Harmonic analysis related to point configurations in large Euclidean subsets

This work was supported in part by the Croatian Science Foundation under the project number HRZZ-IP-2022-10-5116 (FANAP).

Vjekoslav Kovač

(University of Zagreb, Faculty of Science)

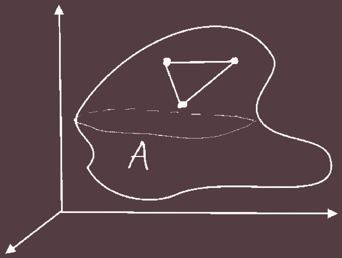
Harmonic Analysis and its Interactions @ Nagoya 2026

Monday, February 16, 2026

Croatian Science Foundation

Euclidean density theorems

They are a part of geometric measure theory, which identifies configurations present in every "large" subset of $\mathbb{R}^n$.



Initiated by Erdős, Székely, Bourgain, Falconer, etc. (1980s).

They could be thought of as "continuous-parameter combinatorics" (Ramsey theory).

Density theorems — What is a “large” set?

A measurable set $A \subseteq [0,1]^n$ is considered *large* if its Lebesgue measure is positive:

$$|A| \gtrsim 1.$$

A measurable set $A \subseteq [0,R]^n$ is considered *large* if its *density* is

$$\delta(A) := \frac{|A|}{R^n} \gtrsim 1.$$

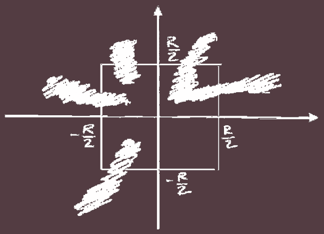
Density theorems — What is a "large" set?

A measurable set $A \subseteq \mathbb{R}^n$ is considered *large* if its *upper density* is positive:

$$\overline{\mathrm{d}}(A) := \limsup_{R\to\infty} \frac{|A \cap ([-R/2, R/2]^n)|}{R^n} > 0,$$

or if its *upper Banach density* is positive:

$$\overline{\delta}(A) := \lim_{R\to\infty} \sup_{x\in\mathbb{R}^n} \frac{|A \cap (x + [0, R]^n)|}{R^n} > 0.$$



Density theorems — Types of desired results

A family of configurations (= patterns):

$$\mathscr{P} = \{P_\lambda : \lambda \in \Lambda\}, \qquad \Lambda \subseteq (0,\infty).$$

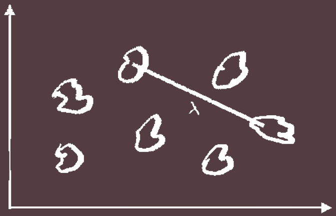
Common types of results:

- A large $A \subseteq \mathbb{R}^n$ contains an (isometric) copy of P_λ for at least one $\lambda \gtrsim 1$.
- A large $A \subseteq \mathbb{R}^n$ contains (isometric) copies of P_λ for all sufficiently large parameters λ.
- A large $A \subseteq [0,1]^n$ contains (isometric) copies of P_λ for an interval $I \subseteq \Lambda$ of parameters λ, with a bound on the length of I depending on $|A|$.

Density theorems — Example

A question by Székely (1982), popularized by Erdős

Does every measurable set $A \subseteq \mathbb{R}^2$ of positive upper density realize all sufficiently large distances between pairs of its points?



Answered affirmatively by:

- Furstenberg, Katznelson, and Weiss (1980s),
- Falconer and Marstrand (1986),
- Bourgain (1986).

Techniques (in this talk)

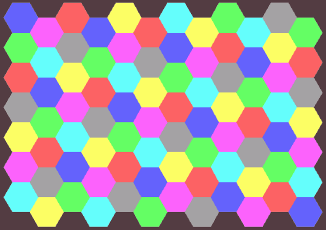
Techniques for positive results

use real linear and multilinear harmonic analysis.

Think classical multipliers, square functions, maximal functions, etc.

Techniques for negative results

are typically some funny colorings.

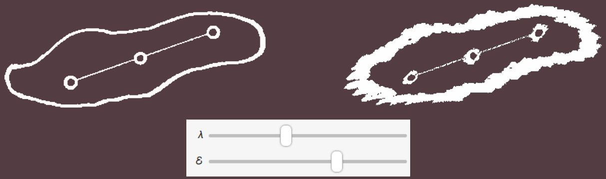


General approach

Bourgain, Lyall–Magyar, Cook–Magyar–Pramanik

"Counting" form:

$$\mathcal{N}_\lambda^0(A) = \lim_{\varepsilon \to 0+} \mathcal{N}_\lambda^\varepsilon(A).$$



$$\underbrace{\mathcal{N}_\lambda^0(A)}_{\text{we want } > 0} = \underbrace{\mathcal{N}_\lambda^1(A)}_{\text{structured part}} + \Big(\underbrace{\mathcal{N}_\lambda^\varepsilon(A) - \mathcal{N}_\lambda^1(A)}_{\text{error part}}\Big) + \Big(\underbrace{\mathcal{N}_\lambda^0(A) - \mathcal{N}_\lambda^\varepsilon(A)}_{\text{uniform part}}\Big)$$

dominant term $\geqslant c(\delta)$

small for all small ε uniformly in λ

small for some λ

General approach

For the *structured part* $\mathcal{N}_\lambda^1$ we need a lower bound

$$\mathcal{N}_\lambda^1 \geqslant c\big(\delta(A)\big)$$

that is uniform in λ, but this should be a simpler/smoother problem.

For the *uniform part* $\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon$ we want

$$\lim_{\varepsilon \to 0} \big|\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon\big| = 0$$

uniformly in λ; this usually leads to some **oscillatory integrals**.

For the *error part* $\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1$ one tries to prove

$$\sum_{j=1}^{J} \big|\mathcal{N}_{\lambda_j}^\varepsilon - \mathcal{N}_{\lambda_j}^1\big| \leqslant C(\varepsilon) o(J)$$

for lacunary scales $\lambda_1 < \cdots < \lambda_J$; this usually leads to some **multilinear singular integrals**.

Vague correspondence

Measure theory:

configurations,
counting forms,

large sets.

Harmonic analysis:

singular integrals,
oscillatory integrals,
Fourier decay (curvature),
maximal functions, etc.,

Lebesgue measure,
surface measure,
Calderón–Zygmund theory
and its generalizations (BHT).

Pair of points

Solved Székely's problem

For every measurable set $A \subseteq \mathbb{R}^2$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A)$ such that $\forall \lambda \in [\lambda_0, \infty)$ $\exists x, y \in A$ with $|x - y| = \lambda$.

Pair of points

Counting form:

$$\mathcal{N}^0_\lambda(A) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_A(x)\mathbb{1}_A(x+v)\,\mathrm{d}\sigma_\lambda(v)\,\mathrm{d}x$$

σ = the circle measure
$\sigma_\lambda(E) := \sigma(\lambda^{-1}E)$
Smoothed counting form:

$$\mathcal{N}^\varepsilon_\lambda(A) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_A(x)\mathbb{1}_A(x+v)(\sigma * \varphi_\varepsilon)_\lambda(v)\,\mathrm{d}v\,\mathrm{d}x$$

- $\mathcal{N}^0_\lambda - \mathcal{N}^\varepsilon_\lambda$ is handled using just the Fourier decay $|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-1/2}$.
- $\mathcal{N}^\varepsilon_\lambda - \mathcal{N}^1_\lambda$ is essentially $\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_A(x)\mathbb{1}_A(x+v)\psi_t(v)\,\mathrm{d}v\,\mathrm{d}x\,\frac{\mathrm{d}t}{t}$ and it is handled using the L–P square function.

Vertex-sets of simplices

$\Delta =$ the set of vertices of a non-degenerate n-dimensional simplex

Bourgain (1986)

For every measurable set $A \subseteq \mathbb{R}^{n+1}$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \Delta)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of $\lambda\Delta$.

Compact formulation, Bourgain (1986)

Take $\delta \in (0, 1/2]$, $A \subseteq [0,1]^{n+1}$ measurable, $|A| \geqslant \delta$. Then the set of "scales"

$$\{\lambda \in (0,1] : A \text{ contains an isometric copy of } \lambda\Delta\}$$

contains an interval of length at least $\big(\exp(\delta^{-C(\Delta,n)})\big)^{-1}$.

Vertex-sets of simplices

Counting form:

$$\mathcal{N}^0_\lambda(A) := \int_{\mathbb{R}^{n+1}} \int_{\mathsf{SO}(n+1,\mathbb{R})} \mathbb{1}_A(x) \Big(\prod_{k=1}^{n} \mathbb{1}_A(x + \lambda U u_k) \Big) \, \mathsf{d}\mu(U) \, \mathsf{d}x$$

Smoothed counting form:

$$\mathcal{N}^\varepsilon_\lambda(A) := \int_{\mathbb{R}^{n+1}} \int_{\mathsf{SO}(n+1,\mathbb{R})} \mathbb{1}_A(x) \Big(\prod_{k=1}^{n} (\mathbb{1}_A * \varphi_{\varepsilon\lambda})(x + \lambda U u_k) \Big) \, \mathsf{d}\mu(U) \, \mathsf{d}x$$

- $\mathcal{N}^0_\lambda - \mathcal{N}^\varepsilon_\lambda$ just uses the decay of $\widehat{\sigma}$.
- $\mathcal{N}^\varepsilon_\lambda - \mathcal{N}^1_\lambda$ just uses L–P theory.

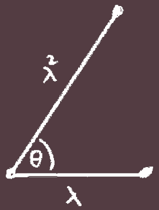
Anisotropically scaled simplices

$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n > 0$,
$u_1, u_2, \ldots, u_n \in \mathbb{R}^n$ unit vectors

K. (2020)

For every measurable set $A \subseteq \mathbb{R}^{n+1}$ satisfying $\overline{\delta}(A) > 0$ there is a positive number $\lambda_0 = \lambda_0(A, a_1, \ldots, a_n, b_1, \ldots, b_n, u_1, \ldots, u_n)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of

$$\{\mathbf{0}, \lambda^{a_1} b_1 u_1, \lambda^{a_2} b_2 u_2, \ldots, \lambda^{a_n} b_n u_n\}.$$



Anisotropically scaled simplices

Counting form:

$$\mathcal{N}_\lambda^0(A) := \int_{\mathbb{R}^{n+1}} \int_{\mathsf{SO}(n+1,\mathbb{R})} \mathbb{1}_A(x) \Big(\prod_{k=1}^{n} \mathbb{1}_A(x + \lambda^{a_k} b_k U u_k) \Big) \,\mathrm{d}\mu(U)\,\mathrm{d}x$$

Smoothed counting form:

$$\mathcal{N}_\lambda^\varepsilon(A) := \int_{\mathbb{R}^{n+1}} \int_{\mathsf{SO}(n+1,\mathbb{R})} \mathbb{1}_A(x) \Big(\prod_{k=1}^{n} (\mathbb{1}_A * \varphi_{(\varepsilon\lambda)^{a_k} b_k})(x+\lambda^{a_k} b_k U u_k) \Big) \,\mathrm{d}\mu(U)\,\mathrm{d}x$$

- Uses anisotropic multilinear C–Z operators:

$$\Lambda(f_0, \ldots, f_n) := \int_{(\mathbb{R}^d)^{n+1}} K(x_1 - x_0, \ldots, x_n - x_0) \Big(\prod_{k=0}^{n} f_k(x_k)\,\mathrm{d}x_k \Big)$$

  Coifman and Meyer (1970s), Grafakos and Torres (2002) meet Stein and Wainger (1978)

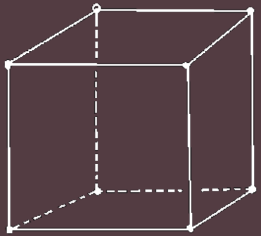
Vertex-sets of boxes

$\square$ = the set of vertices of an n-dimensional rectangular box

Lyall and Magyar (2016, 2019), Durcik and K. (2018, 2020)

For every measurable set $A \subseteq \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = (\mathbb{R}^2)^n$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \square)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of $\lambda\square$ with sides parallel to the distinguished 2-dimensional coordinate planes.

+ compact quantitative version



Vertex-sets of boxes

Counting form:

$$\mathcal{N}_\lambda^0(A) := \int_{(\mathbb{R}^2)^{2n}} \prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} \mathbb{1}_A(x_1 + r_1 y_1, x_2 + r_2 y_2, \ldots, x_n + r_n y_n)$$
$$\mathrm{d}\sigma_\lambda(y_1)\cdots\mathrm{d}\sigma_\lambda(y_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

Smoothed counting form:

$$\mathcal{N}_\lambda^\varepsilon(A) := \int_{(\mathbb{R}^2)^{2n}} \prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} \mathbb{1}_A(x_1 + r_1 y_1, x_2 + r_2 y_2, \ldots, x_n + r_n y_n)$$
$$\times\, (\sigma_\lambda * \mathrm{g}_{\varepsilon\lambda})(y_1)\cdots(\sigma_\lambda * \mathrm{g}_{\varepsilon\lambda})(y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

The error part is (quantitatively most efficiently) estimated using

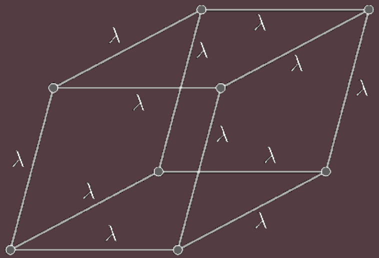
- *entangled singular integral forms* — K. (2010, 2011), Durcik (2014, 2015), Durcik, K., Škreb, Thiele (2016),
- recently a.k.a. *singular Brascamp–Lieb estimates* — Durcik, Thiele (2018, 2019), Durcik, Slavíková, Thiele (2021, 2023).

Hypercube graphs in the plane

$\Gamma_n = 1$-skeleton of an n-dimensional hypercube

K. and Predojević (2023)

For a positive integer n and a measurable set of positive upper Banach density $A \subseteq \mathbb{R}^2$ there exists a number $\lambda_0(A, n) > 0$ such that for every number $\lambda \geqslant \lambda_0(A, n)$ the set A contains an isometric copy of the distance graph $\lambda \cdot \Gamma_n$.



Arithmetic progressions in ℓ^p-norms

Cook, Magyar, and Pramanik (2015)

If $p \neq 1, 2, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(p, d, A) \in (0, \infty)$ such that for every $\lambda \geqslant \lambda_0$ one can find $x, y \in \mathbb{R}^d$ satisfying $x, x+y, x+2y \in A$ and $\|y\|_{\ell^p} = \lambda$.

Durcik and K. (2020)

Take $n \geqslant 3$, $p \neq 1, 2, \ldots, n-1, \infty$, $d \geqslant d_{\mathrm{min}}(n, p)$, $\delta \in (0, 1/2]$, $A \subseteq [0,1]^d$ measurable, $|A| \geqslant \delta$. Then the set of ℓ^p-norms of the gaps of n-term APs in the set A contains an interval of length at least

$$\begin{cases} \big(\exp(\exp(\delta^{-C(n,p,d)}))\big)^{-1} & \text{when } 3 \leqslant n \leqslant 4, \\ \big(\exp(\exp(\exp(\delta^{-C(n,p,d)})))\big)^{-1} & \text{when } n \geqslant 5. \end{cases}$$

Arithmetic progressions in ℓ^p-norms

$$\mathrm{d}\sigma(x) := \delta_0(\|x\|_{\ell^p}^p - 1)\,\mathrm{d}x$$

$$\mathcal{N}_\lambda^0(A) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\prod_{k=0}^{n-1}\mathbb{1}_A(x+ky)\,\mathrm{d}\sigma_\lambda(y)\,\mathrm{d}x$$

- Error part uses (dualized and truncated) multilinear Hilbert transforms:

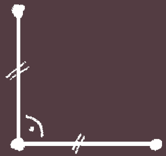
$$\Lambda(f_0,\dots,f_n) = \int_{\mathbb{R}}\int_{[-R,-r]\cup[r,R]}\prod_{k=0}^{n-1} f_k(x+kt)\frac{\mathrm{d}t}{t}\,\mathrm{d}x$$

- Tao (2016) showed $o(\log(R/r))$, Zorin-Kranich (2016)
- Durcik, K., and Thiele (2016) showed $O((\log(R/r))^{1-\varepsilon})$

(Non-rotated) corners in ℓ^p-norms

Durcik, K., and Rimanić (2016)

If $p \neq 1, 2, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d \times \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(p, d, A) \in (0, \infty)$ such that for every $\lambda \geqslant \lambda_0$ one can find $x, y, s \in \mathbb{R}^d$ satisfying $(x, y), (x + s, y), (x, y + s) \in A$ and $\|s\|_{\ell^p} = \lambda$.



(Non-rotated) corners in ℓ^p-norms

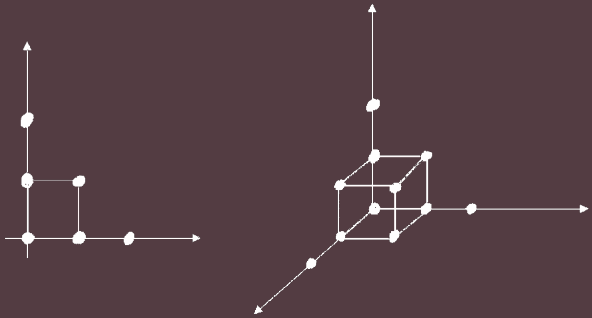
- Error part uses the 2D bilinear square function:

$$S(f,g)(x,y) := \Big(\sum_{k\in\mathbb{Z}} \Big(\int_{\mathbb{R}} f(x+t,y)g(x,y+t)\psi_k(t)\,\mathrm{d}t \Big)^2 \Big)^{1/2}$$

- Durcik, K., Škreb, and Thiele (2016)

Progression-extended boxes in ℓ^p-norms

Durcik and K. (2018)



- Uses some hybrid singular integral forms

Pairs of points along a beam of parabolae

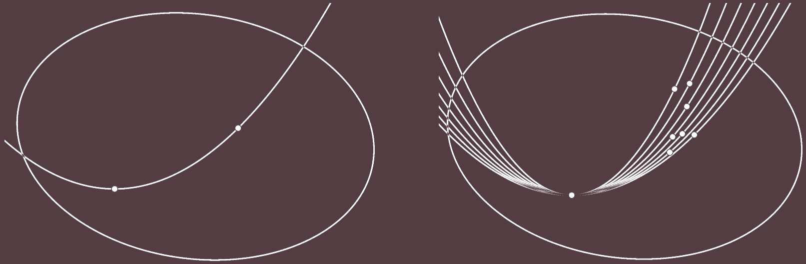
Kuca, Orponen, Sahlsten (2021), Durcik, K., Stipčić (2023)

$\beta \in (0,\infty)$, $\beta \neq 1$, $0 < \delta \leqslant 1/2$, $A \subseteq [0,1]^2$, $|A| \geqslant \delta$
$\implies$ $\exists (x,y) \in A$, $\exists I \subseteq (0,\infty)$ interval s.t.

$$\exp(-\delta^{-C}) \leqslant \inf I < \sup I \leqslant \exp(\delta^{-C}), \quad |I| \geqslant \exp(-\delta^{C}),$$

and that for every $a \in I$ the set A intersects the arc

$$\big\{(x,y) + \big(u, au^{\beta}\big) : \exp(-\delta^{-C}) \leqslant u \leqslant \exp(\delta^{-C})\big\}.$$



Pairs of points along a beam of parabolae

- "Pinned" results need maximal functions!
- Uses Bourgain's generalized circular maximal function associated with non-vanishing-curvature boundaries of convex sets (1986):

$$(Mf)(x) := \sup_{t\in(0,\infty)} \big|(f * \sigma_t)(x)\big|$$

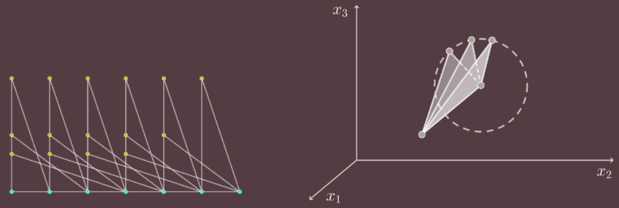
Right-angled simplices of unit volume

$m, n \in \mathbb{N}, \quad 2 \leqslant m \leqslant n$

Graham (1979)

For all finite colorings of $\mathbb{R}^n$ some color-class contains vertices of a right-angled m-dimensional simplex of unit volume.

It is sufficient to color a "large" cube $[0, R]^n$



K. (2024)

$A \subseteq [0, R]^n$, $\delta = |A|/R^n \geqslant \big(C_m/\log R\big)^{1/(9m^2)}$
$\implies$ A contains $m+1$ vertices of a right-angled m-dimensional simplex of unit volume.

Right-angled simplices of unit volume

$\lambda > 0$ a certain (aspect ratio) parameter

Counting form:

$$\mathcal{N}_\lambda^0(A) :=$$

$$\int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \mathbb{1}_A(x,y) \Big(\prod_{k=1}^{m-1} \mathbb{1}_A(x + u_k \mathbb{e}_k, y) \Big) \mathbb{1}_A(x, y+v)$$

$$\mathrm{d}\sigma_{m!|u_1 \cdots u_{m-1}|^{-1}}(v)\, \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k) \Big)\, \mathrm{d}u\, R^{-m-1}\, \mathrm{d}y\, \mathrm{d}x$$

Smoothed counting form:

$$\mathcal{N}_\lambda^\varepsilon(A) :=$$

$$\int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \mathbb{1}_A(x,y) \Big(\prod_{k=1}^{m-1} \mathbb{1}_A(x + u_k \mathbb{e}_k, y) \Big) \mathbb{1}_A(x, y+v)$$

$$(\sigma * \mathbb{g}_\varepsilon)_{m!|u_1 \cdots u_{m-1}|^{-1}}(v) \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k) \Big) \mathrm{d}v \mathrm{d}u\, R^{-m-1} \mathrm{d}y \mathrm{d}x$$

- Uniform part is estimated using the Fourier decay of σ.
- Error part is estimated using basic Littlewood–Paley theory.

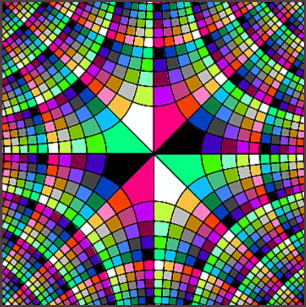
Rectangular boxes of unit volume

Erdős and Graham (1979)

The question is: Is this also true for rectangles?

K. (2024)

There exists a Jordan-measurable coloring of the plane in 25 colors such that no color-class contains the vertices of a rectangle of area 1.



Rectangular boxes of unit volume

K. (2024)

For every $n \in \mathbb{N}$ $\exists$ a finite Jordan-measurable coloring of $\mathbb{R}^n$ s.t., $\forall m \leqslant n$, there is no m-dimensional rectangular box of m-volume equal to 1 with all 2^m vertices colored the same.

$n \geqslant m+1 \implies$ all sufficiently large volumes are attained

K. (2024)

$A \subseteq \mathbb{R}^n$, $\overline{\delta}_n(A) > 0$
$\implies \exists V_0 = V_0(A) > 0 \quad \forall V \geqslant V_0 \quad \exists$ m-dimensional rectangular box of m-volume V with all 2^m vertices in A.

- Previously known for
  - $n \geqslant 5m$ by Durcik and K. (2018),
  - $n \geqslant 2m$ by Lyall and Magyar (2019).
- Still open for $n = m$.

Rectangular boxes of unit volume

Counting form:

$$\mathcal{N}_\lambda^0(A) := \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}}$$
$$\Big(\prod_{(r_1,\ldots,r_m)\in\{0,1\}^m} \mathbb{1}_A(x_1 + r_1 u_1, \ldots, x_{m-1} + r_{m-1} u_{m-1}, y + r_m v) \Big)$$
$$\mathrm{d}\sigma_{\lambda^m |u_1 \cdots u_{m-1}|^{-1}}(v)\, \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k) \Big)\, \mathrm{d}u\, R^{-n}\, \mathrm{d}y\, \mathrm{d}x$$

Smoothed counting form:

$$\mathcal{N}_\lambda^\varepsilon(A) := \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}}$$
$$\Big(\prod_{(r_1,\ldots,r_m)\in\{0,1\}^m} \mathbb{1}_A(x_1 + r_1 u_1, \ldots, x_{m-1} + r_{m-1} u_{m-1}, y + r_m v) \Big)$$
$$(\sigma * \mathrm{g}_\varepsilon)_{\lambda^m |u_1 \cdots u_{m-1}|^{-1}}(v)\, \mathrm{d}v\, \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k) \Big)\, \mathrm{d}u\, R^{-n}\, \mathrm{d}y\, \mathrm{d}x$$

- Error part uses singular Brascamp–Lieb again.

Harmonic analyst's point of view

How are **area** 1 **rectangles** (density results impossible) different from **area** 1 **right-angled triangles** (density results possible in dimensions $n \geqslant 3$)?

Harmonic analyst's point of view — rectangles

$\varphi, \psi \in \mathcal{S}(\mathbb{R})$, $0 \notin \mathsf{supp}(\widehat{\varphi})$ or $0 \notin \mathsf{supp}(\widehat{\psi})$
$(p_1, p_2, p_3, p_4) \in [1, \infty]^4$, $\sum_{k=1}^{4} \frac{1}{p_k} = 1$, $0 < r < R$

Let $C_{r,R}$ be the best constant in:

$$\Big| \int_r^R \int_{\mathbb{R}^4} f_1(x,y) f_2(x,y') f_3(x',y) f_4(x',y') \varphi_t(x-x') \psi_{1/t}(y-y') \, \mathsf{d}y \, \mathsf{d}y' \, \mathsf{d}x \, \mathsf{d}x' \, \frac{\mathsf{d}t}{t} \Big| \leqslant C_{r,R} \prod_{k=1}^{4} \|f_k\|_{\mathsf{L}^{p_k}(\mathbb{R}^2)}.$$

We claim: $C_{r,R} \sim \log(R/r)$ as $R/r \to \infty$.
(Not any better than the trivial estimate obtained from Hölder.)

Harmonic analyst's point of view — rectangles

$M > 0, \quad \mathrm{g}(x) := e^{-\pi x^2}$

$$f_1(x,y) := e^{2\pi \mathbb{i} xy} \mathrm{g}\Big(\frac{x}{M}\Big)\mathrm{g}\Big(\frac{y}{M}\Big)$$

$f_2 := \overline{f_1}, \quad f_3 := \overline{f_1}, \quad f_4 := f_1$

$$RHS \sim C_{r,R} M^2$$

$$\lim_{M\to\infty} \frac{1}{M^2} LHS = \lim_{M\to\infty} \frac{1}{M^2} \int_r^R \int_{\mathbb{R}^4} f_1(x,y) f_2(x,y') f_3(x',y) f_4(x',y') \varphi_t(x-x')\psi_{1/t}(y-y') \, \mathsf{d}(x,x',y,y') \, \frac{\mathsf{d}t}{t}$$

[substitute $u = x - x'$, $v = y - y'$]

$$= \lim_{M\to\infty} \frac{1}{4} \int_r^R \int_{\mathbb{R}^2} e^{2\pi \mathbb{i} uv} \varphi_t(u)\psi_{1/t}(v) \mathrm{g}\Big(\frac{u}{M}\Big)\mathrm{g}\Big(\frac{v}{M}\Big) \, \mathsf{d}(u,v) \, \frac{\mathsf{d}t}{t}$$

$$= \frac{1}{4}\Big(\log\frac{R}{r}\Big) \int_{\mathbb{R}} \widehat{\varphi}(-v)\psi(v) \, \mathsf{d}v$$

Harmonic analyst's point of view — triangles

Let $C'_{r,R}$ be the best constant in: $(p_4 = \infty)$

$$\Big| \int_r^R \int_{\mathbb{R}^4} f_1(x,y) f_2(x,y') f_3(x',y) \varphi_t(x-x') \psi_{1/t}(y-y') \, \mathrm{d}y \, \mathrm{d}y' \, \mathrm{d}x \, \mathrm{d}x' \, \frac{\mathrm{d}t}{t} \Big| \leqslant C'_{r,R} \prod_{k=1}^{3} \|f_k\|_{\mathrm{L}^{p_k}(\mathbb{R}^2)}.$$

A single Cauchy–Schwarz + a square function estimate:

$$C'_{r,R} = O((\log(R/r))^{1/2}).$$

It could be interesting to study boundedness/cancellation of "volume-preserving" or "time reversed" multilinear singular integral operators.

Thank you!

Thank you for your attention!