MULTILINEAR ESTIMATES VIA OPTIMAL CONTROL

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The Bellman function technique

- Inspired by Bellman's work on the optimal control in the 1960s
- Burkholder introduced the trick to harmonic analysis in the 1980s
- Nazarov, Treil, and Volberg started shaping it into a powerful "method" in the 1990s
- The main idea is:
  general estimate
  $\iff$ existence of a function with certain convexity properties
  $\longrightarrow$ methodical search for such function
- It can also be thought of as a "clever induction on scales" with carefully chosen control parameters
- This presentation is a *case study*, rather than an overview

Ellipticity and p-ellipticity

$\Omega \subseteq \mathbb{R}^d$ an open set

$A\colon \Omega \to \mathbb{C}^{d\times d}$ a matrix function with L^∞ coefficients
(Note: non-smooth and complex)

A is *elliptic* if $\exists \lambda, \Lambda \in (0,\infty)$ s.t.

$$\operatorname{Re} \langle A(x)\xi, \xi\rangle_{\mathbb{C}^d} \geqslant \lambda |\xi|^2 \qquad \text{for } \xi \in \mathbb{C}^d, \text{ for a.e. } x \in \Omega$$
$$\big| \langle A(x)\xi, \eta\rangle_{\mathbb{C}^d} \big| \leqslant \Lambda \, |\xi| \, |\eta| \qquad \text{for } \xi, \eta \in \mathbb{C}^d, \text{ for a.e. } x \in \Omega$$

A is *p-elliptic* for $p \in (1,\infty)$ if additionally $\exists \Delta_p \in (0,\infty)$ s.t.

$$\operatorname{Re}\Big\langle A(x)\xi, \, \xi + |1-2/p| \, \bar{\xi} \Big\rangle_{\mathbb{C}^d} \geqslant \Delta_p |\xi|^2 \quad \text{for } \xi \in \mathbb{C}^d, \text{ for a.e. } x \in \Omega$$

Properties of p-elliptic matrix functions

$$\mathcal{A}_p(\Omega) = \text{the class of } p\text{-elliptic matrix functions on } \Omega$$

- $\mathcal{A}_p(\Omega) = \mathcal{A}_{p'}(\Omega)$, where $1/p' + 1/p = 1$
- $\mathcal{A}_p(\Omega)$ increases in $p \in (1, 2]$ and decreases in $p \in [2, \infty)$
- $\{\text{elliptic on } \Omega\} = \mathcal{A}_2(\Omega)$
- $\{\text{real elliptic on } \Omega\} = \bigcap_{p\in(1,\infty)} \mathcal{A}_p(\Omega)$

History of p-ellipticity

$$\mathrm{Re}\left\langle A(x)\xi,\, \xi + |1-2/p|\,\bar{\xi}\right\rangle_{\mathbb{C}^d} \geqslant \Delta_p|\xi|^2 \quad \text{for } \xi \in \mathbb{C}^d, \text{ for a.e. } x \in \Omega$$

- Introduced by Carbonaro and Dragičević (2016)
- A variant of the condition was introduced by Cialdea and Maz'ya (2005) to characterize L^p-contractivity of the generated semigroup
- An equivalent condition was introduced by Dindoš and Pipher (2016) in the context of regularity theory of elliptic PDEs

Divergence-form operators

Boundary conditions reflect the choice of $\mathscr{U}$:

- *Dirichlet*: $\mathscr{U} = H^1_0(\Omega) = \overline{C_c^\infty(\Omega)}$ in $H^1(\Omega)$
- *Neumann*: $\mathscr{U} = H^1(\Omega) = W^{1,2}(\Omega)$
- *mixed*: $\mathscr{U} = \overline{\{u|_\Omega : u \in C_c^\infty(\mathbb{R}^d \backslash \Gamma)\}}$ in $H^1(\Omega)$, $\Gamma \subseteq \partial\Omega$ closed

Divergence-form operator informally: "$L_{A,\mathscr{U}} u = -\operatorname{div}(A\nabla u)$"

Rigorously:

$$\langle L_{A,\mathscr{U}} u, v \rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d} \qquad \text{for } u \in \mathcal{D}(L_{A,\mathscr{U}}),\ v \in \mathscr{U},$$

where $\mathcal{D}(L_{A,\mathscr{U}}) := \{u \in \mathscr{U} : \text{RHS extends boundedly to } \mathrm{L}^2(\Omega)\}$

$(T_t^{A,\mathscr{U}})_{t \geqslant 0}$ is the operator semigroup on $\mathrm{L}^2(\Omega)$ generated by $-L_{A,\mathscr{U}}$

Trilinear embedding

Take $p, q, r \in (1, \infty)$ s.t. $1/p + 1/q + 1/r = 1$

Theorem (Carbonaro, Dragičević, K., and Škreb (2020))

Suppose that $A, B, C : \Omega \to \mathbb{C}^{d\times d}$ are $\max\{p,q,r\}$-elliptic. Then for $f \in (\mathrm{L}^p \cap \mathrm{L}^2)(\Omega)$, $g \in (\mathrm{L}^q \cap \mathrm{L}^2)(\Omega)$ and $h \in (\mathrm{L}^r \cap \mathrm{L}^2)(\Omega)$ we have

$$\int_0^\infty \int_\Omega \left|T_t^{A,\mathscr{U}} f\right| \left|\nabla T_t^{B,\mathscr{V}} g\right| \left|\nabla T_t^{C,\mathscr{W}} h\right| \mathrm{d}x\, \mathrm{d}t \leqslant \mathcal{C} \left\|f\right\|_{\mathrm{L}^p(\Omega)} \left\|g\right\|_{\mathrm{L}^q(\Omega)} \left\|h\right\|_{\mathrm{L}^r(\Omega)}$$

When $\Omega = \mathbb{R}^d$, the same conclusion holds if only: A is p-elliptic, B is q-elliptic and $(1+q/r)$-elliptic, C is r-elliptic and $(1+r/q)$-elliptic

The embedding constant $\mathcal{C}$ only depends on p, q, r and the $*$-ellipticity constants of A, B, C; in particular the estimate is "dimension-free"

Applications of the trilinear embedding

- By "removing absolute values" we, in particular, bound the following *semigroup paraproduct*:

$$\Theta(f,g,h) := \int_0^\infty \int_{\mathbb{R}^d} \psi(t\,L_{A,\mathscr{U}})f\,\varphi(t\,L_{B,\mathscr{V}})g\,\varphi(t\,L_{C,\mathscr{W}})h\,\mathrm{d}x\,\frac{\mathrm{d}t}{t}$$

  for

$$\psi(z) := ze^{-z}, \quad \varphi(z) = e^{-z};$$

  a possible beginning of a *paradifferential calculus* for complex divergence-form operators

Applications of the trilinear embedding

- Auscher, Hofmann, and Martell (2012) proved boundedness of the *"conical" square function*:

$$(\mathcal{CS}^A f)(x) := \left(\iint_{\{|x-y|<\sqrt{t}\}} \left|\nabla(T_t^A f)(y)\right|^2 \, \frac{dy\, dt}{t^{d/2}} \right)^{1/2}$$

  for real elliptic A; our main theorem refines their result
- *Kato–Ponce inequalities*, i.e., *fractional Leibniz rules*:

$$\left\| L^\beta_{A,\mathscr{U}}(fg) \right\|_{L^s(\Omega)} \leqslant C \left(\left\| L^\beta_{A,\mathscr{U}} f \right\|_{L^{p_1}(\Omega)} \|g\|_{L^{q_1}(\Omega)} + \|f\|_{L^{p_2}(\Omega)} \left\| L^\beta_{A,\mathscr{U}} g \right\|_{L^{q_2}(\Omega)} \right)$$

  where $1/s = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, $\beta \in (0, 1/s)$

A toy-problem

For a dyadic interval $I \subset \mathbb{R}$ and a locally integrable function $f\colon \mathbb{R} \to \mathbb{C}$ denote

$$\langle f \rangle_I := \frac{1}{|I|} \int_I f$$

Let us translate the problem from the semigroup setting to a dyadic martingale setting:

$$\begin{array}{rccc}
 & (t, x) & \longleftrightarrow & \text{dyadic interval } I \\
\text{scale:} & t & \longleftrightarrow & |I| \\
\text{position:} & x & \longleftrightarrow & \text{left endpoint of } I \\
 & \int_0^\infty \int_{\mathbb{R}^d} \cdots \, dx \, dt & \longleftrightarrow & \sum_{I \text{ dyadic interval}} |I| \cdots \\
 & (T_t f)(x) & \longleftrightarrow & \langle f \rangle_I \\
 & (\nabla T_t f)(x) & \longleftrightarrow & \frac{1}{2}\left(\langle f \rangle_{I_{right}} - \langle f \rangle_{I_{left}} \right)
\end{array}$$

A toy-problem

The toy-model of our estimate:

$$\sum_{I \text{ dyadic interval}} |I| \left|\langle f\rangle_I\right| \frac{1}{2}\left|\langle g\rangle_{I_{left}} - \langle g\rangle_{I_{right}}\right| \frac{1}{2}\left|\langle h\rangle_{I_{left}} - \langle h\rangle_{I_{right}}\right|$$
$$\leqslant \mathcal{C} \, \|f\|_{L^p(\mathbb{R})} \, \|g\|_{L^q(\mathbb{R})} \, \|h\|_{L^r(\mathbb{R})}$$

This estimate can be shown very easily using:

- the well-known bounds for the dyadic martingale maximal function (in probability: Doob's inequality),
- the well-known bounds for the dyadic square function (in probability: the Burkholder–Davis–Gundy inequalities)

However, we want to give an "optimal control proof," which will translate to our setting (where maximal and square function bounds are no longer available)

The Bellman function restatement

Rewrite the problem in terms of the averages:

$$\Phi_I(f,g,h) := \frac{1}{|I|}\sum_{J\subseteq I} |J| \left|\langle f\rangle_J\right| \frac{1}{2}\left|\langle g\rangle_{J_{left}} - \langle g\rangle_{J_{right}}\right| \frac{1}{2}\left|\langle h\rangle_{J_{left}} - \langle h\rangle_{J_{right}}\right|$$

as the estimate:

$$\Phi_I(f,g,h) \leqslant \mathcal{C}\,\langle |f|^p\rangle_I^{1/p}\,\langle |g|^q\rangle_I^{1/q}\,\langle |h|^r\rangle_I^{1/r} \leqslant \mathcal{C}\left(\tfrac{1}{p}\,\langle |f|^p\rangle_I + \tfrac{1}{q}\,\langle |g|^q\rangle_I + \tfrac{1}{r}\,\langle |h|^r\rangle_I\right)$$

Define the *abstract Bellman function*:

$$\mathcal{B}(u,v,w,U,V,W) := \sup_{f,g,h} \Phi_I(f,g,h),$$

where the supremum is taken over all f,g,h s.t.

$$\langle f\rangle_I = u,\ \langle g\rangle_I = v,\ \langle h\rangle_I = w,\ \langle |f|^p\rangle_I = U,\ \langle |g|^q\rangle_I = V,\ \langle |h|^r\rangle_I = W$$

The Bellman function restatement

($\mathcal{B}$1) *Domain*:

$$u, v, w \in \mathbb{C}, \quad U, V, W \in [0, \infty), \quad |u|^p \leqslant U, \quad |v|^q \leqslant V, \quad |w|^r \leqslant W$$

($\mathcal{B}$2) *Range*:

$$0 \leqslant \mathcal{B}(u, v, w, U, V, W) \leqslant \mathcal{C}\left(\tfrac{1}{p}U + \tfrac{1}{q}V + \tfrac{1}{r}W \right)$$

($\mathcal{B}$3) *Certain concavity*:

$$\begin{aligned} \mathcal{B}(u, v, w, U, V, W) \geqslant & \frac{1}{2}\mathcal{B}(u_1, v_1, w_1, U_1, V_1, W_1) \\ & + \frac{1}{2}\mathcal{B}(u_2, v_2, w_2, U_2, V_2, W_2) \\ & + \frac{1}{2}|u_1 + u_2| \, \frac{1}{2}|v_1 - v_2| \, \frac{1}{2}|w_1 - w_2| \end{aligned}$$

whenever $u = \frac{1}{2}(u_1 + u_2)$, $v = \frac{1}{2}(v_1 + v_2)$, etc. and all three 6-tuples belong to the domain

The Bellman function restatement

Substitute $\triangle u = \frac{1}{2}(u_1 - u_2)$, etc.

Assume that $\mathcal{B}$ is C^1 and "piecewise" C^2:

$(\mathcal{B}3')$ *Infinitesimal version*:

$$-\frac{1}{2}(\underbrace{d^2\mathcal{B}}_{\text{quadratic form}})\underbrace{(u,v,w,U,V,W)}_{\text{at a point}}\underbrace{(\triangle u, \triangle v, \triangle w, \triangle U, \triangle V, \triangle W)}_{\text{on a vector}} \geqslant |u||\triangle v||\triangle w|$$

$$(\mathcal{B}3) \overset{\text{Taylor's formula}}{\Longrightarrow} (\mathcal{B}3')$$

$$(\mathcal{B}3') \overset{\text{convexity of the domain}}{\Longrightarrow} (\mathcal{B}3)$$

The Bellman function restatement

Sufficiency of ($\mathcal{B}$1)–($\mathcal{B}$3)

Apply ($\mathcal{B}$3) n times:

$$
\begin{aligned}
&|I|\,\mathcal{B}\big(\langle f\rangle_I, \langle g\rangle_I, \langle h\rangle_I, \langle f^p\rangle_I, \langle g^q\rangle_I, \langle h^r\rangle_I\big) \\
&\geqslant \sum_{\substack{J\subseteq I\\ |J|=2^{-n}|I|}} |J|\, \underbrace{\mathcal{B}\big(\langle f\rangle_J, \langle g\rangle_J, \langle h\rangle_J, \langle f^p\rangle_J, \langle g^q\rangle_J, \langle h^r\rangle_J\big)}_{\geqslant 0} \\
&\quad + \sum_{\substack{J\subseteq I\\ |J|>2^{-n}|I|}} \big|\langle f\rangle_J\big|\, \frac{1}{2}\big|\langle g\rangle_{J_{left}} - \langle g\rangle_{J_{right}}\big|\, \frac{1}{2}\big|\langle h\rangle_{J_{left}} - \langle h\rangle_{J_{right}}\big|
\end{aligned}
$$

Use ($\mathcal{B}$2) and let $n \to \infty$

Construction of $\mathcal{B}$

From a predating paper by K. and Škreb (2016)

Motivated by ($\mathcal{B}$2) we make the initial ansatz:

$$\mathcal{B}(u,v,w,U,V,W) \;=\; \mathcal{C}\left(\tfrac{1}{p}U + \tfrac{1}{q}V + \tfrac{1}{r}W\right) \;-\; \alpha(u,v,w)$$

Properties ($\mathcal{B}$1), ($\mathcal{B}$2), ($\mathcal{B}$3′) can now be restated as the following properties of α:

($\mathcal{A}$1) *Domain*: $\mathbb{C}^3$

($\mathcal{A}$2) *Range*:

$$0 \leqslant \alpha(u,v,w) \leqslant \widetilde{\mathcal{C}}\left(|u|^p + |v|^q + |w|^r\right)$$

($\mathcal{A}$3′) *Certain convexity*:

$$\Big(\underbrace{d^2\alpha}_{\text{quadratic form}}\Big)\underbrace{(u,v,w)}_{\text{at a point}}\underbrace{(\triangle u, \triangle v, \triangle w)}_{\text{on a vector}} \geqslant 2|u||\triangle v||\triangle w|$$

Construction of $\mathcal{B}$

WLOG assume $q > r$ and make another ansatz motivated by ($\mathcal{A}2$):

$$\alpha(u, v, w) = |u|^p \, \gamma\Big(\underbrace{\frac{|v|^q}{|u|^p}}_{t}, \underbrace{\frac{|w|^r}{|u|^p}}_{s} \Big)$$

Consult the literature:

- a single variable function $\gamma(t)$ was constructed by Nazarov and Treil (1995) and it was used in all bilinear embeddings;
- it is made of three powers of t: $1, t^c, t$;
- try to make $\gamma(t, s)$ out of linear combinations of powers of t, s

Construction of $\mathcal{B}$

$$\gamma(t,s) = \begin{cases} a_1 + b_1 t + c_1 s; & 1 \leqslant s \leqslant t \\ a_2 + b_2 t + c_2 s^{\frac{1}{p'}}; & s \leqslant 1 \leqslant t \\ a_3 + b_3 t^{\frac{1}{p'}} + c_3 s^{\frac{1}{p'}}; & s \leqslant t \leqslant 1 \\ a_4 + b_4 t^{\frac{2}{q}} s^{\frac{1}{r} - \frac{1}{q}} + c_4 s^{\frac{1}{p'}}; & t \leqslant s \leqslant 1 \\ a_5 + b_5 t^{\frac{2}{q}} + c_5 t^{\frac{2}{q}} s^{1-\frac{2}{q}} + d_5 s; & t \leqslant 1 \leqslant s \\ a_6 + b_6 t + c_6 t^{\frac{2}{q}} s^{1-\frac{2}{q}} + d_6 s; & 1 \leqslant t \leqslant s \end{cases}$$

Adjust the coefficients so that γ is C^1

Construction of $\mathcal{B}$

$$\gamma(t,s) = \begin{cases} a + bt + cs; & 1 \leqslant s \leqslant t \\ \frac{a(p-1)-c}{p-1} + bt + \frac{cp}{p-1} s^{\frac{1}{p'}}; & s \leqslant 1 \leqslant t \\ \frac{a(p-1)-(b+c)}{p-1} + \frac{bp}{p-1} t^{\frac{1}{p'}} + \frac{cp}{p-1} s^{\frac{1}{p'}}; & s \leqslant t \leqslant 1 \\ \frac{a(p-1)-(b+c)}{p-1} + \frac{bq}{2} t^{\frac{2}{q}} s^{\frac{1}{r}-\frac{1}{q}} + \frac{2cpr-bp(q-r)}{2r(p-1)} s^{\frac{1}{p'}}; & t \leqslant s \leqslant 1 \\ \frac{2ar(p-1)-b(q+r)}{2r(p-1)} + \frac{bq^2}{2p(q-2)} t^{\frac{2}{q}} + \frac{bq(q-r)}{2r(q-2)} t^{\frac{2}{q}} s^{1-\frac{2}{q}} & \\ \qquad + \frac{2cr-b(q-r)}{2r} s; & t \leqslant 1 \leqslant s \\ a + \frac{bq}{p(q-2)} t + \frac{bq(q-r)}{2r(q-2)} t^{\frac{2}{q}} s^{1-\frac{2}{q}} + \frac{2cr-b(q-r)}{2r} s; & 1 \leqslant t \leqslant s \end{cases}$$

Choose $a, b, c > 0$ appropriately and we are done!

Proof of the TE — special case

Let us illustrate the proof of the trilinear embedding in the very special case (also known previously):
$d = 1$, $\Omega = \mathbb{R}$, $A = B = C = I$ (one-dimensional heat semigroups)

Assume for simplicity that $\mathcal{B} \in \mathrm{C}^2$ (mollify it)

$$u(x,t) := (T_t f)(x), \ \ldots, \ U(x,t) := (T_t |f|^p)(x), \ \ldots$$

$$\mathbf{b}(x,t) := \mathcal{B}\big(u(x,t), v(x,t), w(x,t), U(x,t), V(x,t), W(x,t)\big)$$

$$\Longrightarrow \ \big(\partial_t - \tfrac{1}{2}\partial_x^2\big)\mathbf{b}(x,t) = (\nabla\mathcal{B})(u,v,\ldots)\cdot \underbrace{\big(\partial_t - \tfrac{1}{2}\partial_x^2\big)(u,v,\ldots)}_{=0}$$
$$- \tfrac{1}{2}(d^2\mathcal{B})(u,v,\ldots)(\partial_x u, \partial_x v, \ldots)$$

Using $(\mathcal{B}3')$ we get

$$|u(x,t)|\,|\partial_x v(x,t)|\,|\partial_x w(x,t)| \leqslant \big(\partial_t - \tfrac{1}{2}\partial_x^2\big)\mathbf{b}(x,t)$$

Proof of the TE — special case

Integrating by parts and using $\mathcal{B} \geqslant 0$ we get for $\delta, M > 0$:

$$\int_{\mathbb{R}\times(\delta,M-\delta)} k(x, M-t)\,|u(x,t)|\,|\partial_x v(x,t)|\,|\partial_x w(x,t)|\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \int_{\mathbb{R}} k(x,\delta)\,\mathbf{b}(x, M-\delta)\,\mathrm{d}x$$

Letting $\delta \to 0$ and using $(\mathcal{B}2)$:

$$\int_{\mathbb{R}\times(0,M)} k(x, M-t)\,|u(x,t)|\,|\partial_x v(x,t)|\,|\partial_x w(x,t)|\,\mathrm{d}x\,\mathrm{d}t \;\leqslant\; \mathbf{b}(0, M)$$
$$\leqslant \frac{\mathcal{C}}{\sqrt{2\pi M}}\Big(\frac{1}{p}\int_{\mathbb{R}} |f(y)|^p e^{-\frac{y^2}{2M}}\,\mathrm{d}y + \frac{1}{q}\int_{\mathbb{R}} |g(y)|^q e^{-\frac{y^2}{2M}}\,\mathrm{d}y + \frac{1}{r}\int_{\mathbb{R}} |h(y)|^r e^{-\frac{y^2}{2M}}\,\mathrm{d}y\Big)$$

Proof of the TE — special case

Now we have

$$\int_{\mathbb{R}\times(0,M)} \sqrt{2\pi M}\, k(x, M-t)\ |(T_t f)(x)|\ |(T_t g)'(x)|\ |(T_t h)'(x)|\, \mathrm{d}x\, \mathrm{d}t \leqslant \mathcal{C}\left(\tfrac{1}{p}\|f\|_{\mathrm{L}^p(\mathbb{R})} + \tfrac{1}{q}\|g\|_{\mathrm{L}^q(\mathbb{R})} + \tfrac{1}{r}\|h\|_{\mathrm{L}^r(\mathbb{R})}\right)$$

i.e., homogenizing back,

$$\int_{\mathbb{R}\times(0,M)} \sqrt{2\pi M}\, k(x, M-t)\ |(T_t f)(x)|\ |(T_t g)'(x)|\ |(T_t h)'(x)|\, \mathrm{d}x\, \mathrm{d}t \leqslant \mathcal{C}\, \|f\|_{\mathrm{L}^p(\mathbb{R})}\|g\|_{\mathrm{L}^q(\mathbb{R})}\|h\|_{\mathrm{L}^r(\mathbb{R})}$$

Observe that $\sqrt{2\pi M}\, k(x, M-t) \longrightarrow 1$ as $M \to \infty$ uniformly over $(x,t) \in [-R, R] \times (0, N]$, so let $M \to \infty$, $R \to \infty$, $N \to \infty$ to obtain

$$\int_{\mathbb{R}\times(0,\infty)} |(T_t f)(x)|\,|(T_t g)'(x)|\,|(T_t h)'(x)|\, \mathrm{d}x\, \mathrm{d}t \leqslant \mathcal{C}\, \|f\|_{\mathrm{L}^p(\mathbb{R})}\|g\|_{\mathrm{L}^q(\mathbb{R})}\|h\|_{\mathrm{L}^r(\mathbb{R})}$$

In this particular case we did not use the structure of $\mathcal{B}$

Proof of the TE — general case

What is the relation to p-ellipticity?

For $\alpha : \mathbb{C}^3 \to \mathbb{R}$, $A, B, C \in \mathbb{C}^{d\times d}$, $(u, v, w) \in \mathbb{C}^3$, and $(\zeta, \eta, \xi) \in (\mathbb{C}^d)^3$ we define the *generalized Hessian form of* α *with respect to* (A, B, C),

$$H_{\alpha}^{A,B,C}[(u, v, w); (\zeta, \eta, \xi)],$$

as the standard inner product of

$$\begin{bmatrix} \operatorname{Re} A & -\operatorname{Im} A & & & & \\ \operatorname{Im} A & \operatorname{Re} A & & & & \\ & & \operatorname{Re} B & -\operatorname{Im} B & & \\ & & \operatorname{Im} B & \operatorname{Re} B & & \\ & & & & \operatorname{Re} C & -\operatorname{Im} C \\ & & & & \operatorname{Im} C & \operatorname{Re} C \end{bmatrix} \begin{bmatrix} \operatorname{Re} \zeta \\ \operatorname{Im} \zeta \\ \operatorname{Re} \eta \\ \operatorname{Im} \eta \\ \operatorname{Re} \xi \\ \operatorname{Im} \xi \end{bmatrix} \in (\mathbb{R}^d)^6$$

Proof of the TE — general case

and

$$\left(\mathrm{Hess}(\alpha;(u,v,w))\otimes I_{\mathbb{R}^d}\right)\begin{bmatrix}\mathrm{Re}\,\zeta\\ \mathrm{Im}\,\zeta\\ \mathrm{Re}\,\eta\\ \mathrm{Im}\,\eta\\ \mathrm{Re}\,\xi\\ \mathrm{Im}\,\xi\end{bmatrix}\in(\mathbb{R}^d)^6$$

Here one has to interpret $\mathrm{Hess}(\alpha;(u,v,w))$ as the 6×6 real Hessian matrix of the function

$$\mathbb{R}^6\to\mathbb{R},\quad (u_r,u_i,v_r,v_i,w_r,w_i)\mapsto\alpha(u_r+iu_i,v_r+iv_i,w_r+iw_i)$$

Proof of the TE — general case

Lemma

If $\alpha(u,v,w) := |u|^a|v|^b|w|^c$ *for some* $a,b,c \in [0,\infty\rangle$, *then*

$$H_{\alpha}^{A,B,C}[(u,v,w);(\zeta,\eta,\xi)]$$
$$\geqslant \frac{1}{2}|u|^a|v|^b|w|^c\Big(a^2\Delta_a(A)\Big|\frac{\zeta}{u}\Big|^2 + b^2\Delta_b(B)\Big|\frac{\eta}{v}\Big|^2 + c^2\Delta_c(C)\Big|\frac{\xi}{w}\Big|^2$$
$$-2(\Lambda(A)+\Lambda(B))\Big|\frac{\zeta}{u}\Big|\Big|\frac{\eta}{v}\Big| - 2(\Lambda(A)+\Lambda(C))\Big|\frac{\zeta}{u}\Big|\Big|\frac{\xi}{w}\Big| - 2(\Lambda(B)+\Lambda(C))\Big|\frac{\eta}{v}\Big|\Big|\frac{\xi}{w}\Big|\Big)$$

We see that appropriate p-ellipticity conditions can (potentially) guarantee generalized convexity properties

It is now very convenient (or even crucial?) that the constructed Bellman function $\mathcal{B}$ is made of powers.

Thank you for your attention!