Lower bounds for the L^p norms of some Fourier multipliers

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Lower estimates

Upper vs. lower estimates

Upper estimates for operator norms

$$\|T\|_{\mathrm{L}^p\to\mathrm{L}^p} \leqslant C(T,p)$$

are only as good as we can match them with lower estimates

$$\|T\|_{\mathrm{L}^p\to\mathrm{L}^p} \geqslant c(T,p).$$

The latter might amount to merely constructing an **example** of f s.t.

$$\|f\|_{\mathrm{L}^p} = 1 \quad \text{and} \quad \|Tf\|_{\mathrm{L}^p} \geqslant c(T,p),$$

but concrete examples can be quite complicated.

Sharp estimates

Let $(T^k)_{k\in\mathbb{Z}}$ or $(T^\lambda)_{\lambda\in\mathbb{R}}$ be a one-parameter group of bounded linear operators on every $\mathrm{L}^p(\mathbb{R}^n)$, $p \in (1,\infty)$.

- T^λ are just integer or real powers of T.
- A typical situation with singular integrals.

In this talk the estimates are considered *sharp* if they are of the form

$$c(T)\,\kappa(\lambda,p) \leqslant \|T^\lambda\|_{\mathrm{L}^p\to\mathrm{L}^p} \leqslant C(T)\,\kappa(\lambda,p),$$

$$\|T^\lambda\|_{\mathrm{L}^p\to\mathrm{L}^p} \sim_T \kappa(\lambda,p),$$

i.e., we insist on finding sharp asymptotics simultaneously in λ and p.

Our techniques for lower bounds will occasionally give **exact** constants, but the exact upper bounds are rarely available.

ON UPPER ESTIMATES

$$(S_\Omega f)(x) := \mathrm{p.\,v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(y/|y|)}{|y|^n}\, \mathrm{d}y$$

$$\|S_\Omega\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|\Omega\|_{\mathrm{L}^2(\mathbb{S}^{n-1})}$$

$$\|\Omega\|_{\mathrm{L}\log\mathrm{L}(\mathbb{S}^{n-1})}$$

Christ and Rubio de Francia (1988), Hofmann (1988), Seeger (1996), Tao (1999), etc.

In combination with the easy L^2 bound and the real interpolation this often gives sharp L^p estimates.

In this talk we are only discussing the lower estimates.

The complex Riesz transform

POWERS OF THE COMPLEX RIESZ TRANSFORM

$$R = R_2 + \mathbb{i}R_1, \quad R_1, R_2 = \text{2D Riesz transforms}$$

As Fourier multipliers:

$$(\widehat{R^k f})(\zeta) = \left(\frac{\bar{\zeta}}{|\zeta|}\right)^k \widehat{f}(\zeta); \quad \zeta \in \mathbb{C}.$$

As singular integrals:

$$(R^k f)(z) = \frac{\mathbb{i}^{|k|}|k|}{2\pi}\,\mathrm{p.\,v.}\int_{\mathbb{C}} f(z-w)\frac{(w/|w|)^{-k}}{|w|^2}\,\mathrm{d}A(w).$$

R^2 is the Ahlfors–Beurling operator; its symbol is $\left(\frac{\bar{\zeta}}{|\zeta|}\right)^2 = \frac{\bar{\zeta}}{\zeta}$.

PROBLEM BY IWANIEC AND MARTIN

Problem [Iwaniec and Martin (1996)].

What is the asymptotics of $\|R^k\|_{\mathrm{L}^p(\mathbb{C})\to\mathrm{L}^p(\mathbb{C})}$?

$$p \in (1,\infty), \qquad \frac{1}{p}+\frac{1}{q}=1, \qquad p^* := \max\{p,q\}$$

- Dragičević, Petermichl, and Volberg (2006) and Dragičević (2011) resolved the case of even $k \in \mathbb{Z}\setminus\{0\}$:
$$\|R^k\|_{\mathrm{L}^p(\mathbb{C})\to\mathrm{L}^p(\mathbb{C})} \sim (p^*-1)\,|k|^{1-2/p^*}.$$
- Examples were certain truncations of $|z|^{-2/p}$, which is **not** in $\mathrm{L}^p(\mathbb{C})$.
- Even k are easier because $\quad R^2\colon \partial_{\bar z} f \mapsto \partial_z f$.
- Odd k are harder as we only have $\quad R\colon (-\Delta)^{1/2} f \mapsto -2\mathrm{i}\partial_z f$.

The asymptotics

Theorem [Carbonaro, Dragičević, and K. (2021)].

$$\begin{aligned}\|R^k\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} &\sim (p^* - 1)\,|k|^{2|1/2-1/p|}\\ &\sim p^*\,|k|^{1-2/p^*}\\ \|R^k\|_{L^1(\mathbb{C})\to L^{1,\infty}(\mathbb{C})} &\sim |k|\end{aligned}$$

for $p \in (1,\infty)$ and $k \in \mathbb{Z}\setminus\{0\}$.

In the paper we gave 3 different proofs of the lower estimate.

Theorem [Carbonaro, Dragičević, and K. (2021)].

$$\|R^k\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} \geqslant \frac{\Gamma(1/p)\Gamma(1/q+k/2)}{\Gamma(1/q)\Gamma(1/p+k/2)} \geqslant \frac{1}{2}(p-1)\,k^{1-2/p}$$

for $p \geqslant 2$ and $k \in \mathbb{N}$.

APPROXIMATE EXTREMIZERS

$$f = f_{k,p,\varepsilon}, \quad g = g_{p,\varepsilon} \qquad \text{for } \varepsilon \in (0,1]$$

$$R^k f = g$$

$\widehat{g}$ is a truncation of $|\zeta|^{-2/q}$ $=$ F.T. of $|z|^{-2/p}$

$\widehat{f}$ is a truncation of $\left(\frac{\zeta}{|\zeta|}\right)^k |\zeta|^{-2/q}$ $=$ F.T. of $\left(\frac{z}{|z|}\right)^k |z|^{-2/p}$ (up to multiplicative constants)

Decomposing into Gaussians (Stein's trick):

$$\widehat{g}(\zeta) := \int_\varepsilon^{1/\varepsilon} e^{-\pi t^2 |\zeta|^2} t^{1-2/p}\, \mathrm{d}t; \quad \zeta \in \mathbb{C}$$

$$\widehat{f}(\zeta) := \left(\frac{\zeta}{|\zeta|}\right)^k \int_\varepsilon^{1/\varepsilon} e^{-\pi t^2 |\zeta|^2} t^{1-2/p}\, \mathrm{d}t; \quad \zeta \in \mathbb{C}$$

Approximate extremizers

$$f(z) = \mathbb{i}^k \pi^{-1/p-1/2} \Big(\frac{z}{|z|}\Big)^k |z|^{-2/p} \int_0^{\pi/2} \Big(\underbrace{\int_{\pi(\sin\vartheta)^2\varepsilon^2|z|^2}^{\pi(\sin\vartheta)^2\varepsilon^{-2}|z|^2} x^{1/p-1/2} e^{-x}\,\mathrm{d}x}_{\longrightarrow \Gamma(1/p+1/2) \text{ as } \varepsilon\to 0+} \Big) \frac{\sin k\vartheta\,\mathrm{d}\vartheta}{(\sin\vartheta)^{2/p}}$$

($k \in \mathbb{N}$ odd)

$$\|f\|_{L^p(\mathbb{C})} = 2^{2/p}\pi^{-1/2}\Gamma\Big(\frac{1}{p}+\frac{1}{2}\Big) I_{k,1/p} \Big(\log\frac{1}{\varepsilon}\Big)^{1/p} + O_{k,p}^{\varepsilon\to 0+}(1),$$

where

$$I_{k,\alpha} := \int_0^{\pi/2} \frac{\sin k\vartheta}{(\sin\vartheta)^{2\alpha}}\,\mathrm{d}\vartheta = \frac{\pi^{1/2}}{2}\cdot\frac{\Gamma(1-\alpha)\Gamma(\alpha+k/2)}{\Gamma(\alpha+1/2)\Gamma(1-\alpha+k/2)}$$

Approximate extremizers

$$g(z) = \frac{1}{2}\pi^{-1/p}|z|^{-2/p} \underbrace{\int_{\pi\varepsilon^2|z|^2}^{\pi\varepsilon^{-2}|z|^2} x^{1/p-1}e^{-x}\,\mathrm{d}x}_{\longrightarrow\Gamma(1/p)\text{ as }\varepsilon\to 0+}$$

$$\|g\|_{L^p(\mathbb{C})} = 2^{-(1-2/p)}\Gamma\Big(\frac{1}{p}\Big)\Big(\log\frac{1}{\varepsilon}\Big)^{1/p} + O_p^{\varepsilon\to 0+}(1)$$

Finally,

$$\|R^k\|_{\mathrm{L}^p(\mathbb{C})\to\mathrm{L}^p(\mathbb{C})} \geqslant \lim_{\varepsilon\to 0+}\frac{\|g_{p,\varepsilon}\|_{L^p(\mathbb{C})}}{\|f_{k,p,\varepsilon}\|_{L^p(\mathbb{C})}}$$

gives the desired constant. □

Generalization

Spherical harmonics

Spherical harmonics of degree $j \geqslant 0$ are restrictions to $\mathbb{S}^{n-1}$ of harmonic homogeneous polynomials in n variables of degree j.

An example in $n = 2$ dimensions: $Y(x, y) = (x + iy)^k$, $Y(z) = z^k$, $Y(e^{\mathrm{i}\varphi}) = e^{\mathrm{i}k\varphi}$ for $k \in \mathbb{Z}$.

Why are they important for us?

Bochner (1951): $Y =$ a spherical harmonic of degree j.

$$K(x) = Y\Big(\frac{x}{|x|}\Big)|x|^{-n/p} \implies \widehat{K}(\xi) = \mathrm{i}^{-j}\gamma_{n,j,n/q} Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-n/q}$$

Stein and Weiss (1971):

$$K(x) = \mathrm{p.\,v.}\, Y\Big(\frac{x}{|x|}\Big)|x|^{-n} \implies \widehat{K}(\xi) = \mathrm{i}^{-j}\gamma_{n,j,0} Y\Big(\frac{\xi}{|\xi|}\Big)$$

$$\gamma_{n,j,\alpha} := \pi^{n/2-\alpha}\frac{\Gamma((j+\alpha)/2)}{\Gamma((j+n-\alpha)/2)}$$

A general lower bound for multipliers T_m

Theorem [Bulj and K. (2022)]. Take $p \in [1,2]$, $m =$ a bounded homogeneous measurable symbol, $(Y_j)_{j=0}^{\infty}$ a sequence s.t.

- Y_j is a spherical harmonic on $\mathbb{S}^{n-1}$ of degree j;
- $u := \sum_{j=0}^{\infty} Y_j$ converges in $\mathrm{L}^q(\mathbb{S}^{n-1})$;
- $v := \sum_{j=0}^{\infty} \mathbb{i}^{-j}\gamma_{n,j,n/p} Y_j$ converges in $\mathrm{L}^2(\mathbb{S}^{n-1})$.

For $p > 1$, $q < \infty$:

$$\|T_m\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \geqslant \frac{\gamma_{n,0,n/q}}{\sigma(\mathbb{S}^{n-1})^{1/p}} \frac{|\langle m, v\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}} \geqslant c_n\,(q-1)\,\frac{|\langle m, v\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathrm{L}^q(\mathbb{S}^{n-1})}}.$$

For $p = 1$, $q = \infty$:

$$\|T_m\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \geqslant \frac{c}{n} \frac{|\langle m, v\rangle_{\mathrm{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathrm{L}^\infty(\mathbb{S}^{n-1})}}.$$

SMOOTH TRUNCATIONS

Here is a "quantification" of

$$K(x) = Y\Big(\frac{x}{|x|}\Big)|x|^{-n/p} \implies \widehat{K}(\xi) = \mathbb{i}^{-j}\gamma_{n,j,n/q}Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-n/q}.$$

Lemma. Take $p \in (1, \infty)$, $\varepsilon \in (0, 1/2]$, and a spherical harmonic Y of degree $j \geqslant 0$. One can find a Schwartz function $h = h_{n,p,\varepsilon,Y}$ s.t.

$$\Big\| h(x) - Y\Big(\frac{x}{|x|}\Big)|x|^{-n/p}\mathbb{1}_{\{\varepsilon\leqslant|x|\leqslant 1/\varepsilon\}}(x)\Big\|_{\mathrm{L}^p_x(\mathbb{R}^n)} \lesssim_{n,p,Y} 1,$$

$$\Big\| \widehat{h}(\xi) - \mathbb{i}^{-j}\gamma_{n,j,n/q}Y\Big(\frac{\xi}{|\xi|}\Big)|\xi|^{-n/q}\mathbb{1}_{\{\varepsilon\leqslant|\xi|\leqslant 1/\varepsilon\}}(\xi)\Big\|_{\mathrm{L}^q_\xi(\mathbb{R}^n)} \lesssim_{n,p,Y} 1.$$

Idea of proof: use superpositions of Gaussians similarly as before.

PROOF OF THE THEOREM

Choose a Schwartz function f that differs by $O_{n,p}^{\varepsilon\to 0+}(1)$ from

$$x \mapsto |x|^{-n/p}\mathbb{1}_{\{\varepsilon\leqslant|x|\leqslant 1/\varepsilon\}}(x) \qquad \text{in } \mathrm{L}^p,$$

while $\widehat{f}$ differs by $O_{n,p}^{\varepsilon\to 0+}(1)$ from

$$\xi \mapsto \gamma_{n,0,n/q}|\xi|^{-n/q}\mathbb{1}_{\{\varepsilon\leqslant|\xi|\leqslant 1/\varepsilon\}}(\xi) \qquad \text{in } \mathrm{L}^q.$$

Choose a Schwartz function g that differs by $O_{n,p,m,J}^{\varepsilon\to 0+}(1)$ from

$$x \mapsto \Big(\sum_{j=0}^{J} Y_j\Big(\frac{x}{|x|}\Big)\Big)|x|^{-n/q}\mathbb{1}_{\{\varepsilon\leqslant|x|\leqslant 1/\varepsilon\}}(x) \qquad \text{in } \mathrm{L}^q,$$

while $\widehat{g}$ differs by $O_{n,p,m,J}^{\varepsilon\to 0+}(1)$ from

$$\xi \mapsto \Big(\sum_{j=0}^{J} \mathrm{i}^{-j}\gamma_{n,j,n/p} Y_j\Big(\frac{\xi}{|\xi|}\Big)\Big)|\xi|^{-n/p}\mathbb{1}_{\{\varepsilon\leqslant|\xi|\leqslant 1/\varepsilon\}}(\xi) \qquad \text{in } \mathrm{L}^p.$$

Proof of the theorem

$$\|T_m\|_{\mathsf{L}^p(\mathbb{R}^n)\to\mathsf{L}^p(\mathbb{R}^n)} \geqslant \frac{|\langle T_m f, g\rangle_{\mathsf{L}^2(\mathbb{R}^n)}|}{\|f\|_{\mathsf{L}^p(\mathbb{R}^n)}\|g\|_{\mathsf{L}^q(\mathbb{R}^n)}} = \frac{|\langle m\widehat{f}, \widehat{g}\rangle_{\mathsf{L}^2(\mathbb{R}^n)}|}{\|f\|_{\mathsf{L}^p(\mathbb{R}^n)}\|g\|_{\mathsf{L}^q(\mathbb{R}^n)}}$$
$$= \frac{\gamma_{n,0,n/q}|\langle m, \sum_{j=0}^{J} \mathrm{i}^{-j}\gamma_{n,j,n/p} Y_j\rangle_{\mathsf{L}^2(\mathbb{S}^{n-1})}| + o_{n,p,m,J}^{\varepsilon\to 0+}(1)}{\sigma(\mathbb{S}^{n-1})^{1/p}\|\sum_{j=0}^{J} Y_j\|_{\mathsf{L}^q(\mathbb{S}^{n-1})} + o_{n,p,m,J}^{\varepsilon\to 0+}(1)}$$

First take the limit as $\varepsilon \to 0+$; then take the limit as $J \to \infty$:

$$\|T_m\|_{\mathsf{L}^p(\mathbb{R}^n)\to\mathsf{L}^p(\mathbb{R}^n)} \geqslant \frac{\gamma_{n,0,n/q}}{\sigma(\mathbb{S}^{n-1})^{1/p}} \frac{|\langle m, v\rangle_{\mathsf{L}^2(\mathbb{S}^{n-1})}|}{\|u\|_{\mathsf{L}^q(\mathbb{S}^{n-1})}}.$$

The bound for $\|T_m\|_{\mathsf{L}^1(\mathbb{R}^n)\to\mathsf{L}^{1,\infty}(\mathbb{R}^n)}$ is obtained by real interpolation in the limit as $p \to 1+$; a trick used by Dragičević, Petermichl, and Volberg (2006). □

CHOOSING u AND v

How to choose $u \longleftrightarrow v$ in:

$$\|T_m\|_{\mathsf{L}^p(\mathbb{R}^n)\to\mathsf{L}^p(\mathbb{R}^n)} \geqslant c_n\,(q-1)\,\frac{\left|\langle m, v\rangle_{\mathsf{L}^2(\mathbb{S}^{n-1})}\right|}{\|u\|_{\mathsf{L}^q(\mathbb{S}^{n-1})}}\ ?$$

- Perhaps v should be “in the direction” of m.
- If we took $v = m$, then we would not always know how to estimate $\|u\|_{\mathsf{L}^q(\mathbb{S}^{n-1})}$. Sogge’s estimates for Y_j are too weak here.
- Would this even be optimal?
- The beauty of sub-optimal choices.

Smooth phases

MAZ'YA'S PROBLEM

$$m_\phi^\lambda(\xi) := e^{\mathrm{i}\lambda\phi(\xi/|\xi|)}; \quad \xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$$

$$\phi \in \mathrm{C}^\infty(\mathbb{S}^{n-1}) \text{ real-valued}, \quad \lambda \in \mathbb{R},\ |\lambda| \geqslant 1$$

$$\widehat{(T_\phi^\lambda f)}(\xi) = m_\phi^\lambda(\xi)\widehat{f}(\xi)$$

Problem [Maz'ya (1970s, 2018)]. Prove or disprove for $p \in (1, \infty)$:

$$\|T_\phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{n,p,\phi} |\lambda|^{(n-1)|1/p-1/2|}.$$

The conjecture is reasonable:

- it holds for $n = 1$;
- for $n \geqslant 2$ the Hörmander–Mihlin theorem easily gives the bound with $(n+2)|1/p - 1/2|$ in the exponent;
- it **fails** for $n = 2$; Dragičević, Petermichl, and Volberg (2006).

A SHARP RESULT

Theorem [Bulj and K. (2022)]. Take $n \geqslant 2$, $p \in (1, \infty)$, $\lambda \in \mathbb{R}$, $|\lambda| \geqslant 1$.

$$\|T_\phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{n,\phi} (p^*-1)\,|\lambda|^{n|1/p-1/2|}$$
$$\|T_\phi^\lambda\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \lesssim_{n,\phi} |\lambda|^{n/2}$$

Take $n \geqslant 2$ even. There exists $\phi \in \mathrm{C}^\infty(\mathbb{S}^{n-1})$ s.t. for $p \in (1, \infty)$ and $k \in \mathbb{Z} \setminus \{0\}$:

$$\|T_\phi^k\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \gtrsim_{n,\phi} (p^*-1)\,|k|^{n|1/p-1/2|}$$
$$\|T_\phi^k\|_{\mathrm{L}^1(\mathbb{R}^n)\to\mathrm{L}^{1,\infty}(\mathbb{R}^n)} \gtrsim_{n,\phi} |k|^{n/2}$$

In particular, Maz'ya's problem has a negative answer in all even dimensions n. (Seems to be the case in **all** dimensions $n \geqslant 2$.)

INDEPENDENT WORK

- Stolyarov (2022) independently showed upper estimates

$$\|T_\phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{n,\phi,p} |\lambda|^{n|1/p-1/2|} \quad \text{for } p\in(1,\infty)$$

using a Hardy→Lorentz via Besov result of Seeger (1988) and an advanced interpolation argument.

- Stolyarov (2022) independently showed lower estimates

$$\|T_\phi^\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)\to\mathrm{L}^p(\mathbb{R}^n)} \gtrsim_{n,\phi,p} |\lambda|^{n|1/p-1/2|} \quad \text{for } p\in(1,\infty)$$

and a particular choice of phase ϕ in **every** dimension $n \geqslant 2$.

- Recall that we also care about the sharp dependence on p.

  Stolyarov's bounds $\lesssim_{n,\phi,p}$ and $\gtrsim_{n,\phi,p}$ above are **not** optimal in p.

Proof of the lower bounds

Choose ϕ so that m_ϕ^k, $k \in \mathbb{N}$, coincides with

$$\widetilde{m}^k(\xi) = \prod_{i=1}^{n/2} \left(\frac{\xi_{2i-1} + \mathbb{i}\xi_{2i}}{|\xi_{2i-1} + \mathbb{i}\xi_{2i}|} \right)^k$$

on “most” of the sphere $\mathbb{S}^{n-1}$; avoid singularities.

$$\widetilde{m}^k = \sum_{j=nk/2}^{\infty} \widetilde{Y}_j^{(k)}$$

$$u^{(k)} := \sum_{j=nk/2}^{\infty} \mathbb{i}^j \gamma_{n,j,0} \widetilde{Y}_j^{(k)}$$

$$v^{(k)} := \sum_{j=nk/2}^{\infty} \gamma_{n,j,n/p} \gamma_{n,j,0} \widetilde{Y}_j^{(k)}$$

Proof of the lower bounds

The main difficulty is now in showing:

$$\|u^{(k)}\|_{\mathrm{L}^q(\mathbb{S}^{n-1})} \sim_n k^{-n/2}$$

for $k \in \mathbb{N}$, $q \in [1, \infty]$.

In fact, we can compute:

$$u^{(k)}(\zeta_1, \ldots, \zeta_{n/2}) = \frac{2\pi^{n/2}\mathbb{i}^{nk/2}}{(n/2-1)!} k^{-n/2} \left(\prod_{i=1}^{n/2} \left(\frac{\zeta_i}{|\zeta_i|} \right)^k \right)$$
$$\int_0^\infty \left(\prod_{i=1}^{n/2} \int_0^\infty J_k(2\sqrt{k}\rho t|\zeta_i|) e^{-\rho^2/2k} \rho \,\mathrm{d}\rho \right) \frac{\mathrm{d}t}{t}$$

for every $(\zeta_1, \ldots, \zeta_{n/2}) \in \mathbb{S}^{n-1}$, where J_k are the Bessel functions.

L^q is interpolated between L^1 and L^∞.

PROOF OF THE LOWER BOUNDS

Use our general theorem in combination with:

$$\frac{\left|\langle m_\phi^k, v^{(k)}\rangle_{\mathsf{L}^2(\mathbb{S}^{n-1})}\right|}{\|u^{(k)}\|_{\mathsf{L}^q(\mathbb{S}^{n-1})}} \gtrsim_n \frac{\left|\langle m^k, v^{(k)}\rangle_{\mathsf{L}^2(\mathbb{S}^{n-1})}\right|}{\|u^{(k)}\|_{\mathsf{L}^\infty(\mathbb{S}^{n-1})}} \gtrsim_n \frac{k^{n/p-n}}{k^{-n/2}} = k^{n(1/p-1/2)}$$

for $k \in \mathbb{N},\ p \in [1,2]$. □

The Riesz group

THE RIESZ GROUP

$$\phi(\xi) := \xi_1$$
$$(\widehat{T_\phi^\lambda f})(\xi) = e^{\mathbb{i}\lambda\xi_1/|\xi|}\widehat{f}(\xi)$$

$(T_\phi^\lambda)_{\lambda\in\mathbb{R}}$ is called the *Riesz group* since its infinitesimal generator is negative of the Riesz transform, i.e., $-R_1$, where $(\widehat{R_1 f})(\xi) = -\mathbb{i}\frac{\xi_1}{|\xi|}\widehat{f}(\xi)$.

For $n = 2$:

$$\phi(e^{\mathbb{i}\varphi}) := \cos\varphi$$
$$m_{\cos}^\lambda(re^{\mathbb{i}\varphi}) = e^{\mathbb{i}\lambda\cos\varphi}$$
$$(\widehat{T_{\cos}^\lambda f})(re^{\mathbb{i}\varphi}) = e^{\mathbb{i}\lambda\cos\varphi}\widehat{f}(re^{\mathbb{i}\varphi})$$

Problem by Dragičević, Petermichl, and Volberg

Dragičević, Petermichl, and Volberg (2006):

$$c_\delta\,(p^*-1)\,|k|^{2|1/p-1/2|-\delta} \leqslant \|T^k_{\cos}\|_{\mathrm{L}^p(\mathbb{R}^2)\to\mathrm{L}^p(\mathbb{R}^2)} \leqslant C\,(p^*-1)\,|k|^{2|1/p-1/2|}$$

for $\delta > 0,\ p \in (1,\infty),\ k \in \mathbb{Z}\setminus\{0\}$.

The lower bound disproves Maz'ya's conjecture in $n = 2$ dimensions.

Problem [Dragičević, Petermichl, and Volberg (2006)].
Can one remove the δ?

Theorem [Bulj and K. (2022)]. Take $p \in (1,\infty),\ \lambda \in \mathbb{R},\ |\lambda| \geqslant 1$.

$$\|T^\lambda_{\cos}\|_{\mathrm{L}^p(\mathbb{R}^2)\to\mathrm{L}^p(\mathbb{R}^2)} \sim (p^*-1)\,|\lambda|^{2|1/p-1/2|}$$

$$\|T^\lambda_{\cos}\|_{\mathrm{L}^1(\mathbb{R}^2)\to\mathrm{L}^{1,\infty}(\mathbb{R}^2)} \sim |\lambda|$$

Proof of the lower bounds

The Fourier series expansion of the symbol:

$$m^{\lambda}_{\cos}(e^{\mathrm{i}\varphi}) = e^{\mathrm{i}\lambda\cos\varphi} = J_0(\lambda) + 2\sum_{j=1}^{\infty} \mathrm{i}^j J_j(\lambda)\cos j\varphi,$$

where J_j are the Bessel functions.

$$\begin{aligned}
\cos(\lambda\cos\varphi) &= J_0(\lambda) + 2\sum_{l=1}^{\infty}(-1)^l J_{2l}(\lambda)\cos 2l\varphi\\
\cos(\lambda\sin\varphi) &= J_0(\lambda) + 2\sum_{l=1}^{\infty} J_{2l}(\lambda)\cos 2l\varphi\\
u^{(\lambda)}(e^{\mathrm{i}\varphi}) &:= \cos(\lambda\sin\varphi) - J_0(\lambda)\\
v^{(\lambda)}(e^{\mathrm{i}\varphi}) &:= 2\sum_{l=1}^{\infty}(-1)^l \gamma_{2,2l,2/p} J_{2l}(\lambda)\cos 2l\varphi
\end{aligned}$$

Proof of the lower bounds

Use our general theorem in combination with:

$$\frac{\left|\langle m^{\lambda}_{\cos}, v^{(\lambda)}\rangle_{\mathsf{L}^2(\mathbb{S}^1)}\right|}{\|u^{(\lambda)}\|_{\mathsf{L}^q(\mathbb{S}^1)}} \gtrsim_n \frac{\lambda^{2/p-1}}{1}$$

$$= \lambda^{2(1/p-1/2)}$$

for $\lambda \geqslant 1,\ p \in [1,2]$.

In the numerator we needed the inequality:

$$\sum_{l=1}^{\infty} l^{2/p-1} J_{2l}(\lambda)^2 \gtrsim \lambda^{2/p-1},$$

which is easy to show. $\square$

THE RIESZ GROUP FOR $n \geqslant 3$

$n = 2$

$$\|T^{\lambda}_{\cos}\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \sim (p^* - 1)\,|\lambda|^{2|1/p-1/2|}$$

$n \geqslant 3$

Open problem.
What is the asymptotics for the Riesz group in $n \geqslant 3$ dimensions?

Is it again

$$(p^* - 1)\,|\lambda|^{n|1/p-1/2|}\ ?$$

The case $n = 3$ has applications to the Navier–Stokes equations.

Thank you!