# Improving estimates for discrete polynomial averaging operators

*Vjekoslav Kovač* (U. of Zagreb)

Joint work with

*R. Han, M. T. Lacey, F. Yang* (Georgia Tech), J. Madrid (UCLA)

Croatian Science Foundation

Supported by *HRZZ* UIP-2017-05-4129 (MUNHANAP)

*Zagreb Workshop on Operator Theory*, Zagreb, June 29–30, 2020

# What are averaging operators?

**Finite averages**

$A_N f := \frac{1}{N}\sum_{k=1}^{N} T_k f; \quad N \in \mathbb{N}$

*Assumptions:* $X$ normed space, $T_k : X \to X$ isometries

Obvious property: $||A_N||_{X\to X} \leq 1$

*More general assumptions:* $X$, $Y$ normed spaces, $T_k : X \to Y$, $||T_k||_{X\to Y} = 1$

Again: $||A_N||_{X\to Y} \leq 1$

Can the constant 1 be improved? Sometimes

# Norm-improving property

## Finite averages

$A_N f := \frac{1}{N}\sum_{k=1}^{N} T_k f; \quad N \in \mathbb{N}$

## Occasional additional property

$\|A_N\|_{X\to Y} \leq c(N) < 1, \quad c(N) \to 0$ as $N \to \infty$

Appreciated in harmonic analysis, ergodic theory, etc.

We might want to find the optimal asymptotics of $c(N)$

All depends on the choice of $T_k$

## Example: Translations on $\mathbb{Z}$

$1 \leq p \leq q \leq \infty, \quad a_k \in \mathbb{Z}$

$T_k : \ell^p(\mathbb{Z}) \to \ell^q(\mathbb{Z}), \quad (T_k f)(m) := f(m + a_k)$

**Finite averages on $\mathbb{Z}$**

$(A_N f)(m) = \frac{1}{N} \sum_{k=1}^{N} f(m + a_k); \quad N \in \mathbb{N}, m \in \mathbb{Z}$

Norm-improving property now depends on the choice of numbers $a_k$

Several features comes into play, coming from the Fourier analysis, number theory, etc.

# Discrete polynomial averages

$P\colon \mathbb{Z} \to \mathbb{Z}$ polynomial with integer coefficients of degree $d \geq 2$

**Polynomial averages on $\mathbb{Z}$**

$(A_N f)(m) = \frac{1}{N}\sum_{k=1}^{N} f(m + P(k)); \quad N \in \mathbb{N},\, m \in \mathbb{Z}$

**Theorem (R. Han, V. K., M. T. Lacey, J. Madrid, F. Yang (2019))**

$\|A_N f\|_{\ell^q(\mathbb{Z})} \leq C(P,p,q)\, N^{-d(\frac{1}{p}-\frac{1}{q})}\, \|f\|_{\ell^p(\mathbb{Z})}$

*in a certain range of exponents* $1 \leq p \leq q \leq \infty$

Previously only the case $P(x) = x^2$ was known — R. Han, M. T. Lacey, F. Yang (2019)

# Trying out easy examples

**Theorem**

$(A_N f)(m) = \frac{1}{N}\sum_{k=1}^{N} f(m + P(k))$

$\|A_N f\|_{\ell^q(\mathbb{Z})} \leq C(P, p, q)\, N^{-d(\frac{1}{p}-\frac{1}{q})}\, \|f\|_{\ell^p(\mathbb{Z})}$

Take $f = \mathbb{1}_{\{1,2,3,\ldots,2P(N)\}} \implies$ optimality of the decay $N^{-d(\frac{1}{p}-\frac{1}{q})}$

Take $f = \mathbb{1}_{\{P(1),P(2),\ldots,P(N)\}} \implies$ necessary condition $\frac{d}{q} \geq \frac{d-1}{p}$

Take $f = \mathbb{1}_{\{0\}} \implies$ necessary condition $\frac{d-1}{q} \geq \frac{d}{p} - 1$

# Range of exponents

Case $d = 2$

The theorem holds for $\left\{(p,q)\ :\ \frac{1}{q} \leq \frac{1}{p'},\ \frac{2}{q} > \frac{1}{p'},\ \frac{1}{q} > \frac{2}{p} - 1\right\}$

When $q = p'$ the range specializes to $\boxed{\frac{3}{2} < p \leq 2}$

The range in optimal modulo its boundary

Case $d \geq 3$

The theorem holds for $\left\{(p,q)\ :\ \frac{1}{q} \leq \frac{1}{p'},\ \frac{d^2+d+1}{q} > \frac{d^2+d-1}{p},\ \frac{d^2+d-1}{q} > \frac{d^2+d+1}{p} - 2\right\}$

When $q = p'$ the range specializes to $\boxed{2 - \frac{2}{d^2+d+1} < p \leq 2}$

No estimates outside the range $\left\{(p,q)\ :\ \frac{1}{q} \leq \frac{1}{p'},\ \frac{d}{q} \geq \frac{d-1}{p},\ \frac{d-1}{q} \geq \frac{d}{p} - 1\right\}$

When $q = p'$ this specializes to $\boxed{2 - \frac{1}{d} \leq p \leq 2}$

# Range of exponents

$$(A_N f)(m) = \frac{1}{N} \sum_{k=1}^{N} f(m + P(k))$$

$$\|A_N f\|_{\ell^q(\mathbb{Z})} \leq C(P, p, q)\, N^{-d(\frac{1}{p} - \frac{1}{q})}\, \|f\|_{\ell^p(\mathbb{Z})}$$

Best explained visually

# Sketch of the proof in the case $d \geq 3$

Regard $A_N$ as "projections" of

**Higher-dimensional "universal" polynomial averages**

$$(\widetilde{A}_N f)(m_1, m_2, \ldots, m_d) := \frac{1}{N}\sum_{k=1}^{N} f(m_1 + k, m_2 + k^2, \ldots, m_d + k^d);$$
$$N \in \mathbb{N}, (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$$

We turn to proving

**Theorem**

$$\|\widetilde{A}_N f\|_{\ell^q(\mathbb{Z}^d)} \lesssim_{d,p,q} N^{-\frac{d(d+1)}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{\ell^p(\mathbb{Z}^d)}$$

## Sketch of the proof in the case $d \geq 3$

Write $(\widetilde{A}_N f)(m_1, m_2, \ldots, m_d)$ as

$$\frac{1}{N}\sum_{k=1}^{N}\int_{\mathbb{T}^d}\widehat{f}(t_1, t_2, \ldots, t_d)e^{2\pi i((m_1+k)t_1+(m_2+k^2)t_2+\cdots+(m_d+k^d)t_d)}dt_1dt_2\cdots dt_d$$

$$= \int_{\mathbb{T}^d}\widehat{f}(t_1, t_2, \ldots, t_d)S_N(t_1, t_2, \ldots, t_d)e^{2\pi i(m_1t_1+m_2t_2+\cdots+m_dt_d)}dt_1dt_2\cdots dt_d,$$

where

$S_N(t_1, t_2, \ldots, t_d) := \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i(kt_1+k^2t_2+\cdots+k^dt_d)}$;   $(t_1, t_2, \ldots, t_d) \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ are the normalized *exponential sums*

# Sketch of the proof in the case $d \geq 3$

Take $q = p'$

Apply the *Hausdorff–Young inequality* twice to $\widetilde{A}_N f = \mathscr{F}^{-1}\big((\mathscr{F} f) S_N\big)$:

$$\|\widetilde{A}_N f\|_{\ell^{p'}(\mathbb{Z}^d)} \leq \|\widehat{f} S_N\|_{L^p(\mathbb{T}^d)} \leq \|\widehat{f}\|_{L^{p'}(\mathbb{T}^d)} \|S_N\|_{L^s(\mathbb{T}^d)} \leq \|f\|_{\ell^p(\mathbb{Z}^d)} \|S_N\|_{L^s(\mathbb{T}^d)},$$

where $\frac{1}{s} = \frac{1}{p} - \frac{1}{p'} = \frac{2}{p} - 1$

**Theorem** (*Vinogradov's mean value conj.* – J. Bourgain, C. Demeter, L. Guth (2016))

$\|S_N\|_{L^s(\mathbb{T}^d)} \leq C(d, s) N^{-\frac{d(d+1)}{2s}}$ *for* $d \geq 3$ *and* $s > d(d+1)$

Thank you for your attention!