✓ Large Copies of Large Configurations in ✓ Large Sets

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Incidence Problems in Harmonic Analysis, Geometric Measure Theory, and Ergodic Theory

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Euclidean density theorems (EDTs)

There are patterns in large but otherwise arbitrary structures!

The main idea behind Ramsey theory ($\subseteq$ combinatorics), but also widespread in other areas of mathematics.

Euclidean density theorems (EDTs)

Euclidean density theorems are a mixture of

arithmetic combinatorics e.g., Szemerédi (1975): *a positive density set* $A \subseteq \mathbb{Z}$ *contains arbitrarily long arithmetic progressions.* Kelley–Meka (2023): *if a set* $A \subseteq \{1, \ldots, N\}$ *does not contain a 3-term AP, then* $\lvert A \rvert / N \leq \exp(-c(\log N)^c)$.	AND	**geometric measure theory** e.g., Falconer (1985): *if a set* $A \subseteq \mathbb{R}^d$ *has Hausdorff dimension* $\dim_{\mathscr{H}}(A) > (d+1)/2$, *then its distance set* $\{\lvert x-y \rvert : x, y \in A\}$ *has positive measure.*

Euclidean density theorems (EDTs)

EDTs study large measurable sets.

A measurable set $A \subseteq [0,1]^d$ is considered *large* if

$$|A| > 0;$$

its Lebesgue measure is positive.

A measurable set $A \subseteq \mathbb{R}^d$ is considered *large* if

$$\overline{\delta}(A) := \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|A \cap (x + [0,R]^d)|}{R^d} > 0;$$

its upper Banach density (or some other density) is positive.

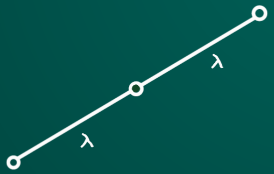
Euclidean density theorems (EDTs)

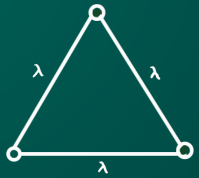
EDTs search inside A for congruent (i.e., isometric) copies of given *configurations* (*patterns*):

$$\mathscr{P} = \{P_\lambda : \lambda \in (0, \infty)\}.$$

$\lambda =$ a certain "size" parameter.

Typically: P_λ is the dilate of a fixed configuration P by a factor of λ.





Two types of desired results

"All large scales" formulation

For every measurable set $A \subseteq \mathbb{R}^d$ satisfying $\overline{\delta}(A) > 0$

$$\exists \lambda_0(\mathscr{P}, A) \quad \forall \lambda \geq \lambda_0 \quad A \text{ contains a congruent copy of } P_\lambda.$$

A rather strong but only qualitative claim.

The number λ_0 depends on more than just the density $\overline{\delta}(A)$.

Two types of desired results

"An interval of scales" formulation

Take $0 < \delta \ll 1$, $A \subseteq [0,1]^d$ measurable, $|A| \geq \delta$.

Then the set of "scales"

$$\{\lambda \in (0,\infty) \,:\, A \text{ contains a congruent copy of } P_\lambda\}$$

contains an interval of length at least $\varepsilon = F_{\mathscr{P},d}(\delta) > 0$.

A weaker but quantitative claim.

Initiates a race to find better dependencies of ε on δ.

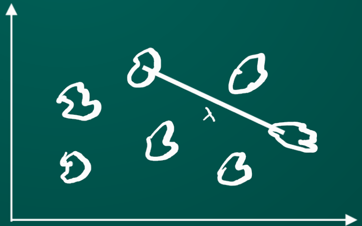
Classical results — a pair of points

A question by Székely (1982). Popularized by Erdős.

Can one find all large dilates of $P = \{0, 1\} \subset \mathbb{R}$ in a large set $A \subseteq \mathbb{R}^2$?

Answered affirmatively by:

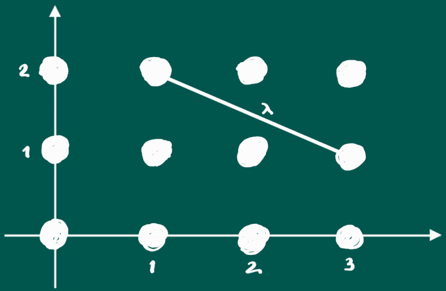
- Furstenberg, Katznelson, and Weiss (1980s),
- Falconer and Marstrand (1986),
- Bourgain (1986).



Classical results

A question by Székely (1982)

For every measurable set $A \subseteq \mathbb{R}^2$ satisfying $\overline{\delta}(A) > 0$ is there a number $\lambda_0 = \lambda_0(A)$ such that for each $\lambda \in [\lambda_0, \infty)$ there exist points $x, x' \in A$ satisfying $|x - x'| = \lambda$?



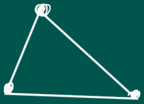
an easy exercise

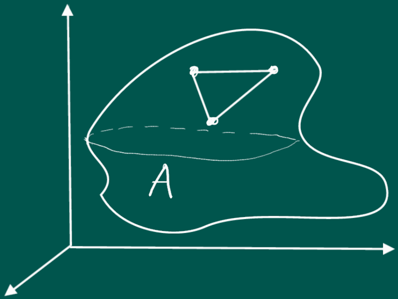
Classical results — simplices

Δ = the set of vertices of a non-degenerate n-dimensional simplex

Theorem (Bourgain (1986))

For every measurable set $A \subseteq \mathbb{R}^{n+1}$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \Delta)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of $\lambda\Delta$.





Classical results — simplices

$\Delta =$ the set of vertices of a non-degenerate n-dimensional simplex.

Theorem (Bourgain (1986))

All large dilates of $\Delta \subset \mathbb{R}^n$ exist in a large set $A \subseteq \mathbb{R}^{n+1}$.
An interval of dilates of length $\left(\exp(\delta^{-C(n,\Delta)})\right)^{-1}$ exists in a large set $A \subseteq [0,1]^{n+1}$.

$C(n,\Delta)$ = a constant depending on n and Δ.
Alternative proofs by Lyall and Magyar (2016, 2018, 2019), K. (2020).
Note the dimensional increase: $\mathbb{R}^n \rightsquigarrow \mathbb{R}^{n+1}$.

Open question #1 (folklore from the 1980s, e.g., Furstenberg)

When $n \geq 2$, does the same hold for $A \subseteq \mathbb{R}^n$?

General point configurations

Open question #2 (folklore from the 1990s, e.g., Graham)

Which point configurations P have all large dilates in large sets $A \subseteq \mathbb{R}^d$ for some (sufficiently large) dimension d?

The most general known positive result:

holds for products of vertex-sets of nondegenerate simplices
$P = \Delta_1 \times \cdots \times \Delta_m$, Lyall and Magyar (2019).

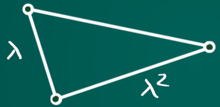
The most general known negative result:

fails for configurations that cannot be inscribed in a sphere,
Graham (1993).

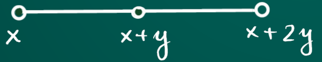
Newer results

We "change the rules" in one of the following ways:

1. want better bounds in the "interval of scales" formulation;
2. consider "anisotropic" dilates of the configuration;



3. measure the configuration size in some ℓ^p for $p \neq 2$;



$\|y\|_{\ell^p} = \lambda$

4. consider "very dense" sets $A \subseteq \mathbb{R}^d$.

General scheme of the approach

Abstracted from: Cook, Magyar, and Pramanik (2017)

$\mathcal{N}_\lambda^0$ = *configuration “counting” form*, identifies the configuration associated with the parameter $\lambda > 0$ (i.e., of “size” λ)

$\mathcal{N}_\lambda^\varepsilon$ = *smoothened counting form*; the picture is blurred up to scale $0 < \varepsilon \leq 1$

The largeness–smoothness multiscale approach:

- $\lambda =$ scale of largeness
- $\varepsilon =$ scale of smoothness

General scheme of the approach (continued)

Decompose:

$$\mathcal{N}_\lambda^0 = \mathcal{N}_\lambda^1 + \left(\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1\right) + \left(\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon\right).$$

$\mathcal{N}_\lambda^1 =$ *structured part,*

$\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1 =$ *error part,*

$\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon =$ *uniform part.*

regularity decomposition

General scheme of the approach (continued)

For the structured part $\mathcal{N}_\lambda^1$ we need a lower bound

$$\mathcal{N}_\lambda^1 \geq c(\delta)$$

that is uniform in λ, but this should be a simpler/smoother problem.

For the uniform part $\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon$ we want

$$\lim_{\varepsilon\to 0} \left|\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon\right| = 0$$

uniformly in λ; this usually leads to some oscillatory integrals.

For the error part $\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1$ one tries to prove

$$\sum_{j=1}^{J} \left|\mathcal{N}_{\lambda_j}^\varepsilon - \mathcal{N}_{\lambda_j}^1\right| \leq C(\varepsilon)o(J)$$

for lacunary scales $\lambda_1 < \cdots < \lambda_J$; this usually leads to some multilinear singular integrals.

General scheme of the approach (continued)

We argue by contradiction. Take sufficiently many lacunary scales $\lambda_1 < \cdots < \lambda_J$ such that $\mathcal{N}^0_{\lambda_j} = 0$ for each j.
The structured part

$$\mathcal{N}^1_{\lambda_j} \geq c(\delta)$$

dominates the uniform part

$$\left|\mathcal{N}^0_{\lambda_j} - \mathcal{N}^\varepsilon_{\lambda_j}\right| \ll 1 \quad \text{(for sufficiently small } \varepsilon\text{)}$$

and the error part

$$\left|\mathcal{N}^\varepsilon_{\lambda_j} - \mathcal{N}^1_{\lambda_j}\right| \ll C(\varepsilon) \quad \text{(for some } j \text{ by pigeonholing)}$$

for at least one index j. This contradicts $\mathcal{N}^0_{\lambda_j} = 0$.

1. Rectangular boxes

Newer results… a warm-up example

$\square$ = the set of vertices of an n-dimensional rectangular box

Theorem (Lyall and Magyar (2019))

For every measurable set $A \subseteq \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = (\mathbb{R}^2)^n$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \square)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of $\lambda\square$ with sides parallel to the distinguished 2-dimensional coordinate planes.

Previous particular cases by:

Lyall and Magyar (2016), for $n = 2$;

Durcik and K. (2018), general n, but in $(\mathbb{R}^5)^n$.

1. Rectangular boxes — quantitative strengthening

Fix $b_1, \ldots, b_n > 0$ (box sidelengths).

Theorem (Durcik and K. (2020))

For $0 < \delta \le 1/2$ and measurable $A \subseteq ([0,1]^2)^n$ with $|A| \ge \delta$ there exists an interval $I = I(A, b_1, \ldots, b_n) \subseteq (0,1]$ of length at least

$$\big(\exp(\delta^{-C(n)})\big)^{-1}$$

s. t. for every $\lambda \in I$ one can find $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}^2$ satisfying

$$(x_1 + r_1 y_1, x_2 + r_2 y_2, \ldots, x_n + r_n y_n) \in A \ \text{ for } (r_1, \ldots, r_n) \in \{0,1\}^n;$$

$$|y_i| = \lambda b_i \ \text{ for } i = 1, \ldots, n.$$

This improves the bound of Lyall and Magyar (2019) of the form $\big(\exp(\exp(\cdots\exp(C(n)\delta^{-3\cdot 2^n})\cdots))\big)^{-1}$ (a tower of height n).

1. Rectangular boxes

$\sigma =$ circle measure in $\mathbb{R}^2$, $f = \mathbb{1}_A$
Configuration counting form:

$$\mathcal{N}_\lambda^0(f) := \int_{(\mathbb{R}^2)^{2n}} \Big(\prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} f(x_1+r_1y_1,\ldots,x_n+r_ny_n) \Big) \Big(\prod_{k=1}^{n} dx_k \, d\sigma_{\lambda b_k}(y_k) \Big)$$

$\mathcal{N}_\lambda^\varepsilon$ can be obtained by "heating up" $\mathcal{N}_\lambda^0$.

$$\mathbb{g} = \text{standard Gaussian}, \quad \mathbb{k} = \Delta \mathbb{g}.$$

The approach benefits from the *heat equation*

$$\frac{\partial}{\partial t}\big(\mathbb{g}_t(x)\big) = \frac{1}{2\pi t}\mathbb{k}_t(x).$$

1. Rectangular boxes

Smoothened counting form:

$$\mathcal{N}_\lambda^\varepsilon(f) := \int_{(\mathbb{R}^2)^{2n}} \Big(\cdots\Big)\Big(\prod_{k=1}^{n}(\sigma * \mathrm{g}_\varepsilon)_{\lambda b_k}(y_k)\,\mathrm{d}x_k\,\mathrm{d}y_k\Big)$$

$$= \int_{(\mathbb{R}^2)^{2n}} \mathscr{F}(\mathbf{x})\Big(\prod_{k=1}^{n}(\sigma * \mathrm{g}_\varepsilon)_{\lambda b_k}(x_k^0 - x_k^1)\Big)\,\mathrm{d}\mathbf{x}$$

$$\mathscr{F}(\mathbf{x}) := \prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} f(x_1^{r_1},\ldots,x_n^{r_n}), \quad \mathrm{d}\mathbf{x} := \mathrm{d}x_1^0\,\mathrm{d}x_1^1\,\mathrm{d}x_2^0\,\mathrm{d}x_2^1\cdots\mathrm{d}x_n^0\,\mathrm{d}x_n^1$$

1. Rectangular boxes — error part (continued)

From $\sum_{j=1}^{J} \left| \mathcal{N}_{\lambda_j}^{\varepsilon}(\mathbb{1}_B) - \mathcal{N}_{\lambda_j}^{1}(\mathbb{1}_B) \right|$ we are lead to study

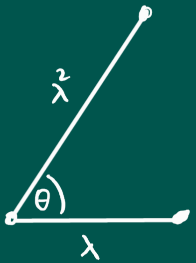
$$\Theta_K\big((f_{r_1,\ldots,r_n})_{(r_1,\ldots,r_n)\in\{0,1\}^n}\big)$$
$$:= \int_{(\mathbb{R}^d)^{2n}} \prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} f_{r_1,\ldots,r_n}(x_1 + r_1 y_1, \ldots, x_n + r_n y_n)\Big)$$
$$K(y_1, \ldots, y_n)\Big(\prod_{k=1}^{n} \mathrm{d}x_k\, \mathrm{d}y_k\Big)$$

Entangled multilinear singular integral forms with cubical structure: Durcik (2014); K. (2010); Durcik and Thiele (2018: entangled Brascamp–Lieb).

2. Anisotropic configurations

Polynomial generalizations?

- There are no triangles with sides λ, λ^2, and λ^3 for large λ
- One can look for triangles with sides λ, λ^2 and a fixed angle θ between them.



2. Anisotropic configurations

We will be working with *anisotropic power-type scalings*

$$(x_1, \dots, x_n) \mapsto (\lambda^{a_1} b_1 x_1, \dots, \lambda^{a_n} b_n x_n).$$

Here $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0$ are fixed parameters.

Crucial observation: the heat equation is unaffected by a power-type change of the time variable

$$\frac{\partial}{\partial t}\left(\mathbb{g}_{t^a b}(x)\right) = \frac{a}{2\pi t}\,\mathbb{k}_{t^a b}(x).$$

2.1 Anisotropic right-angled simplices

Theorem (K. (2020))

For every measurable set $A \subseteq \mathbb{R}^{n+1}$ *satisfying* $\overline{\delta}(A) > 0$ *there is a number* $\lambda_0 = \lambda_0(A, a_1, \ldots, a_n, b_1, \ldots, b_n)$ *such that for each* $\lambda \in [\lambda_0, \infty)$ *one can find a point* $x \in \mathbb{R}^{n+1}$ *and mutually orthogonal vectors* $y_1, y_2, \ldots, y_n \in \mathbb{R}^{n+1}$ *satisfying*

$$\{x, x + y_1, x + y_2, \ldots, x + y_n\} \subseteq A$$

and

$$|y_i| = \lambda^{a_i} b_i; \quad i = 1, 2, \ldots, n.$$

2.1 Anisotropic simplices

Pattern counting form:

$$\mathcal{N}^{0}_{\lambda}(f) := \int_{\mathbb{R}^{n+1}} \int_{\mathrm{SO}(n+1,\mathbb{R})} f(x) \Big(\prod_{k=1}^{n} f(x + \lambda^{a_k} b_k U \mathbb{e}_k) \Big) \, \mathrm{d}\mu(U) \, \mathrm{d}x$$

Smoothened counting form:

$$\mathcal{N}^{\varepsilon}_{\lambda}(f) := \int_{\mathbb{R}^{n+1}} \int_{\mathrm{SO}(n+1,\mathbb{R})} f(x) \Big(\prod_{k=1}^{n} (f * \mathbb{g}_{(\varepsilon\lambda)^{a_k} b_k})(x + \lambda^{a_k} b_k U \mathbb{e}_k) \Big) \, \mathrm{d}\mu(U) \, \mathrm{d}x$$

2.1 Anisotropic simplices — error part

From $\sum_{j=1}^{J} \left| \mathcal{N}_{\lambda_j}^{\varepsilon}(\mathbb{1}_B) - \mathcal{N}_{\lambda_j}^{1}(\mathbb{1}_B) \right|$ we are lead to study

$$\Lambda_K(f_0, \ldots, f_n) := \int_{(\mathbb{R}^d)^{n+1}} K(x_1 - x_0, \ldots, x_n - x_0) \left(\prod_{k=0}^{n} f_k(x_k)\, \mathrm{d}x_k \right)$$

Multilinear C–Z operators: Coifman and Meyer (1970s); Grafakos and Torres (2002).

Here K is a C–Z kernel, but with respect to the quasinorm associated with our anisotropic dilation structure.

2.2 Anisotropic boxes

Theorem (K. (2020))

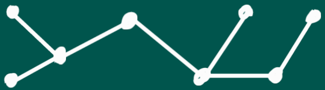
For every measurable set $A \subseteq (\mathbb{R}^2)^n$ *satisfying* $\overline{\delta}(A) > 0$ *there is a number* $\lambda_0 = \lambda_0(A, a_1, \ldots, a_n, b_1, \ldots, b_n)$ *such that for each* $\lambda \in [\lambda_0, \infty)$ *one can find points* $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}^2$ *satisfying*

$$\big\{(x_1 + r_1 y_1, x_2 + r_2 y_2, \ldots, x_n + r_n y_n) : (r_1, \ldots, r_n) \in \{0, 1\}^n\big\} \subseteq A$$

and

$$|y_i| = \lambda^{a_i} b_i; \quad i = 1, 2, \ldots, n.$$

2.3 Anisotropic trees



$\mathcal{T} = (V, E)$ be a finite tree with vertices V and edges E

Theorem (K. (2020))

For every measurable set $A \subseteq \mathbb{R}^2$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \mathcal{T}, a_1, \ldots, a_n, b_1, \ldots, b_n)$ such that for each $\lambda \in [\lambda_0, \infty)$ one can find a set of points

$$\{x_v : v \in V\} \subseteq A$$

satisfying

$$|x_u - x_v| = \lambda^{a_k} b_k \quad \text{for each edge } k \in E \text{ joining vertices } u, v \in V.$$

This is an anisotropic variant of the result by Lyall and Magyar (2018) on certain distance graphs.

2.3 Anisotropic configurations — error part

One would like to handle more general distance graphs (generalizing simplices, boxes, and trees).

For the error part there is a lot of potential in applying entangled multilinear singular integrals associated with bipartite graphs or r-partite r-regular hypergraphs.

The only cases studied so far are:

the so-called "twisted paraproduct operator,"

K. (2010.); Durcik and Roos (2018);

the operators with cubical structure,

Durcik (2014, 2015); Durcik and Thiele (2018).

Dyadic models of these operators are significantly easier:

K. (2011); Stipčić (2019).

3. Arithmetic progressions

Reformulation of Szemerédi's theorem

For $n \geq 3$ and $d \geq 1$ there exists a constant $C(n,d)$ such that for $0 < \delta \leq 1/2$ and a measurable set $A \subseteq [0,1]^d$ with $|A| \geq \delta$ one has

$$\int_{[0,1]^d} \int_{[0,1]^d} \prod_{k=0}^{n-1} \mathbb{1}_A(x+ky)\,dy\,dx$$

$$\geq \begin{cases} \big(\exp(\delta^{-C(n,d)})\big)^{-1} & \text{when } 3 \leq n \leq 4, \\ \big(\exp(\exp(\delta^{-C(n,d)}))\big)^{-1} & \text{when } n \geq 5. \end{cases}$$

Follows from the best known type of bounds in Szemerédi's theorem:

~~$n = 3$, log by Heath-Brown (1987)~~ Kelley-Meka (2023)

$n = 4$, log by Green and Tao (2017)

$n \geq 5$, log log by Gowers (2001)

3. Arithmetic progressions in other ℓ^p-norms

Bourgain's counterexample applies.

Cook, Magyar, and Pramanik (2015) decided to measure gap lengths in the ℓ^p-norm for $p \neq 1, 2, \infty$.

Theorem (Cook, Magyar, and Pramanik (2015))

If $p \neq 1, 2, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(p, d, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ one can find $x, y \in \mathbb{R}^d$ satisfying $x, x + y, x + 2y \in A$ and $\|y\|_{\ell^p} = \lambda$.

Open question #3 (Cook, Magyar, and Pramanik (2015))

Is it possible to lower the dimensional threshold all the way to $d = 2$ or $d = 3$?

3. Arithmetic progressions

Open question #4 (Durcik, K., and Rimanić)

Prove or disprove: if $n \geq 4$, $p \neq 1, 2, \ldots, n-1, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(n, p, d, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ one can find $x, y \in \mathbb{R}^d$ satisfying $x, x+y, \ldots, x+(n-1)y \in A$ and $\|y\|_{\ell^p} = \lambda$.

It is necessary to assume $p \neq 1, 2, \ldots, n-1, \infty$.

In $(0,\infty)$: $x^2 - 2(x+y)^2 + (x+2y)^2 = 2y^2$ $\quad \ell^2$

$-x^3 + 3(x+y)^3 - 3(x+2y)^3 + (x+3y)^3 = 6y^3$ $\quad \ell^3$

3. Arithmetic progressions — compact formulation

Theorem (Durcik and K. (2020))

Take $n \geq 3$, $p \neq 1, 2, \ldots, n-1, \infty$, $d \geq D(n,p)$, $\delta \in (0, 1/2]$, $A \subseteq [0,1]^d$ *measurable*, $|A| \geq \delta$. *Then the set of* ℓ^p*-norms of the gaps of n-term APs in the set A contains an interval of length at least*

$$\begin{cases} \big(\exp(\exp(\delta^{-C(n,p,d)}))\big)^{-1} & \text{when } 3 \leq n \leq 4, \\ \big(\exp(\exp(\exp(\delta^{-C(n,p,d)})))\big)^{-1} & \text{when } n \geq 5. \end{cases}$$

Modifying Bourgain's example $\longrightarrow$ sharp regarding the values of p

One can take $D(n,p) = 2^{n+3}(n+p)$ $\longrightarrow$ certainly not sharp

3. Arithmetic progressions

$\sigma(x) = \delta(\|x\|_{\ell^p}^p - 1)$ = a measure supported on the unit sphere in the ℓ^p-norm

$$\mathcal{N}_\lambda^0(A) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\prod_{k=0}^{n-1}\mathbb{1}_A(x+ky)\,d\sigma_\lambda(y)\,dx$$

$$\mathcal{N}_\lambda^0(A) > 0 \implies (\exists x, y)\big(x, x+y, \ldots, x+(n-1)y \in A,\ \|y\|_{\ell^p} = \lambda\big)$$

$$\mathcal{N}_\lambda^\varepsilon(A) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\prod_{k=0}^{n-1}\mathbb{1}_A(x+ky)(\sigma_\lambda * \varphi_{\varepsilon\lambda})(y)\,dy\,dx$$

for a smooth $\varphi \geq 0$ with $\int_{\mathbb{R}^d}\varphi = 1$

3. Arithmetic progressions — back to the error part

The most interesting part for us is the error part.

We need to estimate

$$\sum_{j=1}^{J} \kappa_j \Big(\mathcal{N}^{\varepsilon}_{\lambda_j}(A) - \mathcal{N}^{1}_{\lambda_j}(A) \Big)$$

for arbitrary scales $\lambda_j \in (2^{-j}, 2^{-j+1}]$ and arbitrary complex signs κ_j, with a bound that is sub-linear in J.

3. Arithmetic progressions — error part

It can be expanded as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\prod_{k=0}^{n-1}\mathbb{1}_A(x+ky)K(y)\,\mathrm{d}y\,\mathrm{d}x,$$

where

$$K(y) := \sum_{j=1}^{J}\kappa_j\big((\sigma_{\lambda_j} * \varphi_{\varepsilon\lambda_j})(y) - (\sigma_{\lambda_j} * \varphi_{\lambda_j})(y)\big)$$

is a translation-invariant Calderón–Zygmund kernel.

3. Arithmetic progressions — error part

If $d = 1$ and $K(y)$ is a truncation of $1/y$, then this becomes the (dualized and truncated) multilinear Hilbert transform,

$$\int_{\mathbb{R}} \int_{[-R,-r]\cup[r,R]} \prod_{k=0}^{n-1} f_k(x+ky) \frac{\mathrm{d}y}{y} \, \mathrm{d}x.$$

- When $n \geq 4$, no L^p-bounds uniform in r, R are known.
- Tao (2016) showed a bound of the form $o(J)$, where $J \sim \log(R/r)$ is the "number of scales" involved.
- Reproved and generalized by Zorin-Kranich (2016).
- Durcik, K., and Thiele (2016) showed a bound $O(J^{1-\varepsilon})$.

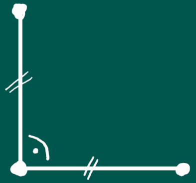
4. Some other arithmetic configurations

Allowed symmetries play a major role.

Note a difference between:

the so-called *corners*: $(x, y), (x + s, y), (x, y + s)$ (harder),

isosceles right triangles: $(x, y), (x + s, y), (x, y + t)$
with $||s||_{\ell^2} = ||t||_{\ell^2}$ (easier).

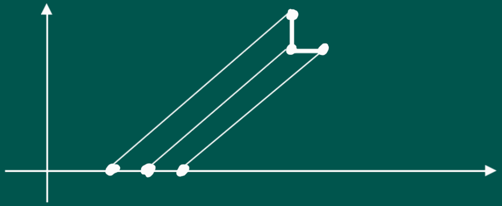


4.1 Corners

Theorem (Durcik, K., and Rimanić (2016))

If $p \neq 1, 2, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d \times \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(p, d, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ one can find $x, y, s \in \mathbb{R}^d$ satisfying $(x, y), (x + s, y), (x, y + s) \in A$ and $\|s\|_{\ell^p} = \lambda$.

Generalizes the result of Cook, Magyar, and Pramanik (2015) via the skew projection $(x, y) \mapsto y - x$.

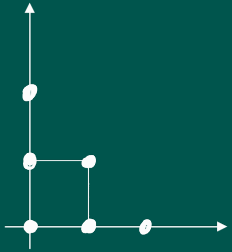


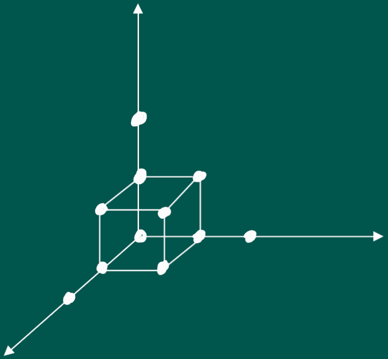
4.2 AP-extended boxes

Consider the configuration in $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$ consisting of:

$$(x_1 + k_1 s_1,\ x_2 + k_2 s_2,\ \dots,\ x_n + k_n s_n), \quad k_1, k_2, \dots, k_n \in \{0, 1\},$$

$$(x_1 + 2s_1, x_2, \dots, x_n),\ (x_1, x_2 + 2s_2, \dots, x_n),\ \dots,\ (x_1, x_2, \dots, x_n + 2s_n).$$





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Fix $b_1, \ldots, b_n > 0$ and $p \neq 1, 2, \infty$.

Theorem (Durcik and K. (2018))

There exists a dimensional threshold $d_{\min}$ such that for any $d_1, d_2, \ldots, d_n \geq d_{\min}$ and any measurable set A with $\overline{\delta}(A) > 0$ one can find $\lambda_0 > 0$ with the property that for any $\lambda \geq \lambda_0$ the set A contains the above 3AP-extended box with $\|s_i\|_{\ell^p} = \lambda b_i$, $i = 1, 2, \ldots, n$.

4.3 Corner-extended boxes

Consider the config. in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n} \times \mathbb{R}^{d_n}$ consisting of:

$$(x_1 + k_1 s_1, \ldots, x_n + k_n s_n, y_1, y_2, \ldots, y_n), \quad k_1, k_2, \ldots, k_n \in \{0, 1\},$$

$$(x_1, x_2, \ldots, x_n, y_1 + s_1, y_2, \ldots, y_n), \ldots, (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n + s_n).$$

Fix $b_1, \ldots, b_n > 0$ and $p \neq 1, 2, \infty$.

Theorem (Durcik and K. (2018))

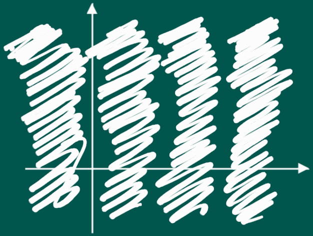
There exists a dimensional threshold $d_{\min}$ such that for any $d_1, \ldots, d_n \geq d_{\min}$ and any measurable set A with $\overline{\delta}(A) > 0$ one can find $\lambda_0 > 0$ with the property that for any $\lambda \geq \lambda_0$ the set A contains the above corner-extended box with $\|s_i\|_{\ell^p} = \lambda b_i$, $i = 1, 2, \ldots, n$.

5. Very dense sets

Theorem (Falconer, K., and Yavicoli (2020))

If $d \geq 2$ and $A \subseteq \mathbb{R}^d$ is measurable with $\overline{\delta}(A) > 1 - \frac{1}{n-1}$, then for ***every*** *$n$-point configuration P there exists $\lambda_0 > 0$ s. t. for every $\lambda \geq \lambda_0$ the set A contains an isometric copy of λP.*

The result would be trivial for $\overline{\delta}(A) > 1 - \frac{1}{n}$ and rotations would not even be needed there.



5. Very dense sets — lower bound

What can one say about the lower bound for such density threshold (depending on n)?

Let us return to arithmetic progressions!

Theorem (Falconer, K., and Yavicoli (2020))

For all $n, d \geq 2$ there exists a measurable set $A \subseteq \mathbb{R}^d$ of density at least

$$1 - \frac{10 \log n}{n^{1/5}}$$

s.t. there are arbitrarily large values of λ for which A contains no congruent copy of $\lambda\{0, 1, \dots, n-1\}$.

5. Very dense sets

Open question #5 (Falconer, K., and Yavicoli)

What is the smallest $0 \leq \delta_{\min}(d, n) < 1$ such that every measurable set $A \in \mathbb{R}^d$ of upper density $> \delta_{\min}(d, n)$ contains all sufficiently large scale similar copies of all n-point configurations?

Previous results give

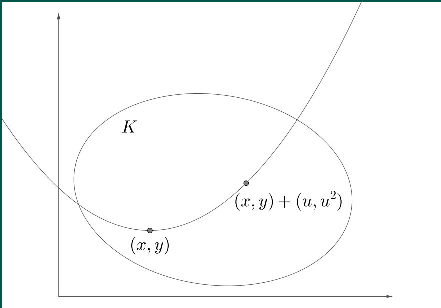
$$1 - \frac{10 \log n}{n^{1/5}} \leq \delta_{\min}(d, n) \leq 1 - \frac{1}{n-1}.$$

Is it possible to improve either one of the two asymptotic bounds $1 - O(n^{-1/5} \log n)$ and $1 - O(n^{-1})$ as $n \to \infty$?

6. A strong-type Furstenberg–Sárközy theorem

Theorem (Kuca, Orponen, and Sahlsten (2021))

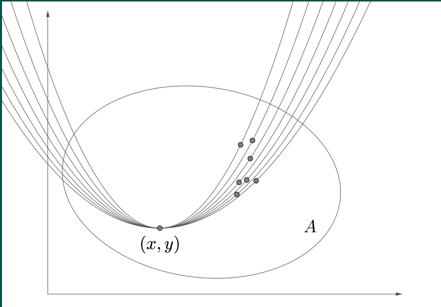
There exists $\varepsilon > 0$ s.t. every compact set $K \subseteq \mathbb{R}^2$ of dimension $\dim_{\mathscr{H}}(A) > 2 - \varepsilon$ contains a pair of points of the form (x, y), $(x, y) + (u, u^2)$.



6. A strong-type Furstenberg–Sárközy theorem

Theorem (Durcik, K., and Stipčić (2023))

A positive measure set $A \subseteq [0,1]^2$ contains a point $(x_0, y_0) \in A$ s.t. A nontrivially intersects parabolas $y - y_0 = a(x - x_0)^2$ for a whole interval $I \subseteq (0, \infty)$ of parameters $a \in I$. For a positive (upper Banach) density set $A \subseteq \mathbb{R}^2$ the interval I can be arbitrarily large on the logarithmic scale.



6. A strong-type Furstenberg–Sárközy theorem

Open question #6 (Durcik, K., and Stipčić)

For a positive (upper Banach) density set $A \subseteq \mathbb{R}^2$, can one find a point $(x_0, y_0) \in A$ s.t. A nontrivially intersects **all** parabolas $y - y_0 = a(x - x_0)^2$?

Future work

What else is there to do?

Thank you!

HW: Try one of the open problems.