Multilinear singular integrals with entangled structure

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Examples of multilinear singular integrals

One-dimensional examples:

- $T(f_1, \ldots, f_n)(x) :=$
  $$\text{p.v.} \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_n)\, f_1(y_1) \cdots f_n(y_n)\, dy_1 \cdots dy_n$$
  (*multilinear C-Z operators*)
- $$T(f, g)(x) := \text{p.v.} \int_{\mathbb{R}} \frac{1}{t}\, f(x-t)g(x+t)dt$$
  (*bilinear Hilbert transform*)
- $$T(f, g, h)(x) := \text{p.v.} \int_{\mathbb{R}} \frac{1}{t}\, f(x-t)g(x+t)h(x+2t)dt$$
  (*trilinear Hilbert transform*, still unresolved)

Examples of multilinear singular integrals

Higher-dimensional complications (already in $\mathbb{R}^2$):

- $T(F_1, \ldots, F_n)(x) :=$
  $$\text{p.v.} \int_{(\mathbb{R}^2)^n} K(x, y_1, \ldots, y_n)\, F_1(y_1) \cdots F_n(y_n)\, dy_1 \cdots dy_n$$
  (still *multilinear C-Z operators*)

- $T(F, G)(x, y) :=$
  $$\text{p.v.} \int_{\mathbb{R}^2} K(s, t)\, F\big((x, y) + A(s, t)\big)\, G\big((x, y) + B(s, t)\big)\, ds dt$$
  (*2D bilinear Hilbert transform*)

- $T(F, G)(x, y) := \text{p.v.} \int_{\mathbb{R}} \frac{1}{t} F(x+t, y) G(x, y+t) dt$
  (*triangular Hilbert transform*, still unresolved)

Entangled structure

Object of our study:

Multilinear singular integral forms with functions that partially share variables

Schematically:

$$\Lambda(F_1, F_2, \ldots) = \int_{\mathbb{R}^n} F_1(x_1, x_2)\, F_2(x_1, x_3) \ldots \\ K(x_1, \ldots, x_n)\, dx_1 dx_2 dx_3 \ldots dx_n$$

K = singular kernel
$F_1, F_2, \ldots$ = functions on $\mathbb{R}^2$

Entangled structure — Generalized modulation invariance

An alternative viewpoint: *generalized modulation invariances*

$$\int_{\mathbb{R}^n} F_1(x_1,x_2)\, F_2(x_1,x_3)\ \dots\ K(x_1,\dots,x_n)\, dx_1 dx_2 dx_3 \dots dx_n$$

$$= \int_{\mathbb{R}^n} e^{2\pi i a x_1} F_1(x_1,x_2)\, e^{-2\pi i a x_1} F_2(x_1,x_3)\ \dots\ K(x_1,\dots,x_n)\, dx_1 dx_2 dx_3 \dots dx_n$$

$$= \int_{\mathbb{R}^n} \varphi(x_1) F_1(x_1,x_2)\, \frac{1}{\varphi(x_1)} F_2(x_1,x_3)\ \dots\ K(x_1,\dots,x_n)\, dx_1 dx_2 dx_3 \dots dx_n$$

$\rightsquigarrow$ Wave packet decompositions are not efficient

Entangled structure — Estimates

Our goal: L^p estimates

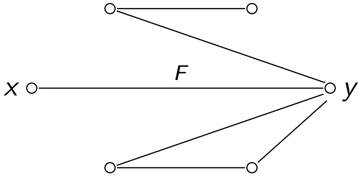
$$|\Lambda(F_1, F_2, \ldots, F_k)| \lesssim \|F_1\|_{L^{p_1}} \|F_2\|_{L^{p_2}} \ldots \|F_k\|_{L^{p_k}}$$

in a nonempty open subrange of

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} = 1$$

Entangled structure — Graph structure

Structure determined by K and a simple undirected graph on $x_1, x_2, \ldots$



$$\Lambda(F, \ldots) = \int \cdots F(x, y) \cdots dx\, dy \cdots$$

Entangled structure — Example: Triangular HT

$$T(F,G)(x,y) := \text{p.v.} \int_{\mathbb{R}} F(x+t,y)G(x,y+t)\frac{dt}{t}$$

No positive or negative results are known!

It has to be difficult:

A conjectured bound is invariant under affine transformations
Implies bounds for

$$\Lambda(F,G,H) = \int_{\mathbb{R}^2} \text{p.v.} \int_{\mathbb{R}} F(\mathbf{x}+t\mathbf{v_1})\, G(\mathbf{x}+t\mathbf{v_2})\, H(\mathbf{x}+t\mathbf{v_3})\, \frac{dt}{t}\, d\mathbf{x}$$

uniformly over all choices of non-collinear vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbb{R}^2$

Entangled structure — Example: Triangular HT

Implies bounds for

$$\Lambda(F,G,H) = \int_{\mathbb{R}^2} \text{p.v.} \int_{\mathbb{R}} F(\mathbf{x}+t\mathbf{v_1})\, G(\mathbf{x}+t\mathbf{v_2})\, H(\mathbf{x}+t\mathbf{v_3})\, \frac{dt}{t}\, d\mathbf{x}$$

uniformly over all choices of non-collinear vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbb{R}^2$

In the limit towards the degenerate case

$$\mathbf{v_1} = (\beta_1, 0),\ \mathbf{v_2} = (\beta_2, 0),\ \mathbf{v_3} = (\beta_3, 0)$$

we would recover uniform bounds for the BHT

$$\Lambda(f,g,h) = \int_{\mathbb{R}} \text{p.v.} \int_{\mathbb{R}} f(x+t\beta_1)\, g(x+t\beta_2)\, h(x+t\beta_3)\, \frac{dt}{t}\, dx$$

$\rightsquigarrow$ Lacey and Thiele (1997, 1999), Thiele (2002)

Entangled structure — Example: Triangular HT

$$\Lambda(F,G,H) = \int_{\mathbb{R}^2} \text{p.v.} \int_{\mathbb{R}} F(x+t,y)G(x,y+t)H(x,y)\frac{dt}{t}\,dxdy$$

By taking

$$F(x,y) = f(x),\;\; G(x,y) = g(x)e^{iN(x)y},\;\; H(x,y) = h(x)e^{-iN(x)y}$$

we obtain bounds for the linearized Carleson operator

$$(Cf)(x) = \int_{\mathbb{R}} f(x+t)e^{iN(x)t}\frac{dt}{t}$$

$$\Lambda(F,G,H) = \langle Cf, gh \rangle$$

$\rightsquigarrow$ Carleson (1966)

Entangled structure — Example: Triangular HT

$$T(F,G)(x,y) := \text{p.v.}\int_{\mathbb{R}} F(x+t,y)G(x,y+t)\frac{dt}{t}$$

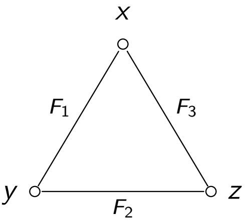
Substitute: $z = -x - y - t$,
$F_1(x,y) = H(x,y),\ F_2(y,z) = F(-y-z,y),\ F_3(z,x) = G(x,-x-z)$

$$\Lambda(F,G,H) = \left\langle T(F,G), H \right\rangle$$
$$= \int_{\mathbb{R}^3} F_1(x,y)\, F_2(y,z)\, F_3(z,x)\, \frac{-1}{x+y+z}\, dxdydz$$

Entangled structure — Example: Triangular HT

$$\Lambda(F_1, F_2, F_3) = \int_{\mathbb{R}^3} F_1(x, y)\, F_2(y, z)\, F_3(z, x)\, \frac{-1}{x + y + z}\, dxdydz$$

Associated graph:

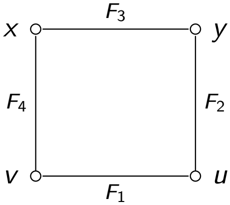


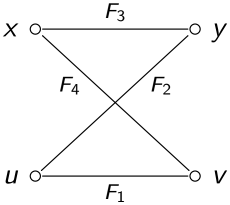
Note: this graph is not bipartite

Quadrilinear variant:

$$\Lambda(F_1, F_2, F_3, F_4) = \int_{\mathbb{R}^4} F_1(u,v)F_2(u,y)F_3(x,y)F_4(x,v)\, K(u,v,x,y)\, dudvdxdy$$

Associated graph:





Note: this graph is bipartite

Scope of our techniques

We specialize to:

- bipartite graphs
- multilinear Calderón-Zygmund kernels K
- "perfect" dyadic models

Dyadic T(1) — Perfect dyadic conditions

$m, n =$ positive integers

$$D := \{(\underbrace{x, \ldots, x}_{m}, \underbrace{y, \ldots, y}_{n}) : x, y \in \mathbb{R}\}$$

the "diagonal" in $\mathbb{R}^{m+n}$

Perfect dyadic Calderón-Zygmund kernel $K \colon \mathbb{R}^{m+n} \to \mathbb{C}$,
Auscher, Hofmann, Muscalu, Tao, Thiele (2002):

- $|K(x_1, \ldots, x_m, y_1, \ldots, y_n)|$
  $\lesssim \big(\sum_{i_1 < i_2} |x_{i_1} - x_{i_2}| + \sum_{j_1 < j_2} |y_{j_1} - y_{j_2}| \big)^{2-m-n}$
- K is constant on $(m+n)$-dimensional dyadic cubes disjoint from D
- K is bounded and compactly supported

Dyadic T(1) — Bipartite structure

$E \subseteq \{1, \dots, m\} \times \{1, \dots, n\}$

G = simple bipartite undirected graph on
$\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$
x_i—$y_j \;\Leftrightarrow\; (i,j) \in E$

$|E|$-linear singular form:

$$\Lambda\big((F_{i,j})_{(i,j)\in E}\big) := \int_{\mathbb{R}^{m+n}} K(x_1, \dots, x_m, y_1, \dots, y_n) \prod_{(i,j)\in E} F_{i,j}(x_i, y_j)\, dx_1 \dots dx_m dy_1 \dots dy_n$$

Assume: there are no isolated vertices in G
$\rightsquigarrow$ avoids degeneracy

Dyadic T(1) — Adjoints

There are $|E|$ mutually adjoint $(|E|-1)$-linear operators $T_{u,v}$, $(u,v) \in E$:

$$\Lambda\big((F_{i,j})_{(i,j)\in E}\big) = \int_{\mathbb{R}^2} T_{u,v}\big((F_{i,j})_{(i,j)\neq(u,v)}\big)\, F_{u,v}$$

Explicitly:

$$\begin{aligned} &T_{u,v}\big((F_{i,j})_{(i,j)\in E\setminus\{(u,v)\}}\big)(x_u, y_v) \\ &= \int_{\mathbb{R}^{m+n-2}} K(x_1,\ldots,x_m,y_1,\ldots,y_n) \\ &\qquad\qquad \prod_{(i,j)\in E\setminus\{(u,v)\}} F_{i,j}(x_i,y_j)\ \prod_{i\neq u} dx_i \prod_{j\neq v} dy_j \end{aligned}$$

Dyadic T(1) — A T(1)-type theorem

Theorem. "Entangled" T(1) — K. and Thiele (2011, 2013)

(a) For $m, n \geq 2$ and a graph G there exist positive integers $d_{i,j}$ such that $\sum_{(i,j)\in E} \frac{1}{d_{i,j}} > 1$ and the following holds. If

$$|\Lambda(\mathbf{1}_Q, \ldots, \mathbf{1}_Q)| \lesssim |Q|, \quad Q \textit{ dyadic square},$$

$$\| T_{u,v}(\mathbf{1}_{\mathbb{R}^2}, \ldots, \mathbf{1}_{\mathbb{R}^2}) \|_{\mathrm{BMO}(\mathbb{R}^2)} \lesssim 1, \quad (u,v) \in E,$$

then

$$\big|\Lambda\big((F_{i,j})_{(i,j)\in E}\big)\big| \lesssim \prod_{(i,j)\in E} \|F_{i,j}\|_{\mathrm{L}^{p_{i,j}}(\mathbb{R}^2)}$$

for exponents $p_{i,j}$ s.t. $\sum_{(i,j)\in E} \frac{1}{p_{i,j}} = 1,\ d_{i,j} < p_{i,j} \leq \infty$.

(b) Conversely, the estimate for some choice of exponents implies the conditions.

Dyadic T(1) — A T(1)-type theorem, reformulation

Theorem. "Entangled" T(1) — K. and Thiele (2011, 2013)

For $m, n \geq 2$ and a graph G there exist positive integers $d_{i,j}$ such that $\sum_{(i,j)\in E} \frac{1}{d_{i,j}} > 1$ and the following holds. If

$$\big\| T_{u,v}(\mathbf{1}_Q, \ldots, \mathbf{1}_Q)\big\|_{\mathrm{L}^1(Q)} \lesssim |Q|, \quad Q \text{ dyadic square}, \ (u,v) \in E,$$

then

$$\big|\Lambda\big((F_{i,j})_{(i,j)\in E}\big)\big| \lesssim \prod_{(i,j)\in E} \|F_{i,j}\|_{\mathrm{L}^{p_{i,j}}(\mathbb{R}^2)}$$

for exponents $p_{i,j}$ s.t. $\sum_{(i,j)\in E} \frac{1}{p_{i,j}} = 1$, $d_{i,j} < p_{i,j} \leq \infty$.

Dyadic T(1) — Proof outline

The only nonstandard part — sufficiency of the testing conditions

Outline of the proof:

- decomposition into paraproducts
- a stopping time argument for reducing global estimates to local estimates
- cancellative paraproducts with ℓ^∞ coefficients
  - "most" cases of graphs G
    $\rightsquigarrow$ $d_{i,j}$ related to sizes of connected components of G
    $\rightsquigarrow$ stuctural induction + Bellman function technique
  - exceptional cases of graphs G
- non-cancellative paraproducts with BMO coefficients
  $\rightsquigarrow$ reduction to cancellative paraproducts
- counterexample for $m = 1$ or $n = 1$

Dyadic T(1) — Decomposition into paraproducts

$\mathbf{h}_I = \mathbf{1}_{I_{\text{left}}} - \mathbf{1}_{I_{\text{right}}}$ L^∞-normalized Haar wavelet

Kernel decomposition:

$$K(x_1,\ldots,x_m,y_1,\ldots,y_n)$$
$$= \sum_{\substack{I_1\times\ldots\times I_m\times J_1\times\ldots\times J_n \\ \mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\,\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}\in\{\mathbf{1},\mathbf{h}\} \\ \mathbf{a}^{(i)},\,\mathbf{b}^{(j)} \textit{ not all equal } \mathbf{1}}} \nu^{\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}}_{I_1\times\ldots\times I_m\times J_1\times\ldots\times J_n} \mathbf{a}^{(1)}_{I_1}(x_1)\ldots\mathbf{a}^{(m)}_{I_m}(x_m)\mathbf{b}^{(1)}_{J_1}(y_1)\ldots\mathbf{b}^{(n)}_{J_n}(y_n)$$

[*perfect dyadic condition*]

$$= \sum_{\substack{I\times J \\ \mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\,\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}\in\{\mathbf{1},\mathbf{h}\} \\ \mathbf{a}^{(i)},\,\mathbf{b}^{(j)} \textit{ not all equal } \mathbf{1}}} \lambda^{\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}}_{I\times J} |I|^{2-m-n} \mathbf{a}^{(1)}_{I}(x_1)\ldots\mathbf{a}^{(m)}_{I}(x_m)\mathbf{b}^{(1)}_{J}(y_1)\ldots\mathbf{b}^{(n)}_{J}(y_n)$$

Dyadic T(1) — Decomposition into paraproducts

Decomposition of Λ:

$$\Lambda = \sum_{\substack{\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\,\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}\in\{\mathbf{1},\mathbf{h}\} \\ \mathbf{a}^{(i)},\,\mathbf{b}^{(j)} \textit{ are not all equal } \mathbf{1}}} \Theta_E^{\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}}$$

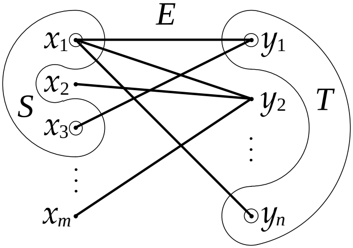
It remains to bound each individual "entangled" paraproduct $\Theta_E^{\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(m)},\mathbf{b}^{(1)},\ldots,\mathbf{b}^{(n)}}$

Two types:

- cancellative $\rightsquigarrow$ require only ℓ^∞ coefficients $\lambda_{I\times J}$
- non-cancellative $\rightsquigarrow$ require BMO coefficients $\lambda_{I\times J}$

Dyadic T(1) — Decomposition into paraproducts

$$S := \left\{ i \in \{1, \ldots, m\} : \mathbf{a}^{(i)} = \mathbf{h} \right\}$$
$$T := \left\{ j \in \{1, \ldots, n\} : \mathbf{b}^{(j)} = \mathbf{h} \right\}$$



Dyadic T(1) — Decomposition into paraproducts

$$S := \{i \in \{1, \ldots, m\} : \mathbf{a}^{(i)} = \mathbf{h}\}$$
$$T := \{j \in \{1, \ldots, n\} : \mathbf{b}^{(j)} = \mathbf{h}\}$$

Cancellative subtypes:

(C1) $\max\{|S|, |T|\} \geq 2$

(C2) $S = \{i\},\ T = \{j\}$, and $(i, j) \notin E$

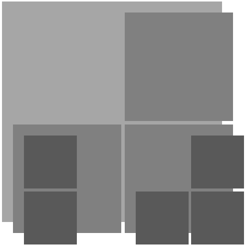
Non-cancellative subtypes:

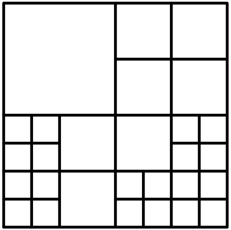
(NC1) $|S| + |T| = 1$

(NC2) $S = \{i\},\ T = \{j\}$, and $(i, j) \in E$

Dyadic T(1) — Reduction to local estimates

$\mathcal{C}$ = dyadic squares in $\mathbb{R}^2$





$\mathcal{T}$ = a finite convex tree of dyadic squares
$Q_{\mathcal{T}}$ = *root* or *tree-top* of $\mathcal{T}$
$\mathcal{L}(\mathcal{T})$ = leaves of $\mathcal{T}$, squares that are not contained in $\mathcal{T}$ but their parents are

Dyadic T(1) — Paraproduct estimates

Bounds for cancellative paraproducts

(a) $\|\lambda\|_{\ell^\infty} := \sup_{Q\in\mathcal{C}} |\lambda_Q| \lesssim 1$

(b) $\Theta_{\mathcal{T}}\big((F_{i,j})_{(i,j)\in E}\big) \lesssim \|\lambda\|_{\ell^\infty}\, |Q_{\mathcal{T}}| \prod_{(i,j)\in E} \max_{Q\in\mathcal{T}\cup\mathcal{L}(\mathcal{T})} \big[F_{i,j}^{d_{i,j}}\big]_Q^{1/d_{i,j}}$

Bounds for non-cancellative paraproducts

(a) $\|\lambda\|_{bmo} := \sup_{Q_0\in\mathcal{C}} \Big(\frac{1}{|Q_0|} \sum_{Q\in\mathcal{C}\,:\,Q\subseteq Q_0} |Q||\lambda_Q|^2\Big)^{1/2} \lesssim 1$

(b) $\Theta_{\mathcal{T}}\big((F_{i,j})_{(i,j)\in E}\big) \lesssim \|\lambda\|_{bmo}\, |Q_{\mathcal{T}}| \prod_{(i,j)\in E} \max_{Q\in\mathcal{T}\cup\mathcal{L}(\mathcal{T})} \big[F_{i,j}^{d_{i,j}}\big]_Q^{1/d_{i,j}}$

Bellman functions in harmonic analysis

- "Invented" by Burkholder (1980s)
- Developed by Nazarov, Treil, and Volberg (1990s)
- We primarily keep the "induction on scales" idea

A broad class of interesting dyadic objects can be reduced to bounding expressions of the form

$$\Lambda_{\mathcal{T}}(F_1, \ldots, F_\ell) = \sum_{Q \in \mathcal{T}} |Q| \, \mathcal{A}_Q(F_1, \ldots, F_\ell)$$

$\mathcal{T}$ = a finite convex tree of dyadic squares
$\mathcal{A}_Q(F_1, \ldots, F_\ell)$ = some "scale-invariant" quantity depending on $F_1, \ldots, F_\ell$ and $Q \in \mathcal{T}$

Dyadic T(1) — Calculus of finite differences

$$\mathcal{B} = \mathcal{B}_Q(F_1, \ldots, F_\ell)$$

First order difference of $\mathcal{B}$:

$$\Box\mathcal{B} = \Box\mathcal{B}_Q(F_1, \ldots, F_\ell)$$

$$\begin{aligned}\Box\mathcal{B}_{I\times J} := {} & \tfrac{1}{4}\mathcal{B}_{I_{\mathrm{left}}\times J_{\mathrm{left}}} + \tfrac{1}{4}\mathcal{B}_{I_{\mathrm{left}}\times J_{\mathrm{right}}} \\ & + \tfrac{1}{4}\mathcal{B}_{I_{\mathrm{right}}\times J_{\mathrm{left}}} + \tfrac{1}{4}\mathcal{B}_{I_{\mathrm{right}}\times J_{\mathrm{right}}} - \mathcal{B}_{I\times J}\end{aligned}$$

Dyadic T(1) — Calculus of finite differences

Suppose: $|\mathcal{A}| \le \Box\mathcal{B}$, i.e.

$$|\mathcal{A}_Q(F_1, \dots, F_\ell)| \le \Box\mathcal{B}_Q(F_1, \dots, F_\ell)$$

for all $Q \in \mathcal{T}$ and nonnegative bounded measurable $F_1, \dots, F_\ell$

$$|Q|\,|\mathcal{A}_Q(F_1, \dots, F_\ell)| \le \sum_{\widetilde{Q} \textit{ is a child of } Q} |\widetilde{Q}|\,\mathcal{B}_{\widetilde{Q}}(F_1, \dots, F_\ell) - |Q|\,\mathcal{B}_Q(F_1, \dots, F_\ell)$$

$$|\Lambda_{\mathcal{T}}(F_1, \dots, F_\ell)| \le \sum_{Q \in \mathcal{L}(\mathcal{T})} |Q|\,\mathcal{B}_Q(F_1, \dots, F_\ell) - |Q_{\mathcal{T}}|\,\mathcal{B}_{Q_{\mathcal{T}}}(F_1, \dots, F_\ell)$$

$\mathcal{B}$ = a *Bellman function* for $\Lambda_{\mathcal{T}}$

Entangled structure — Open problems

General bipartite graphs G

How to obtain boundedness of

$$\Lambda\big((F_{i,j})_{(i,j)\in E}\big) := \int_{\mathbb{R}^{m+n}} K(x_1,\ldots,x_m,y_1,\ldots,y_n) \prod_{(i,j)\in E} F_{i,j}(x_i,y_j)\,dx_1\ldots dx_m dy_1\ldots dy_n$$

at least for some (class of) continuous singular kernels K,

- when the graph G is bipartite, has at least 4 edges, and K is a C-Z kernel,
- when the graph G is a triangle and $K(x,y,z) = \frac{1}{x+y+z}$?

$\rightsquigarrow$ Still far from a complete T(1)-type theorem

Appl. #1 — 2D BHT

A variant of the 2D bilinear Hilbert transform

$$T(F,G)(x,y) = \text{p.v.}\int_{\mathbb{R}^2} K(s,t)\, F\big((x,y)-A(s,t)\big)\, G\big((x,y)-B(s,t)\big)\, ds\, dt$$

Introduced by Demeter and Thiele and bounded for "most" cases of $A, B \in \mathrm{M}_2(\mathbb{R})$ (2008)

Essentially the only case that was left out:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Appl. #1 — 2D BHT

As a multiplier:

$$T(F,G)(x,y) = \int_{\mathbb{R}^4} \mu(\xi_1,\xi_2,\eta_1,\eta_2) e^{2\pi i (x(\xi_1+\eta_1)+y(\xi_2+\eta_2))} \widehat{F}(\xi_1,\xi_2)\widehat{G}(\eta_1,\eta_2)\, d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

$$\mu(\xi_1,\xi_2,\eta_1,\eta_2) = m\big(A^\tau(\xi_1,\xi_2) + B^\tau(\eta_1,\eta_2)\big), \qquad m = \widehat{K}$$

$$m \in C^\infty\big(\mathbb{R}^2 \setminus \{(0,0)\}\big)$$

$$\big|\partial_{\tau_1}^{\alpha_1}\partial_{\tau_2}^{\alpha_2} m(\tau_1,\tau_2)\big| \le C_{\alpha_1,\alpha_2}(|\tau_1|+|\tau_2|)^{-\alpha_1-\alpha_2}$$

Note: $\mu(\xi_1,\xi_2,\eta_1,\eta_2)$ is singular along the 2-plane $A^\tau(\xi_1,\xi_2) + B^\tau(\eta_1,\eta_2) = (0,0)$

Appl. #1 — Remaining case of 2D BHT

$$T(F,G)(x,y) = \int_{\mathbb{R}^4} m(\xi_1,\eta_2) e^{2\pi i (x(\xi_1+\eta_1)+y(\xi_2+\eta_2))} \widehat{F}(\xi_1,\xi_2)\widehat{G}(\eta_1,\eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2$$

Theorem. L^p estimate — Bernicot (2010), K. (2010)

$$\|T(F,G)\|_{\mathrm{L}^r} \leq C_{p,q,r} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^q}$$

for $1 < p, q < \infty$, $0 < r < 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Theorem. Sobolev estimate — Bernicot and K. (2013)

If $\operatorname{supp} m \subseteq \left\{(\xi_1,\eta_2) : |\xi_1| \leq c\,|\eta_2|\right\}$, then

$$\|T(F,G)\|_{\mathrm{L}^r_y(\mathrm{W}^{s,r}_x)} \leq C_{p,q,r,s} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{W}^{s,q}}$$

for $s \geq 0$, $1 < p, q < \infty$, $1 < r < 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Appl. #1 — Remaining case of 2D BHT

$$T(F,G)(x,y) = \int_{\mathbb{R}^2} F(x-s,y)\,G(x,y-t)\,K(s,t)\,ds\,dt$$

Outline of the proof:

Substitute $u = x - s$, $v = y - t$:

$$\Lambda(F,G,H) = \big\langle T(F,G), H \big\rangle$$
$$= \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)K(x-u,y-v)\,dudvdxdy$$

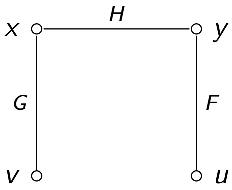
Non-translation-invariant generalization:

$$\Lambda(F,G,H) = \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)K(u,v,x,y)\,dudvdxdy$$

Appl. #1 — Remaining case of 2D BHT

$$\Lambda(F,G,H) = \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)K(u,v,x,y)\, dudvdxdy$$

Graph associated with its structure:



Appl. #1 — Remaining case of 2D BHT

$$\Lambda(F,G,H) = \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)K(u,v,x,y)\, dudvdxdy$$

Perform cone decomposition of the symbol $m = \widehat{K}$:

$$m = \sum_j m^{[j]}$$
$$m^{[j]}(\xi_1,\eta_2) = \sum_{k\in\mathbb{Z}} \widehat{\varphi_k^{[j]}}(\xi_1)\,\widehat{\psi_k^{[j]}}(\eta_2)$$

Appl. #1 — Ordinary paraproduct

Dyadic version

$$T_{\mathrm{d}}(f,g) := \sum_{k\in\mathbb{Z}} (\mathbb{E}_k f)(\Delta_k g)$$

$\mathbb{E}_k f := \sum_{|I|=2^{-k}} \left(\frac{1}{|I|}\int_I f\right) \mathbf{1}_I, \quad \Delta_k g := \mathbb{E}_{k+1} g - \mathbb{E}_k g$

Continuous version

$$T_{\mathrm{c}}(f,g) := \sum_{k\in\mathbb{Z}} (\mathrm{P}_{\varphi_k} f)(\mathrm{P}_{\psi_k} g)$$

$\mathrm{P}_{\varphi_k} f := f * \varphi_k, \quad \mathrm{P}_{\psi_k} g := g * \psi_k$

φ, ψ Schwartz, $\quad \mathrm{supp}(\hat{\psi}) \subseteq \{\xi \in \mathbb{R} : \frac{1}{2} \leq |\xi| \leq 2\}$

$\varphi_k(t) := 2^k \varphi(2^k t), \quad \psi_k(t) := 2^k \psi(2^k t)$

Appl. #1 — Twisted paraproduct

Dyadic version

$$T_{\mathrm{d}}(F,G) := \sum_{k\in\mathbb{Z}} (\mathbb{E}_k^{(1)} F)(\Delta_k^{(2)} G)$$

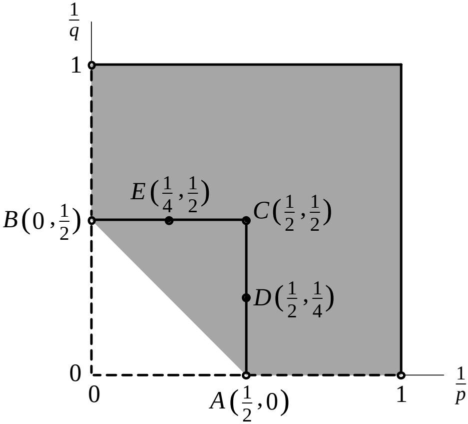
$\mathbb{E}_k^{(1)}$ martingale averages in the 1st variable
$\Delta_k^{(2)}$ martingale differences in the 2nd variable

Continuous version

$$T_{\mathrm{c}}(F,G) := \sum_{k\in\mathbb{Z}} (\mathrm{P}_{\varphi_k}^{(1)} F)(\mathrm{P}_{\psi_k}^{(2)} G)$$

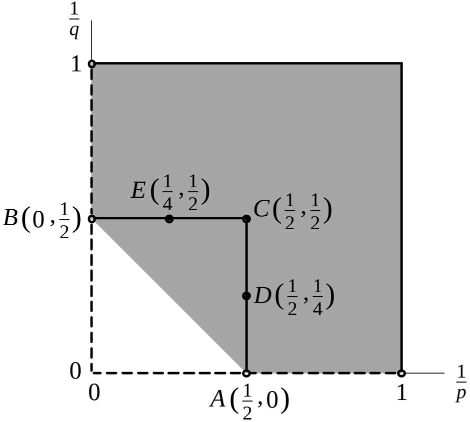
$\mathrm{P}_{\varphi_k}^{(1)}$, $\mathrm{P}_{\psi_k}^{(2)}$ L-P projections in the 1st and the 2nd variable
$(\mathrm{P}_{\varphi_k}^{(1)} F)(x,y) := \int_{\mathbb{R}} F(x-t,y)\varphi_k(t)dt$
$(\mathrm{P}_{\psi_k}^{(2)} G)(x,y) := \int_{\mathbb{R}} G(x,y-t)\psi_k(t)dt$

Appl. #1 — Twisted paraproduct, estimates



- the shaded region – the strong estimate
- two solid sides of the square – the weak estimate
- two dashed sides of the square – no estimates
- the white region – unresolved

Appl. #1 — Proof outline



Dyadic version T_d

- $\triangle ABC$ – a special case of **dyadic entangled results** (bipartite, 3 edges)
- the rest of the shaded region – fiberwise C-Z decomp. $\rightsquigarrow$ Bernicot (2010)
- dashed segments – counterexamples
- D, E – an alternative (more direct) proof

Continuous version T_c

- comparing with the dyadic version

Appl. #1 — Transition to cont. version

Assume: $\psi_k = \phi_{k+1} - \phi_k$ for some ϕ Schwartz, $\int_{\mathbb{R}} \phi = 1$
The general case is then obtained by composing with a bounded Fourier multiplier in the second variable

Calderón (1960s), Jones, Seeger, and Wright (2008)

If φ is Schwartz and $\int_{\mathbb{R}} \varphi = 1$, then the square function

$$\mathcal{S}F := \Big(\sum_{k \in \mathbb{Z}} \big| \mathrm{P}_{\varphi_k} F - \mathbb{E}_k F \big|^2 \Big)^{1/2}$$

satisfies

$$\|\mathcal{S}F\|_{\mathrm{L}^p(\mathbb{R})} \lesssim_p \|F\|_{\mathrm{L}^p(\mathbb{R})}$$

for $1 < p < \infty$.

Proposition

$$\big\| T_{\mathrm{c}}(F, G) - T_{\mathrm{d}}(F, G) \big\|_{\mathrm{L}^{pq/(p+q)}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^q}$$

Appl. #1a — General dilations

$\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$

$(\delta_t^{(1)})_{t>0}$, $(\delta_t^{(2)})_{t>0}$ groups of dilations:

- $\delta_t^{(j)} \colon \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$ is a linear transformation,
- $\delta_1^{(j)}$ is the identity and $\delta_{st}^{(j)} = \delta_s^{(j)} \delta_t^{(j)}$ for $s, t > 0$,
- $\langle 0, +\infty\rangle \times \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$, $(t, x) \mapsto \delta_t^{(j)} x$ is continuous,
- $\lim_{t\to 0} \delta_t^{(j)} x = \mathbf{0}$

Equivalently, for some $A_j \in \mathrm{M}_{d_j}(\mathbb{R})$, $\mathrm{Re}(\sigma(A_j)) > 0$

$$\delta_t^{(j)} = t^{A_j} = e^{(\log t) A_j}, \quad t > 0,\ j = 1, 2$$

Example: non-isotropic dilations on $\mathbb{R}^n$,

$$\delta_t(x_1, \ldots, x_n) := \left(t^{a_1} x_1, \ldots, t^{a_n} x_n\right)$$

Appl. #1a — General dilations

Take two Schwartz functions $\varphi^{(j)}\colon \mathbb{R}^{d_j} \to \mathbb{C}$, $j = 1, 2$ normalized by $\int_{\mathbb{R}^{d_j}} \varphi^{(j)}(x)dx = 1$

Denote the dilates:

$$\varphi_t^{(j)}(x) := \left(\det \delta_t^{(j)}\right) \varphi^{(j)}\left(\delta_t^{(j)} x\right), \quad x \in \mathbb{R}^{d_j},\ t > 0,\ j = 1, 2$$

Fix $0 < \alpha, \beta < 1$

$$(\mathrm{P}_k^{(1)} F)(x, y) := \int_{\mathbb{R}^{d_1}} F(x - u, y)\varphi_{\alpha^k}^{(1)}(u)du$$
$$(\mathrm{P}_k^{(2)} F)(x, y) := \int_{\mathbb{R}^{d_2}} F(x, y - v)\varphi_{\beta^k}^{(2)}(v)dv$$

Appl. #1a — General dilations

General-dilation twisted paraproduct:

$$T_{\alpha,\beta}(F,G) := \sum_{k\in\mathbb{Z}} \big(\mathrm{P}^{(1)}_k F\big)\big(\mathrm{P}^{(2)}_{k+1} G - \mathrm{P}^{(2)}_k G\big)$$

Theorem. K. and Škreb (2014)

There exist parameters $0<\alpha,\beta<1$, depending only on the dilation structure, such that the estimate

$$\big\|T_{\alpha,\beta}(F,G)\big\|_{\mathrm{L}^r(\mathbb{R}^d)} \leq C\|F\|_{\mathrm{L}^p(\mathbb{R}^d)}\|G\|_{\mathrm{L}^q(\mathbb{R}^d)}$$

holds whenever $1<r<2<p,q<\infty$ and $1/r=1/p+1/q$, with a constant C depending on p,q,r, α,β, $\varphi^{(1)},\varphi^{(2)}$, and the dilation groups.

Appl. #1a — General dilations

Sketch of the proof:

- generalize the needed dyadic estimate to more general martingales
- consider the corresponding space of homogeneous type $\rightsquigarrow$ Stein and Wainger (1978)
- apply the estimate to martingales on dyadic cubes constructed by Christ (1990)
- use the full generality of the Jones-Seeger-Wright square function

Appl. #2 — Several commuting transformations

Multiple averages

$$\mathrm{M}_n(f_1, f_2, \ldots, f_r) := \frac{1}{n}\sum_{k=0}^{n-1}(f_1 \circ T_1^k)(f_2 \circ T_2^k)\cdots(f_r \circ T_r^k)$$

$(X, \mathcal{F}, \mu)$ a probability space
$T_1, T_2, \ldots, T_r \colon X \to X$ commuting, measure preserving:

$$T_i T_j = T_j T_i, \quad \mu\big(T_i^{-1}(E)\big) = \mu(E) \text{ for } E \in \mathcal{F}$$

$f_1, f_2, \ldots, f_r \in \mathrm{L}^\infty$

Motivation:
Furstenberg and Katznelson (1978)
(multidimensional Szemerédi's theorem)

Appl. #2 — Several commuting transformations

Multiple averages

$$\mathrm{M}_n(f_1, f_2, \ldots, f_r) := \frac{1}{n}\sum_{k=0}^{n-1}(f_1 \circ T_1^k)(f_2 \circ T_2^k)\cdots(f_r \circ T_r^k)$$

Convergence in L^2 as $n \to \infty$:

- $r = 1 \rightsquigarrow$ von Neumann (1930)
- $r = 2 \rightsquigarrow$ Conze and Lesigne (1984)
- $r \geq 3 \rightsquigarrow$ Tao (2008)
- generalizations
  $\rightsquigarrow$ Austin (2010); Walsh (2011); Zorin-Kranich (2011)

Appl. #2 — Several commuting transformations

Multiple averages

$$\mathrm{M}_n(f_1, f_2, \ldots, f_r) := \frac{1}{n}\sum_{k=0}^{n-1}(f_1 \circ T_1^k)(f_2 \circ T_2^k)\cdots(f_r \circ T_r^k)$$

Convergence a.e. as $n \to \infty$:

- $r = 1 \rightsquigarrow$ Birkhoff (1931)
- $r \geq 2 \rightsquigarrow$ a long-standing open problem $\rightsquigarrow$ Calderón?
- $r = 2$ and $T_2 = T_1^m$, $m \in \mathbb{Z} \rightsquigarrow$ Bourgain (1990)
- many other partial results

We **do not** discuss a.e. convergence here

Appl. #2 — Connection with entangled forms

A real-analytic approach to bilinear ergodic averages (suggested by Demeter, Muscalu, Thiele):

$$\mathrm{M}_n(f_1, f_2)(\omega) := \frac{1}{n} \sum_{k=0}^{n-1} f_1(T_1^k \omega) f_2(T_2^k \omega), \quad \omega \in \Omega$$

$$\mathrm{A}_T(F_1, F_2)(x, y) := \frac{1}{T} \int_0^T F_1(x+t, y) F_2(x, y+t) dt, \quad (x, y) \in \mathbb{R}^2$$

Appl. #2 — Connection with entangled forms

$$\mathrm{A}_T(F_1, F_2)(x, y) := \frac{1}{T}\int_0^T F_1(x+t, y)F_2(x, y+t)dt, \quad (x, y) \in \mathbb{R}^2$$

We can suspect the entangled structure from:

$$\begin{aligned}
&\big\|\mathrm{A}_T(F_1, F_2)\big\|_{\mathrm{L}^2}^2 \\
&= \int_{\mathbb{R}^4} F_1(x+s, y)F_2(x, y+s)F_1(x+t, y)F_2(x, y+t) \\
&\qquad\qquad T^{-2}\mathbf{1}_{[0,T]}(s)\mathbf{1}_{[0,T]}(t)dsdtdxdy
\end{aligned}$$

substitute $u = x + y + s,\ v = x + y + t$

$$\begin{aligned}
&= \int_{\mathbb{R}^4} \widetilde{F}_1(u, y)\widetilde{F}_2(u, x)\widetilde{F}_1(v, y)\widetilde{F}_2(v, x) \\
&\qquad\qquad T^{-2}\mathbf{1}_{[0,T]}(u-x-y)\mathbf{1}_{[0,T]}(v-x-y)dudvdxdy
\end{aligned}$$

a 4-cycle of variables!

Appl. #2 — Several commuting transformations

Single averages

$$\mathrm{M}_n(f) := \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$$

Quantify L^2 convergence of the sequence:

- control the number of jumps **in the norm**
- bound the norm-variation

Appl. #2 — Several commuting transformations

Single averages

$$\mathrm{M}_n(f) := \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$$

Jones, Ostrovskii, and Rosenblatt (1996) $p = 2$;
Jones, Kaufman, Rosenblatt, and Wierdl (1998) $p \geq 2$

$$\sup_{n_0<n_1<\cdots<n_m} \sum_{j=1}^{m} \left\| \mathrm{M}_{n_{j-1}}(f) - \mathrm{M}_{n_j}(f) \right\|_{\mathrm{L}^p}^p \leq C_p \left\| f \right\|_{\mathrm{L}^p}^p$$

Consequence:

$\left(\mathrm{M}_n(f)\right)_{n=0}^{\infty}$ has $O\left(\varepsilon^{-p}\|f\|_{\mathrm{L}^p}^p\right)$ jumps of size $\geq \varepsilon$ in the L^p norm

Avigad and Rute (2013) $\rightsquigarrow$ in more general Banach spaces

Appl. #2 — Several commuting transformations

Double averages

$$\mathrm{M}_n(f, g) := \frac{1}{n} \sum_{k=0}^{n-1} (f \circ S^k)(g \circ T^k)$$

Avigad and Rute (2012) $\rightsquigarrow$ asked for any (reasonable/explicit) quantitative estimates of norm convergence of multiple averages

Partial results:

- $T = S^m$, $m \in \mathbb{Z}$ $\rightsquigarrow$ Bourgain (1990); Demeter (2007)
- variational estimate for the dyadic model of BHT $\rightsquigarrow$ Do, Oberlin, and Palsson (2012)

Appl. #2 — The result: Commuting $\mathbb{A}^\omega$ actions

"Cantor group" model of the integers

$$\mathbb{A}^\omega := \bigoplus_{k=1}^{\infty} \mathbb{A} = \mathbb{A} \oplus \mathbb{A} \oplus \cdots$$

for some finite abelian group $\mathbb{A}$

Typically: $\mathbb{A} = \mathbb{Z}/d\mathbb{Z}$

$$\mathbb{A}^\omega = \left\{ a = (a_k)_{k=1}^{\infty} \ : \ (\exists k_0)(\forall k > k_0)(a_k = 0) \right\}$$

Følner sequence $(F_n)_{n=1}^{\infty}$: $\lim_{n\to\infty} \frac{|(a+F_n)\triangle F_n|}{|F_n|} = 0$

The most natural choice:

$$F_n = \left\{ a = (a_k)_{k=1}^{\infty} \ : \ (\forall k > n)(a_k = 0) \right\} \cong \mathbb{A}^n$$

Appl. #2 — The result: Commuting $\mathbb{A}^\omega$ actions

"Cantor group" model of the integers

$$\mathbb{A}^\omega := \bigoplus_{k=1}^{\infty} \mathbb{A} = \mathbb{A} \oplus \mathbb{A} \oplus \cdots$$

for some finite abelian group $\mathbb{A}$

$S = (S^a)_{a\in\mathbb{A}^\omega}$ and $T = (T^a)_{a\in\mathbb{A}^\omega}$ commuting measure preserving $\mathbb{A}^\omega$-actions on a probability space $(X, \mathcal{F}, \mu)$:

- $S^a, T^a \colon X \to X$ are measurable
- $S^{\mathbf{0}} = T^{\mathbf{0}} = \mathrm{id}$, $S^aS^b = S^{a+b}$, $T^aT^b = T^{a+b}$, $S^aT^b = T^bS^a$
- $\mu(S^aE) = \mu(E) = \mu(T^aE)$ for $a \in \mathbb{A}^\omega$, $E \in \mathcal{F}$

Appl. #2 — The result: Commuting $\mathbb{A}^\omega$ actions

Toy-model of double averages

$$\mathrm{M}_n(f,g) := \frac{1}{|F_n|}\sum_{a\in F_n}(f\circ S^a)(g\circ T^a)$$

Such multiple averages have already appeared in the literature:

- general countable amenable group $\rightsquigarrow$ Bergelson, McCutcheon, and Zhang (1997); Zorin-Kranich (2011); Austin (2013)
- "powers" of the same action of $(\mathbb{Z}/p\mathbb{Z})^\omega$, p prime $\rightsquigarrow$ Bergelson, Tao, and Ziegler (2013)

Note: All known convergence results are only qualitative or extremely weakly quantitative in nature

Appl. #2 — The result: Commuting $\mathbb{A}^\omega$ actions

Toy-model of double averages

$$\mathrm{M}_n(f,g) := \frac{1}{|F_n|}\sum_{a\in F_n}(f\circ S^a)(g\circ T^a)$$

Theorem. K. (2014) $p \geq 2$

$$\sup_{n_0<n_1<\cdots<n_m}\sum_{j=1}^{m}\left\|\mathrm{M}_{n_{j-1}}(f,g)-\mathrm{M}_{n_j}(f,g)\right\|_{\mathrm{L}^p}^p \;\leq\; C_p\,\|f\|_{\mathrm{L}^{2p}}^p\|g\|_{\mathrm{L}^{2p}}^p$$

Consequences:

$\|f\|_{\mathrm{L}^{2p}}=\|g\|_{\mathrm{L}^{2p}}=1 \;\Rightarrow O\big(\varepsilon^{-p}\big)$ ε-jumps in L^p, $p\geq 2$

$\|f\|_{\mathrm{L}^\infty}=\|g\|_{\mathrm{L}^\infty}=1 \;\Rightarrow O(\varepsilon^{-\max\{p,2\}})$ ε-jumps in L^p, $p\geq 1$

Appl. #2 — Open problems

Open problem: double averages, $\mathbb{Z}_+$-actions or $\mathbb{Z}$-actions

$$\mathrm{M}_n(f,g) := \frac{1}{n}\sum_{k=0}^{n-1}(f\circ S^k)(g\circ T^k)$$

$$\sup_{n_0<n_1<\cdots<n_m}\sum_{j=1}^{m}\left\|\mathrm{M}_{n_{j-1}}(f,g)-\mathrm{M}_{n_j}(f,g)\right\|_{\mathrm{L}^p}^p \le C_p\,\|f\|_{\mathrm{L}^{2p}}^p\|g\|_{\mathrm{L}^{2p}}^p$$

Open problem: triple averages, "powers" of a single $\mathbb{A}^\omega$-action

$$\mathrm{M}_n(f,g,h) := \frac{1}{|F_n|}\sum_{a\in F_n}(f\circ T^{c_1 a})(g\circ T^{c_2 a})(h\circ T^{c_3 a}),\quad c_1,c_2,c_3\in\mathbb{Z}$$

$$\sup_{n_0<\cdots<n_m}\sum_{j=1}^{m}\left\|\mathrm{M}_{n_{j-1}}(f,g,h)-\mathrm{M}_{n_j}(f,g,h)\right\|_{\mathrm{L}^p}^p \le C_p\,\|f\|_{\mathrm{L}^{3p}}^p\|g\|_{\mathrm{L}^{3p}}^p\|h\|_{\mathrm{L}^{3p}}^p$$

Thank you!

Thank you for your attention!