Sharp estimates for Gowers norms on discrete cubes

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Motivation from combinatorics

$A \subset \mathbb{Z}^d$ finite set

Additive energy [Tao and Vu (2006)]:

$$E_2(A) := \left|\{(a_1, a_2, a_3, a_4) \in A^4 \ : \ a_1 - a_2 = a_3 - a_4\}\right|$$

Generalizations by Schoen and Shkredov (2013), Shkredov (2014), de Dios Pont, Greenfeld, Ivanisvili, and Madrid (2021).

Higher energies:

$$\widetilde{E}_k(A) := \left|\{(a_1, a_2, \ldots, a_{2k-1}, a_{2k}) \in A^{2k} \ : \ a_1 - a_2 = \cdots = a_{2k-1} - a_{2k}\}\right|$$

k-additive energies:

$$E_k(A) := \left|\{(a_1, \ldots, a_{2k}) \in A^{2k} \ : \ a_1 + \cdots + a_k = a_{k+1} + \cdots + a_{2k}\}\right|$$

Counting k-parallelotopes in A [Shkredov (2014)]:

$$P_k(A) := \Big|\Big\{(a, h_1, \ldots, h_k) \in (\mathbb{Z}^d)^{k+1} \ : a + \epsilon_1 h_1 + \cdots + \epsilon_k h_k \in A \text{ for every } (\epsilon_1, \ldots, \epsilon_k) \in \{0,1\}^k\Big\}\Big|$$

Motivation from combinatorics

What can be said if additionally

$$A \subseteq \{0,1\}^d \subset \mathbb{Z}^d?$$

Kane and Tao (2017):

$$E_2(A) \leqslant |A|^{\log_2 6}.$$

De Dios Pont, Greenfeld, Ivanisvili, and Madrid (2021):

$$\widetilde{E}_k(A) \leqslant |A|^{\log_2(2^k+2)} \qquad \text{for } k \geqslant 2.$$

De Dios Pont, Greenfeld, Ivanisvili, and Madrid (2021), K. (2022):

$$E_k(A) \leqslant |A|^{\log_2 \binom{2k}{k}} \qquad \text{for } k \geqslant 2.$$

All these exponents are seen to be sharp by taking $A = \{0,1\}^d$.

Motivation from combinatorics

What can be said if

$$A \subseteq \{0, 1, 2, \ldots, n-1\}^d \subset \mathbb{Z}^d?$$

Already for $n = 3$ the sharp estimate is

$$E_2(A) \leqslant |A|^{2.7207109973\ldots}$$

and the exponent is just some strange number.

Shao (2024) studied the optimal exponent t_n in

$$A \subseteq \{0, 1, \ldots, n-1\}^d \implies E_2(A) \leqslant |A|^{t_n}$$

and showed

$$3 - \left(1 + o^{n\to\infty}(1)\right)\frac{3\log_2 3 - 4}{2\log_2 n} \leqslant t_n \leqslant 3 - \frac{c}{\log_2 n}$$

Main combinatorial results

We study the gen. energies P_k. Let $t_{k,n}$ be the optimal exponent in

$$A \subseteq \{0, 1, \ldots, n-1\}^d \implies P_k(A) \leqslant |A|^{t_{k,n}}$$

Three theorems. [Beker, Crmarić, and K. (2024)]:

$$t_{k,2} = \log_2(2k+2)$$

For a fixed $k \geqslant 2$ and $n \to \infty$:

$$k+1-\left(1+o_k^{n\to\infty}(1)\right)\frac{(k+1)\log_2(k+1)-2k}{2\log_2 n} \leqslant t_{k,n} \leqslant k+1-\frac{c}{\log_2 n}$$

For a fixed $n \geqslant 2$ and $k \to \infty$:

$$t_{k,n} = \frac{(n-1)\log_2(2k) - \log_2(n-1)!}{H_{n-1}} + o_n^{k\to\infty}(1),$$

where $H_m := -\sum_{j=0}^{m} \frac{\binom{m}{j}}{2^m} \log_2 \frac{\binom{m}{j}}{2^m}$ is the entropy of $B(m, 1/2)$.

Numerous sharp inequalities in analysis are known for

$$f\colon \mathbb{Z}^d \to \mathbb{C}.$$

(Think of Hausdorff–Young, Young's convolution ineq., etc.)

What changes if

$$\operatorname{supp} f \subseteq \{0, 1, 2, \ldots, n-1\}^d?$$

Can they be refined?

Example.

Sharp Hausdorff–Young on $\mathbb{Z}$:

$$\|\widehat{f}\|_{\mathrm{L}^4(\mathbb{T})} \leqslant \|f\|_{\ell^{4/3}(\mathbb{Z})}$$

Shao (2024):

$$\operatorname{supp} f \subseteq \{0, 1, \ldots, n-1\} \implies \|\widehat{f}\|_{\mathrm{L}^4(\mathbb{T})} \leqslant \|f\|_{\ell^{q_n}(\mathbb{Z})},$$

where

$$q_n = \frac{4}{3 - c/\log_2 n} > \frac{4}{3}.$$

Gowers uniformity norms [Gowers (2001)]:

$$\|f\|_{\mathsf{U}^k} := \Big(\sum_{a,h_1,\dots,h_k} \prod_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} \mathcal{C}^{\epsilon_1+\cdots+\epsilon_k} f(a+\epsilon_1 h_1+\cdots+\epsilon_k h_k) \Big)^{1/2^k}$$

where $\mathcal{C}\colon z \mapsto \bar{z}$.

Note:

$$P_k(A) = \|\mathbb{1}_A\|_{\mathsf{U}^k}^{2^k}$$

Sharp estimate:

$$\|f\|_{\mathsf{U}^k(\mathbb{Z}^d)} \leqslant \|f\|_{\ell^{p_k}(\mathbb{Z}^d)}$$

for $p_k = 2^k/(k+1)$.

Auxilliary result

Let $p_{k,n}$ be the optimal exponent in

$$\operatorname{supp} f \subseteq \{0, 1, \ldots, n-1\}^d \implies \|f\|_{\mathsf{U}^k(\mathbb{Z}^d)} \leqslant \|f\|_{\ell^{p_{k,n}}(\mathbb{Z}^d)}$$

We will see that

$$p_{k,n} t_{k,n} = 2^k.$$

Proposition. [Beker, Crmarić, and K. (2024)] $k, n \geqslant 2$, $p, t > 0$, $pt = 2^k$. The following are equivalent.

(1) $\|f\|_{\mathrm{U}^k} \leqslant \|f\|_{\ell^p}$ holds for every $f\colon \mathbb{Z} \to [0, \infty)$ supported in $\{0, 1, \ldots, n-1\}$.

(2) $\|f\|_{\mathrm{U}^k} \leqslant \|f\|_{\ell^p}$ holds for every $d \geqslant 0$ and every $f\colon \mathbb{Z}^d \to \mathbb{C}$ supported in $\{0, 1, \ldots, n-1\}^d$.

(3) $P_k(A) \leqslant |A|^t$ holds for every $d \geqslant 0$ and every $A \subseteq \{0, 1, \ldots, n-1\}^d$.

(4) There exists $C \in (0, \infty)$ such that $P_k(A) \leqslant C|A|^t$ holds for every $d \geqslant 0$ and every $A \subseteq \{0, 1, \ldots, n-1\}^d$.

We are essentially imitating the proof by de Dios Pont, Greenfeld, Ivanisvili, and Madrid (2021), who studied E_k and $\| \underbrace{f * f * \cdots * f}_{k \text{ times}} \|_{\ell^2}^2$.

Proof of (1) $\implies$ (2)

The assumption (1) is: for $g\colon \mathbb{Z} \to [0, \infty)$, $\operatorname{supp} g \subseteq \{0, 1, \ldots, n-1\}$

$$\sum_{b,l_1,\ldots,l_k \in \mathbb{Z}} \prod_{(\epsilon_1,\ldots,\epsilon_k)\in\{0,1\}^k} g(b + \epsilon_1 l_1 + \cdots + \epsilon_k l_k) \leqslant \Big(\sum_{b=0}^{n-1} g(b)^p \Big)^{2^k/p}$$

We prove the claim (2) by the induction on d.
Induction step: for each $b \in \mathbb{Z}$ define

$$f_b\colon \mathbb{Z}^{d-1} \to \mathbb{C}, \quad f_b(a) := f(a, b).$$

From the induction hypothesis:

$$\|f_b\|_{\mathrm{U}^k} \leqslant \|f_b\|_{\ell^p}$$

Write the LHS as the Gowers inner product:

$$\|f\|_{\mathrm{U}^k}^{2^k} = \sum_{b,l_1,\ldots,l_k\in\mathbb{Z}} \Big\langle \big(f_{b+\epsilon_1 l_1+\cdots+\epsilon_k l_k}\big)_{(\epsilon_1,\ldots,\epsilon_k)\in\{0,1\}^k} \Big\rangle_{\mathrm{U}^k}$$

Proof of (1) $\implies$ (2)

The Gowers–Cauchy–Schwarz inequality yields

$$\begin{aligned}\|f\|_{\mathrm{U}^k}^{2^k} &\leqslant \sum_{b,l_1,\dots,l_k\in\mathbb{Z}} \prod_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} \|f_{b+\epsilon_1 l_1+\cdots+\epsilon_k l_k}\|_{\mathrm{U}^k} \\ &\leqslant \sum_{b,l_1,\dots,l_k\in\mathbb{Z}} \prod_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} \|f_{b+\epsilon_1 l_1+\cdots+\epsilon_k l_k}\|_{\ell^p}\end{aligned}$$

It remains to apply

$$\sum_{b,l_1,\dots,l_k\in\mathbb{Z}} \prod_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} g(b+\epsilon_1 l_1+\cdots+\epsilon_k l_k) \leqslant \Big(\sum_{b=0}^{n-1} g(b)^p\Big)^{2^k/p}$$

with

$$g(b) := \|f_b\|_{\ell^p}$$

to conclude

$$\|f\|_{\mathrm{U}^k}^{2^k} \leqslant \Big(\sum_{b=0}^{n-1} \|f_b\|_{\ell^p}^p\Big)^{2^k/p} = \|f\|_{\ell^p}^{2^k}.$$

The assumption (4) can be restated:

$$A \subseteq \{0, 1, \ldots, n-1\}^d \implies \|\mathbb{1}_A\|_{\mathsf{U}^k} \leqslant D|A|^{t/2^k} = D\|\mathbb{1}_A\|_{\ell^p}$$

Decompose a general $f\colon \mathbb{Z}^d \to [0, \infty)$, $\operatorname{supp} f \subseteq \{0, 1, \ldots, n-1\}^d$ as

$$f = \sum_{i=0}^{N} \underbrace{\frac{M}{2^i}\mathbb{1}_{A_i}}_{f_i} + f',$$

where $M = \|f\|_{\ell^\infty}$, $N = \lceil d \log_2 n \rceil$, and $0 \leqslant f' < 2^{-N}M$. From

$$\|f_i\|_{\mathsf{U}^k} = \frac{M}{2^i}\|\mathbb{1}_{A_i}\|_{\mathsf{U}^k} \leqslant D\frac{M}{2^i}\|\mathbb{1}_{A_i}\|_{\ell^p} = D\|f_i\|_{\ell^p} \leqslant D\|f\|_{\ell^p},$$

$$\|f'\|_{\mathsf{U}^k} \leqslant M(n^d)^{(k+1)/2^k - 1} \leqslant M \leqslant \|f\|_{\ell^p},$$

we get

$$\|f\|_{\mathsf{U}^k} \leqslant \sum_{i=0}^{N} \|f_i\|_{\mathsf{U}^k} + \|f'\|_{\mathsf{U}^k} \leqslant D(3 + d\log_2 n)\|f\|_{\ell^p}.$$

Proof of (4) $\implies$ (1)

We remove the constant using the tensor power trick:

$$f(a_1, a_2, \ldots, a_d) = g(a_1)g(a_2)\cdots g(a_d)$$

$$\implies \|f\|_{\mathrm{U}^k} = \|g\|_{\mathrm{U}^k}^d, \quad \|f\|_{\ell^p} = \|g\|_{\ell^p}^d$$

$$\implies \|g\|_{\mathrm{U}^k} \leqslant \big(D(3 + d\log_2 n)\big)^{1/d}\|g\|_{\ell^p}$$

Letting $d \to \infty$ we obtain

$$\|g\|_{\mathrm{U}^k} \leqslant \|g\|_{\ell^p}.$$

One only needs to prove

$$\sum_{b,l_1,\ldots,l_k\in\mathbb{Z}} \prod_{(\epsilon_1,\ldots,\epsilon_k)\in\{0,1\}^k} g(b+\epsilon_1 l_1+\cdots+\epsilon_k l_k) \leqslant \Big(\sum_{b=0}^{1} g(b)^{2^k/t}\Big)^t$$

for $g\colon \mathbb{Z}\to[0,\infty)$, $\operatorname{supp} g\subseteq\{0,1\}$ and

$$t=\log_2(2k+2).$$

Denoting

$$x=g(0)^{2^k/t}, \quad y=g(1)^{2^k/t}$$

this simplifies as

$$x,y\in[0,\infty) \implies x^t+y^t+2kx^{t/2}y^{t/2}\leqslant(x+y)^t$$

A nice calculus exercise!

Proof of the asymptotics as $n \to \infty$

The lower bound

$$k+1-\left(1+o_k^{n\to\infty}(1)\right)\frac{(k+1)\log_2(k+1)-2k}{2\log_2 n}\leqslant t_{k,n}$$

means

$$\liminf_{n\to\infty}\left((t_{k,n}-k-1)\log_2 n\right)\geqslant -\frac{k+1}{2}\log_2(k+1)+k.$$

The Gowers norms on $\mathbb{R}$:

$$\|f\|_{\mathrm{U}^k(\mathbb{R})} := \left(\int_{\mathbb{R}^{k+1}} \prod_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} \mathcal{C}^{\epsilon_1+\cdots+\epsilon_k} f(x+\epsilon_1 h_1+\cdots+\epsilon_k h_k) \, \mathrm{d}x \, \mathrm{d}h_1 \cdots \mathrm{d}h_k \right)^{1/2^k}$$

Eisner and Tao (2012):

$$\|g\|_{\mathrm{U}^k(\mathbb{R})} \leqslant C_k \|g\|_{\mathrm{L}^{2^k/(k+1)}(\mathbb{R})}$$

with

$$C_k = \frac{2^{k/2^k}}{(k+1)^{(k+1)/2^{k+1}}}$$

and the equality is attained for Gaussians (among other extremizers).

Proof of the asymptotics as $n \to \infty$

$$f_{M,n}(m) := \begin{cases} \exp\left(-4M^2\left(\frac{m}{n}-\frac{1}{2}\right)^2\right) & \text{for } m \in \{0,1,2,\dots,n-1\}, \\ 0 & \text{otherwise} \end{cases}$$

is plugged into

$$\|f_{M,n}\|_{\mathrm{U}^k} \leqslant \|f_{M,n}\|_{\ell^{p_{k,n}}}.$$

Denoting $g(x) := e^{-x^2}$ we get

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \|f_{M,n}\|_{\mathrm{U}^k}^{2^k} = \frac{1}{(2M)^{k+1}} \|g\mathbb{1}_{[-M,M]}\|_{\mathrm{U}^k(\mathbb{R})}^{2^k},$$

$$\limsup_{n\to\infty} \frac{1}{n^{t_{k,n}}} \|f_{M,n}\|_{\ell^{p_{k,n}}}^{2^k} \leqslant \frac{1}{(2M)^{k+1}} \|g\mathbb{1}_{[-M,M]}\|_{\mathrm{L}^{2^k/(k+1)}(\mathbb{R})}^{2^k},$$

so take logarithms, subtract, and let $M \to \infty$:

$$\liminf_{n\to\infty} \big((t_{k,n} - k - 1)\log_2 n\big) \geqslant 2^k \log_2 \frac{\|g\|_{\mathrm{U}^k(\mathbb{R})}}{\|g\|_{\mathrm{L}^{2^k/(k+1)}(\mathbb{R})}} = 2^k \log_2 C_k$$

Proof of the asymptotics as $n \to \infty$

The upper bound

$$t_{k,n} \leqslant k + 1 - \frac{c}{\log_2 n}$$

will follow from Shao's result (2024) and

$$t_{k+1,n} \leqslant t_{k,n} + 1.$$

This is an easy consequence of a recursive formula

$$\|f\|_{\mathrm{U}^{k+1}}^{2^{k+1}} = \sum_{h \in \mathbb{Z}^d} \left\| \overline{f(\cdot + h)} f(\cdot) \right\|_{\mathrm{U}^k}^{2^k}$$

and Young's convolution inequality.

Recall that the *Shannon entropy* of

$$X \sim \begin{pmatrix} \cdots & 0 & 1 & 2 & \cdots & n-1 & \cdots \\ \cdots & q_0 & q_1 & q_2 & \cdots & q_{n-1} & \cdots \end{pmatrix}$$

is

$$\mathrm{H}(X) := -\sum_j q_j \log_2 q_j.$$

We denoted

$$H_m := \mathrm{H}(B(m, 1/2)).$$

Proof of the asymptotics as $k \to \infty$

For the lower bound

$$\liminf_{k\to\infty}\Big(t_{k,n} - \frac{(n-1)\log_2(2k) - \log_2(n-1)!}{H_{n-1}}\Big) \geqslant 0$$

start with

$$\sum_{a,h_1,\dots,h_k\in\mathbb{Z}} \prod_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} f(a+\epsilon_1 h_1+\cdots+\epsilon_k h_k) \leqslant \Big(\sum_{j=0}^{n-1} f(j)^{2^k/t}\Big)^t$$

for $t = t_{k,n}$ and only observe the mutually equal terms obtained by taking:

- $a \in \{0, 1, \dots, n-1\}$ arbitrary,
- precisely a of the numbers $h_1, \dots, h_k$ equal -1,
- precisely $n-1-a$ of the numbers $h_1, \dots, h_k$ equal 1,
- precisely $k-n+1$ of the numbers $h_1, \dots, h_k$ equal 0.

Proof of the asymptotics as $k \to \infty$

$$\binom{k}{n-1} 2^{n-1} \prod_{j=0}^{n-1} f(j)^{\binom{n-1}{j} 2^{k-n+1}} \leqslant \left(\sum_{j=0}^{n-1} f(j)^{2^k / t_{k,n}} \right)^{t_{k,n}}.$$

Now take

$$f(j) := \left(\frac{\binom{n-1}{j}}{2^{n-1}} \right)^{t_{k,n}/2^k}$$

$$\implies \binom{k}{n-1} 2^{n-1} \prod_{j=0}^{n-1} \left(\frac{\binom{n-1}{j}}{2^{n-1}} \right)^{\binom{n-1}{j} t_{k,n}/2^{n-1}} \leqslant \left(\sum_{j=0}^{n-1} \frac{\binom{n-1}{j}}{2^{n-1}} \right)^{t_{k,n}} = 1.$$

Taking logarithms,

$$\underbrace{\sum_{j=0}^{n-2} \log_2(k-j)}_{(n-1)\log_2 k + o_n^{k\to\infty}(1)} - \log_2(n-1)! + (n-1) + \underbrace{\sum_{j=0}^{n-1} \frac{\binom{n-1}{j} t_{k,n}}{2^{n-1}} \log_2 \frac{\binom{n-1}{j}}{2^{n-1}}}_{-H_{n-1} t_{k,n}} \leqslant 0$$

Proof of the asymptotics as $k \to \infty$

For the upper bound

$$\limsup_{k\to\infty}\left(t_{k,n} - \frac{(n-1)\log_2(2k) - \log_2(n-1)!}{H_{n-1}}\right) \leqslant 0$$

split the same inequality as

$$\sum_{j=0}^{n-1} f(j)^{2^k} + \sum_{l=1}^{n-1} \sum_{(a,h_1,\dots,h_l)\in T_{n,l}} \binom{k}{l} \prod_{(\epsilon_1,\dots,\epsilon_l)\in\{0,1\}^l} f(a+\epsilon_1 h_1+\cdots+\epsilon_l h_l)^{2^{k-l}} \leqslant \left(\sum_{j=0}^{n-1} f(j)^{2^k/t}\right)^t,$$

$$\begin{aligned} T_{n,l} := \Big\{ (a,h_1,\dots,h_l) \in \mathbb{Z}\times(\mathbb{Z}\setminus\{0\})^l \ :\ & \\ & 0 \leqslant a+\epsilon_1 h_1+\cdots+\epsilon_l h_l \leqslant n-1 \\ & \text{for every } (\epsilon_1,\dots,\epsilon_l)\in\{0,1\}^l \Big\}. \end{aligned}$$

Proof of the asymptotics as $k \to \infty$

Substituting $g(j) = f(j)^{2^k/t}$ we reduce the inequality to

$$\sum_{j=0}^{n-1} g(j)^t + \sum_{l=1}^{n-1} \sum_{(a,h_1,\ldots,h_l)\in T_{n,l}} \binom{k}{l} \prod_{(\epsilon_1,\ldots,\epsilon_l)\in\{0,1\}^l} g(a+\epsilon_1 h_1+\cdots+\epsilon_l h_l)^{t/2^l} \leqslant 1$$

for every $g\colon \{0,1,\ldots,n-1\} \to [0,\infty)$ such that $\sum_{j=0}^{n-1} g(j) = 1$.

We are going to prove this for any given $0 < \delta < 1$,

$$t = \frac{(n-1)\log_2(2k) - \log_2(n-1)!}{H_{n-1}} + \delta,$$

and sufficiently large k.

Proof of the asymptotics as $k \to \infty$

The terms $g(j)^t$ are handled separately, by splitting into 2 cases, etc. Every other (more interesting) term is of the form

$$\binom{k}{l}\left(g(0)^{q_0} g(1)^{q_1} \cdots g(n-1)^{q_{n-1}}\right)^t,$$

where $(q_0, q_1, \ldots, q_{n-1})$ is the distribution on the set $\{0, 1, \ldots, n-1\}$ of

$$h_1 X_1 + \cdots + h_l X_l,$$

where $X_1, X_2, X_3, \ldots$ are independent symmetric Bernoulli trials.

$$1 = \sum_{j=0}^{n-1} g(j) \geqslant \sum_{\substack{0 \leqslant j \leqslant n-1 \\ q_j \neq 0}} q_j \frac{g(j)}{q_j} \geqslant \prod_{\substack{0 \leqslant j \leqslant n-1 \\ q_j \neq 0}} \left(\frac{g(j)}{q_j}\right)^{q_j} = \frac{\prod_{j=0}^{n-1} g(j)^{q_j}}{\prod_{j=0}^{n-1} q_j^{q_j}}$$

That way we have obtained

$$\prod_{j=0}^{n-1} g(j)^{q_j} \leqslant 2^{-\mathrm{H}(q_0,\ldots,q_{n-1})},$$

so

$$\binom{k}{l}\left(g(0)^{q_0} g(1)^{q_1} \cdots g(n-1)^{q_{n-1}}\right)^t \leqslant \binom{k}{l} 2^{-\mathrm{H}(h_1 X_1 + \cdots + h_l X_l)t}$$

for $(a, h_1, \ldots, h_l) \in T_{n,l}$.

The RHS is at most

$$\begin{cases} 2^{-n+1} \cdot 2^{-H_{n-1}\delta} & \text{for } l = n-1, \\ O_n^{k\to\infty}\left(k^{l-(n-1)\mathrm{H}(h_1X_1+\cdots+h_lX_l)/H_{n-1}}\right) & \text{for } 1 \leqslant l \leqslant n-2. \end{cases}$$

Proof of the asymptotics as $k \to \infty$

Summing over $T_{n,n-1}$:

$$\sum_{(a,h_1,\dots,h_{n-1})\in T_{n,n-1}} \binom{k}{n-1} \prod_{(\epsilon_1,\dots,\epsilon_{n-1})\in\{0,1\}^{n-1}} g(a+\epsilon_1 h_1+\dots+\epsilon_{n-1}h_{n-1})^{t/2^{n-1}} \leqslant \underbrace{2^{-H_{n-1}\delta}}_{<1}$$

Summing over $T_{n,l}$, $1 \leqslant l \leqslant n-2$:

$$\sum_{l=1}^{n-2} \sum_{(a,h_1,\dots,h_l)\in T_{n,l}} \binom{k}{l} \prod_{(\epsilon_1,\dots,\epsilon_l)\in\{0,1\}^l} g(a+\epsilon_1 h_1+\dots+\epsilon_l h_l)^{t/2^l} = o_n^{k\to\infty}(1)$$

as soon as

$$l-(n-1)\frac{\mathrm{H}(h_1X_1+\dots+h_lX_l)}{H_{n-1}}<0$$

$$\iff \quad \frac{\mathrm{H}(h_1X_1+\dots+h_lX_l)}{l} > \frac{H_{n-1}}{n-1}$$

Lemma on minimal entropy

Lemma.

For $n \geqslant 2$, $1 \leqslant l \leqslant n-1$, $h_1, \ldots, h_l \in \mathbb{Z}\backslash\{0\}$ s.t. $|h_1|+\cdots+|h_l| \leqslant n-1$ we have

$$\frac{\mathrm{H}(h_1 X_1 + \cdots + h_l X_l)}{l} \geqslant \frac{H_{n-1}}{n-1},$$

with eq. attained only when $l = n-1$ and $|h_1| = \cdots = |h_{n-1}| = 1$.

Sketch of the proof. From the precise bounds by Adell, Lekuona, and Yu (2010),

$$\frac{1}{2}\log_2 \frac{e\pi m}{2} - \frac{1}{4m} < H_m < \frac{1}{2}\log_2 \frac{e\pi m}{2} + \frac{1}{10m},$$

one easily gets

$$l < n-1 \implies \frac{H_l}{l} > \frac{H_{n-1}}{n-1},$$

so one can maximize the LHS with a fixed number of terms l.

Sharper lemma on minimal entropy

Lemma.

For $m \in \mathbb{N}$, $h_1, \ldots, h_m \in \mathbb{Z} \setminus \{0\}$ we have

$$\mathrm{H}(h_1 X_1 + \cdots + h_m X_m) \geqslant H_m,$$

with equality attained only when $|h_1| = \cdots = |h_m|$.

Sketch of the proof. First reduce to the case when $h_1, \ldots, h_m > 0$.
$X = h_1 X_1 + \ldots + h_m X_m$, $Y \sim B(m, 1/2)$, $X \not\sim Y$
p_X, p_Y their probability mass functions
$p_X^{\downarrow}, p_Y^{\downarrow}$ their decreasing rearrangements
Karamata's inequality (1932) applies to the strictly concave function

$$\psi\colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} -x \log_2 x & \text{for } x > 0, \\ 0 & \text{for } x = 0 \end{cases}$$

and makes it is enough to see that $p_Y^{\downarrow}$ majorizes $p_X^{\downarrow}$ and $p_X^{\downarrow} \neq p_Y^{\downarrow}$.

Sharper lemma on minimal entropy

Note that

$$\mathcal{A}_z := \left\{ A \subseteq \{1, 2, \dots, m\} : \sum_{j \in A} h_j = z \right\}$$

is an antichain for every $z \in \mathbb{Z}$.

By Erdős' generalization of Sperner's theorem (1945) or by the Yamamoto–Bollobás–Lubell–Meshalkin inequality (1954, …, 1966) we have

$$\begin{aligned} &|\mathcal{A}_{z_1}| + |\mathcal{A}_{z_2}| + \cdots + |\mathcal{A}_{z_N}| \\ &\leqslant \text{maximal size of } N \text{ disjoint antichains in } \mathcal{P}(\{1, 2, \dots, m\}) \\ &= \underbrace{\binom{m}{\lfloor m/2 \rfloor} + \binom{m}{\lfloor m/2 \rfloor + 1} + \binom{m}{\lfloor m/2 \rfloor - 1} + \binom{m}{\lfloor m/2 \rfloor + 2} + \cdots}_{N} \end{aligned}$$

for $N = 1, 2, \dots$. Dividing by 2^m we get $p_X^{\downarrow} \preceq p_Y^{\downarrow}$.

Thank you for your attention!