraprodukt

$$T(f, g)(x, y) := p.v. \int_{\mathbb{R}^2} K(s, t)\, f(x - s, y) g(x, y - t)\, ds\, dt$$

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$

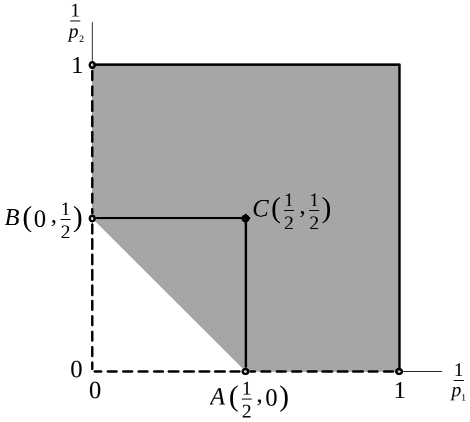
Motivacija:

Ocjene za $\triangle$HT $\Rightarrow$ ocjene za zapetljani paraprodukt
(barem za neke jezgre K, Stein-Weissova metoda rotacija)

Jedini slučaj 2D bilinearne Hilbertove transformacije kojeg nisu bili razriješili Demeter i Thiele, 2010.

6. Zapetljani paraprodukt

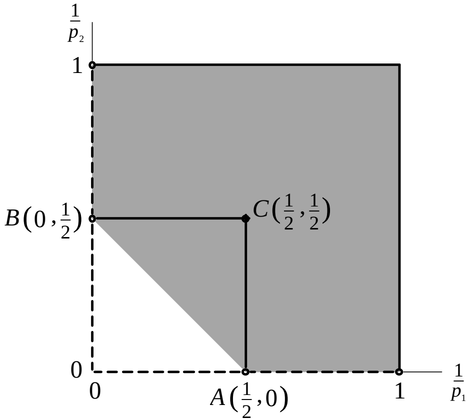
$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$



- osjenčano područje – jaka ocjena na $L^{p_1} \times L^{p_2}$
- podebljane stranice kvadrata – slaba ocjena
- crtkane stranice kvadrata – ne vrijede ocjene
- bijelo područje – nerazriješeno

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v.\int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$



- $\triangle ABC$ – direktan dokaz, K., 2010.
- ostatak osjenčanog područja – uvjetni dokaz, Bernicot, 2010.
- iscrtkane stranice – protuprimjeri, K., 2010.

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y)g(x,y-t)\, ds\, dt$$

Tehnike

Iz aritmetičke kombinatorike

Gowersov $\Box$-skalarni produkt i $\Box$-norma, 1998.

$$[f_1, f_2, f_3, f_4]_{\Box(I\times J)} :=$$
$$\frac{1}{|I\times J|^2} \int_I \int_I \int_J \int_J f_1(u,v) f_2(x,v) f_3(u,y) f_4(x,y)\, du\, dx\, dv\, dy$$
$$\|f\|_{\Box(I\times J)} := [f,f,f,f]^{1/4}_{\Box(I\times J)}$$

U Gowersovom "kvantitativnom" dokazu Szemeredijevog teorema i Škredovljevom teoremu o uglovima

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$

Tehnike

Iz Bellmanove teorije kontrole

Ako želimo kontrolirati (višeskalnu) veličinu $\sum_{n=0}^{N-1} \mathcal{A}_n$, onda je dovoljno konstruirati veličinu $\mathcal{B}_n$ koju možemo kontrolirati i koja zadovoljava $|\mathcal{A}_n| \le \mathcal{B}_{n+1} - \mathcal{B}_n$

Teleskopiranje:

$$\Big| \sum_{n=0}^{N-1} \mathcal{A}_n \Big| \le \mathcal{B}_N - \mathcal{B}_0$$

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$

Poopćenje #1

$E \subseteq \{1,\ldots,m\} \times \{1,\ldots,n\}$

G = jednostavni bipartitni neusmjereni graf na
vrhovima $\{x_1,\ldots,x_m\}$ i $\{y_1,\ldots,y_n\}$
x_i—$y_j \;\Leftrightarrow\; (i,j) \in E$

Zapetljani Calderón-Zygmundov operator ($|E|$-linearna forma):

$$\Lambda\big((f_{i,j})_{(i,j)\in E}\big) := \int_{\mathbb{R}^{m+n}} K(x_1,\ldots,x_m,y_1,\ldots,y_n) \prod_{(i,j)\in E} f_{i,j}(x_i,y_j)\, dx_1 \ldots dx_m dy_1 \ldots dy_n$$

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y)g(x,y-t)\, ds\, dt$$

Poopćenje #1

Zapetljani T(1) teorem (K. i Thiele, 2013.)

Potpuna karakterizacija omeđenosti za dijadske multilinearne forme gornjeg oblika

Poopćenje na funkcije više varijabli i forme pridružene hipergrafovima: Stipčić, 2019.

Parcijalni rezultati u realnom (ne-dijadskom) slučaju: Durcik, 2014., 2015., Durcik i Thiele, 2018.

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$

Poopćenje #2

L^p ocjene ... Soboljevljeve ocjene

Bilinearni množitelj:

$$T(f,g)(x,y) := \int_{\mathbb{R}^4} m(\xi_1,\eta_2) e^{2\pi i(x(\xi_1+\eta_1)+y(\xi_2+\eta_2))} \widehat{f}(\xi_1,\xi_2)\widehat{g}(\eta_1,\eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

pri čemu simbol $m(\xi_1,\xi_2,\eta_1,\eta_2) = m(\xi_1,\eta_2)$ zadovoljava

$$\left|\partial_{\xi_1}^{\alpha_1}\partial_{\eta_2}^{\alpha_2} m(\xi_1,\eta_2)\right| \leq C_{\alpha_1,\alpha_2}(|\xi_1|+|\eta_2|)^{-\alpha_1-\alpha_2}$$

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y) g(x,y-t)\, ds\, dt$$

Poopćenje #2

$$T(f,g)(x,y) := \int_{\mathbb{R}^4} m(\xi_1,\eta_2) e^{2\pi i(x(\xi_1+\eta_1)+y(\xi_2+\eta_2))} \widehat{f}(\xi_1,\xi_2)\widehat{g}(\eta_1,\eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

Rezultat (Bernicot i K., 2013.)

Ako je $\operatorname{supp} m \subseteq \{(\xi_1,\eta_2) : |\xi_1| \leq c\,|\eta_2|\}$, tada vrijedi ocjena

$$\|T(f,g)\|_{L^r_y(\mathrm{W}^{s,r}_x)} \leq C_{p,q,r,s} \|f\|_{L^p} \|g\|_{\mathrm{W}^{s,q}}$$

za $s \geq 0$, $1 < p,q < \infty$, $1 < r < 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

6. Zapetljani paraprodukt

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} K(s,t)\, f(x-s,y)g(x,y-t)\, ds\, dt$$

Primjena

Fundiranje stohastičkog integrala oblika

$$\int_0^t H_s d(X_s Y_s)$$

$(X_s)_{s\geq 0}$ i $(Y_s)_{s\geq 0}$ martingali obzirom na različite filtracije
(svaka profinjuje vjerojatnosni prostor $\Omega_1 \times \Omega_2$ u svojoj koordinati)

(K. i Škreb, 2013.)

7. Dvoparametarska bilinearna Hilbertova transformacija

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} f(x-s, y-t)\, g(x+s, y+t)\, \frac{ds}{s} \frac{dt}{t}$$

7. Dvoparametarska bilinearna Hilbertova transformacija

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} f(x-s, y-t)\, g(x+s, y+t)\, \frac{ds}{s}\frac{dt}{t}$$

Razlog za oprez! “Previše singularna”

Protuprimjer (Muscalu, Pipher, Tao i Thiele, 2004.)

T ne zadovoljava nikoje L^p ocjene

8. Singularni integrali s determinantnom jezgrom

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} f(u,v)\, g(x+u, y+v)\, \frac{du\, dv}{\det \begin{pmatrix} x & u \\ y & v \end{pmatrix}}$$

8. Singularni integrali s determinantnom jezgrom

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} f(u,v)\, g(x+u, y+v)\, \frac{du\, dv}{\det \begin{pmatrix} x & u \\ y & v \end{pmatrix}}$$

Motivacija: Invarijantne trilinearne forme u teoriji reprezentacija — za grupu $SL(3,\mathbb{R})$

$$\Lambda(f,g,h) = p.v. \int_{(\mathbb{R}^2)^2} f(u,v)\, g(x+u, y+v)\, h(x,y) \frac{du\, dv\, dx\, dy}{\det \begin{pmatrix} x & u \\ y & v \end{pmatrix}}$$

8. Singularni integrali s determinantnom jezgrom

$$T(f,g)(x,y) := p.v. \int_{\mathbb{R}^2} f(u,v)\, g(x+u, y+v)\, \frac{du\, dv}{\det \begin{pmatrix} x & u \\ y & v \end{pmatrix}}$$

Još "singularniji", ali ipak omeđen

Rezultat (Gressman, He, K., Street, Thiele i Yung, 2015.)

$$\|T(f,g)\|_{L^r} \leq C_{p,q,r} \|f\|_{L^p} \|g\|_{L^q}$$

čim je $1 < r < 2 < p, q < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ i to su sve L^p ocjene

Ideja dokaza. Superpozicija bilinearnih Hilbertovih transformacija

Kada predajem uvijek razmišljam o načinu na koji bih volio da sam osobno naučio to gradivo.

Elias Menachem Stein (1931.–2018.)

Hvala na pažnji!

Hvala na pažnji!