Coloring and density theorems for configurations of a given volume

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Vjekoslav Kovač (University of Zagreb, Faculty of Science)

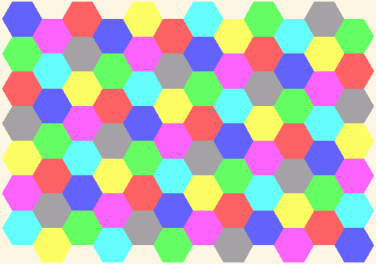
Recent advances in Harmonic Analysis, Malaga

Where does the topic belong?

At the intersection of:

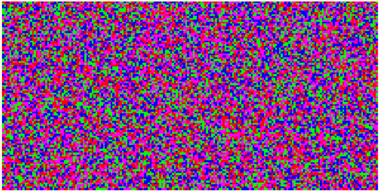
- combinatorics
- geometry
- analysis

> ***Coloring theorems are a part of the Euclidean Ramsey theory,***
>
> which identifies monochromatic configurations present in every finite coloring of $\mathbb{R}^n$



Systematic study initiated by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus (1970s)

A *finite coloring* of $S \subseteq \mathbb{R}^n$
= any partition of S into finitely many *color-classes* $\mathscr{C}_1, \ldots, \mathscr{C}_r$



A coloring is *measurable* if $\mathscr{C}_j$ are Lebesgue-measurable

A coloring is *Jordan-measurable* if $\mathscr{C}_j$ have boundaries of measure 0

Coloring theorems — Types of desired results

We search for (congruent) monochromatic copies of a configuration (= pattern) from a given family:

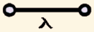
$$\mathscr{P} = \{P_\lambda : \lambda \in \Lambda\}, \qquad \Lambda \subseteq (0, \infty)$$

Common types of results:

- An arbitrary coloring of $\mathbb{R}^n$ or $[0, R]^n$ contains a monochromatic copy of P_λ for some parameter λ
- An arbitrary coloring of $\mathbb{R}^n$ or $[0, R]^n$ contains monochromatic copies of P_λ for all values of λ

A question by Rosenfeld (1994), popularized by Erdős

Does every finite coloring of $\mathbb{R}^2$ contain a pair of equally colored points at an odd distance from each other?



$$\lambda \in 2\mathbb{N} - 1$$

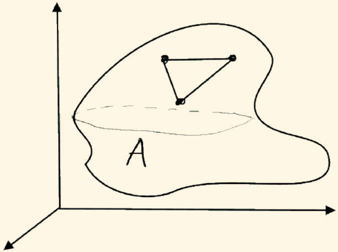
Answered affirmatively by James Davies (2022)

Still open when $2\mathbb{N} - 1$ is replaced with

- either $\{n! : n \in \mathbb{N}\}$ (Kahle),
- or $\{2^n : n \in \mathbb{N}\}$ (Soifer)

(Euclidean) density theorems

> ***Density theorems are a part of geometric measure theory,***
> which identifies configurations present in every “large” subset of $\mathbb{R}^n$



Initiated by Erdős, Székely, Bourgain, Falconer, etc. (1980s)

A measurable set $A \subseteq [0,1]^n$ is considered *large* if its Lebesgue measure is positive:

$$|A| > 0$$

A measurable set $A \subseteq [0,R]^n$ is considered *large* if its *density* is

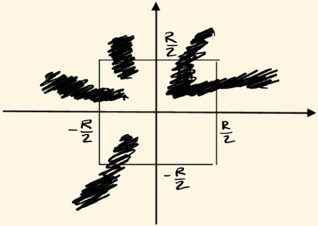
$$\delta = \frac{|A|}{R^n} \gtrsim 1$$

A measurable set $A \subseteq \mathbb{R}^n$ is considered *large* if its *upper density* is positive:

$$\overline{\mathrm{d}}_n(A) := \limsup_{R\to\infty} \frac{|A \cap ([-R/2, R/2]^n)|}{R^n} > 0,$$

or if its *upper Banach density* is positive:

$$\overline{\delta}_n(A) := \lim_{R\to\infty} \sup_{x\in\mathbb{R}^n} \frac{|A \cap (x + [0, R]^n)|}{R^n} > 0$$



Density theorems — Types of desired results

A family of configurations (= patterns):

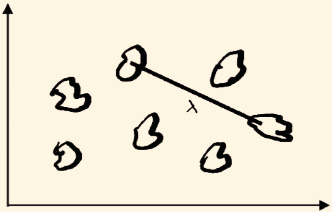
$$\mathscr{P} = \{P_\lambda : \lambda \in \Lambda\}, \qquad \Lambda \subseteq (0, \infty)$$

Other common types of results:

- A large $A \subseteq \mathbb{R}^n$ contains copies of P_λ for all sufficiently large parameters λ
- A large $A \subseteq [0, 1]^n$ contains copies of P_λ for an interval $I \subseteq \Lambda$ of parameters λ, with a bound on the length of I depending on $|A|$

A question by Székely (1982), popularized by Erdős

Does every set $A \subseteq \mathbb{R}^2$ of positive upper density realize all sufficiently large distances between pairs of its points?



Answered affirmatively by:

- Furstenberg, Katznelson, and Weiss (1980s),
- Falconer and Marstrand (1986),
- Bourgain (1986)

Coloring vs. density

Connections between the two worlds

a positive density result $\Longrightarrow$ a positive measurable coloring result
a negative measurable coloring result $\Longrightarrow$ a negative density result

$\mathscr{C}_1, \ldots, \mathscr{C}_r$ a measurable coloring of $\mathbb{R}^n$

$$\Longrightarrow \quad \overline{\delta}_n(\mathscr{C}_1) + \cdots + \overline{\delta}_n(\mathscr{C}_r) \geqslant \overline{\delta}_n(\mathbb{R}^n) = 1$$

$$\Longrightarrow \quad \overline{\delta}_n(\mathscr{C}_j) \geqslant \frac{1}{r} > 0 \quad \text{for at least one index } 1 \leqslant j \leqslant r$$

Techniques (in this talk)

Techniques for positive results

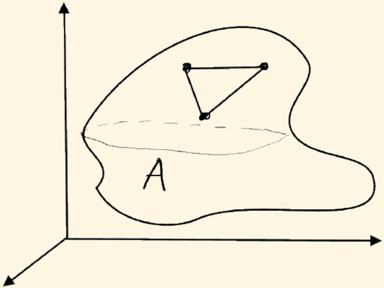
use real (linear and multilinear) harmonic analysis to prove density theorems

Techniques for negative results

are typically funny colorings

Vertex-sets of simplices

Pioneering work by Bourgain (1986)



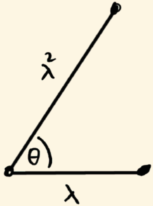
- Uses Littlewood–Paley theory, i.e., square function estimates:

$$(Sf)(x) := \left(\sum_{k\in\mathbb{Z}} \left| (f * \psi_k)(x) \right|^2 \right)^{1/2}$$

Biased selection of previously studied problems

Anisotropically scaled simplices

K. (2020)



- Uses anisotropic multilinear C–Z operators:

$$\Lambda(f_0, \dots, f_n) := \int_{(\mathbb{R}^d)^{n+1}} K(x_1 - x_0, \dots, x_n - x_0) \left(\prod_{k=0}^{n} f_k(x_k)\, \mathrm{d}x_k \right)$$

- Coifman and Meyer (1970s), Grafakos and Torres (2002) meet Stein and Wainger (1978)

Biased selection of previously studied problems

Arithmetic progressions in ℓ^p-norms

Cook, Magyar, and Pramanik (2015)
Durcik and K. (2020)

- Uses (dualized and truncated) multilinear Hilbert transforms:

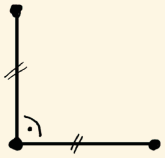
$$\Lambda(f_0, \ldots, f_n) = \int_{\mathbb{R}} \int_{[-R,-r]\cup[r,R]} \prod_{k=0}^{n} f_k(x + kt) \frac{\mathrm{d}t}{t} \, \mathrm{d}x$$

- Tao (2016) showed $o(\log(R/r))$, Zorin-Kranich (2016)
- Durcik, K., and Thiele (2016) showed $O((\log(R/r))^{1-\varepsilon})$

Biased selection of previously studied problems

(Non-rotated) corners in ℓ^p-norms

Durcik, K., and Rimanić (2016)



- Uses the 2D bilinear square function:

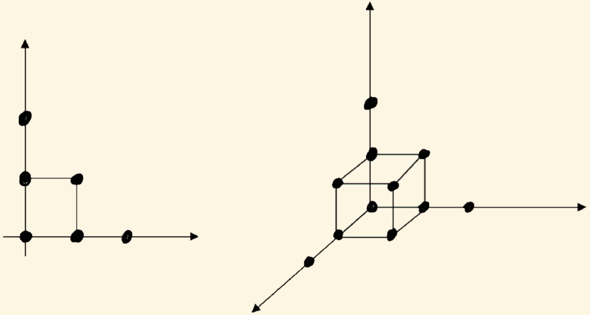
$$S(f,g)(x,y) := \left(\sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}} f(x+t,y)g(x,y+t)\psi_k(t)\,\mathrm{d}t \right)^2 \right)^{1/2}$$

- Durcik, K., Škreb, and Thiele (2016)

Biased selection of previously studied problems

Progression-extended boxes in ℓ^p-norms

Durcik and K. (2018)

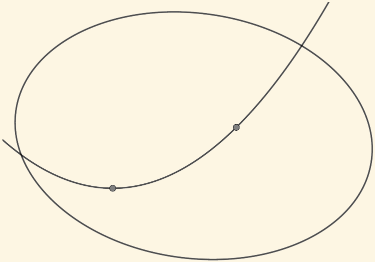


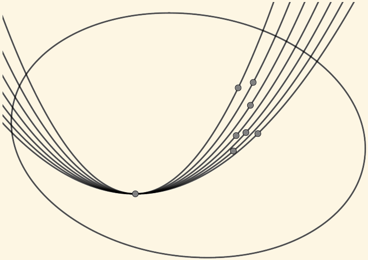
- Uses some hybrid singular integral forms

Biased selection of previously studied problems

Pairs of points along a parabola (or a beam of parabolae)

Kuca, Orponen, and Sahlsten (2021)
Durcik, K., and Stipčić (2023)

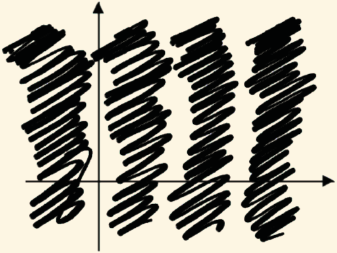




- Uses Bourgain's generalized circular maximal function (1986):

$$(Mf)(x) := \sup_{t\in(0,\infty)} \left|(f * \sigma_t)(x)\right|$$

> ***Similar copies of arbitrary finite configurations in very dense sets***
> Falconer, K., and Yavicoli (2020)



- Uses the method of rotations + Diophantine approximations: quantitative equidistribution of quadratic sequences modulo 1

Configurations of a given area/volume

We will discuss:

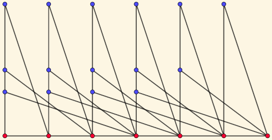
- triangles and simplices
- rectangles and rectangular boxes
- parallelograms and parallelotopes

$m, n \in \mathbb{N}, \; 2 \leqslant m \leqslant n$

Graham (1979)

For all finite colorings of $\mathbb{R}^n$ some color-class contains vertices of a right-angled m-dimensional simplex of unit volume

It is sufficient to color a “large” cube $[0, R]^n$ in r colors



Open problem (essentially Graham, 1979)

Is there a reasonable lower bound (i.e., not of the Ackermann type) on the number $R = R(r)$?

$m \geqslant 2,\ \ n \geqslant m+1$

Theorem (K., 2024)

(a) $R > 1,\ A \subseteq [0,R]^n,\ \delta = \frac{|A|}{R^n} \geqslant \left(\frac{C_m}{\log R}\right)^{1/(9m^2)}$

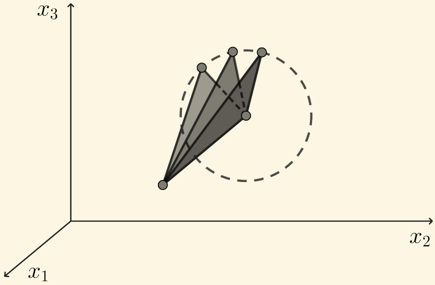
$\implies$ A contains $m+1$ vertices of a right-angled m-dimensional simplex of unit volume

(b) $R \geqslant \exp(C_m r^{9m^2}),\ \ [0,R]^n$ is measurably colored in r colors

$\implies$ there exists a right-angled m-dimensional simplex of unit volume with monochromatic vertices

Assume $n = m + 1$

E.g., $m = 2$, $n = 3$



$\theta = m^{-1} 2^{-m^2-m-1} \delta^{m+1}$, $\quad \lambda > 0$ a certain (aspect ratio) parameter

Configuration-counting form:

$$\mathcal{N}^0_\lambda(A; R) :=$$
$$\int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \mathbb{1}_A(x, y) \Big(\prod_{k=1}^{m-1} \mathbb{1}_A(x + u_k \mathbb{e}_k, y) \Big) \mathbb{1}_A(x, y + v)$$
$$d\sigma_{m!|u_1 \cdots u_{m-1}|^{-1}}(v)\, \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k) \Big)\, du\, R^{-m-1}\, dy\, dx$$

$\sigma =$ the normalized circle measure on $\mathbb{S}^1 \subseteq \mathbb{R}^2$

$\mathcal{N}^0_\lambda(A; R) =$ a certain density of a subcollection of axes-aligned right-simplices inside A

A great idea developed by
Bourgain (1986), ..., Cook, Magyar, and Pramanik (2015)
is a *smoothed counting form* defined for $\varepsilon \in (0,1]$:

$$\mathcal{N}_{\lambda}^{\varepsilon}(A;R) :=$$

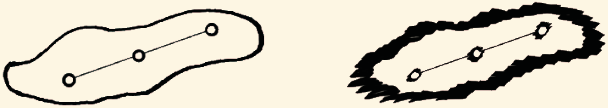
$$\int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \mathbb{1}_A(x,y) \Big(\prod_{k=1}^{m-1} \mathbb{1}_A(x+u_k \mathbb{e}_k, y) \Big) \mathbb{1}_A(x, y+v)$$

$$(\sigma * \mathbb{g}_{\varepsilon})_{m!|u_1\cdots u_{m-1}|^{-1}}(v) \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k) \Big) \mathrm{d}v \mathrm{d}u \, R^{-m-1} \mathrm{d}y \mathrm{d}x$$

We use the normalized Gaussian $\mathbb{g}$ for this purpose to apply the heat equation (Durcik and K., 2020)

Interlude — A general scheme

$$\mathcal{N}^0_\lambda(A;R) = \lim_{\varepsilon \to 0+} \mathcal{N}^\varepsilon_\lambda(A;R)$$



$$\underbrace{\mathcal{N}^0_\lambda(A;R)}_{\text{we want } >0} = \underbrace{\mathcal{N}^1_\lambda(A;R)}_{\text{structured part}} + \Big(\underbrace{\mathcal{N}^\varepsilon_\lambda(A;R) - \mathcal{N}^1_\lambda(A;R)}_{\text{error part}}\Big) + \Big(\underbrace{\mathcal{N}^0_\lambda(A;R) - \mathcal{N}^\varepsilon_\lambda(A;R)}_{\text{uniform part}}\Big)$$

dominant term $\geqslant c(\delta)$

small for all small ε uniformly in λ

small for some λ

For the structured part $\mathcal{N}_\lambda^1$ we need a lower bound

$$\mathcal{N}_\lambda^1 \geqslant c(\delta)$$

that is uniform in λ, but this should be a simpler/smoother problem

For the uniform part $\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon$ we want

$$\lim_{\varepsilon \to 0} \left|\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon\right| = 0$$

uniformly in λ; this usually leads to some **oscillatory integrals**

For the error part $\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1$ one tries to prove

$$\sum_{j=1}^{J} \left|\mathcal{N}_{\lambda_j}^\varepsilon - \mathcal{N}_{\lambda_j}^1\right| \leqslant C(\varepsilon) o(J)$$

for lacunary scales $\lambda_1 < \cdots < \lambda_J$; this usually leads to some **multilinear singular integrals**

Lemma

$R, \lambda \in (0, \infty),\ \ R^{-1/(m-1)} \leqslant \lambda \leqslant R,\ \ A \subseteq [0, R]^{m+1},\ \ \delta = |A|/R^{m+1}$

$$\implies \mathcal{N}^1_\lambda(A; R) \gtrsim \delta^{(m+1)(2m-1)}$$

Estimated by cutting $[0, R]^{m+1}$ into pieces of size

$$\lambda \times \cdots \times \lambda \times \lambda^{-m+1} \times \lambda^{-m+1}$$

and using basic enumerative combinatorics

Lemma

$\lambda, R \in (0, \infty), \;\; \varepsilon \in (0, 1], \;\; A \subseteq [0, R]^{m+1}$

$$\implies \left|\mathcal{N}^0_\lambda(A; R) - \mathcal{N}^\varepsilon_\lambda(A; R)\right| \lesssim \varepsilon^{1/2}$$

Estimated using the Fourier decay of σ:

$$|\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1/2}$$

$$f(x,y;u) := \mathbb{1}_A(x,y)\prod_{k=1}^{m-1}\mathbb{1}_A(x+u_k\mathbb{e}_k,y)$$

$$\mathbb{g}(x) := e^{-\pi|x|^2},\ \mathbb{k} := \Delta\mathbb{g}$$

The heat equation:

$$(\sigma*\mathbb{g}_\tau)(v)-(\sigma*\mathbb{g}_\varepsilon)(v) = -\int_\tau^\varepsilon(\sigma*\mathbb{k}_t)(v)\,\frac{\mathrm{d}t}{2\pi t}$$

$$\begin{aligned}&\mathcal{N}_\lambda^\tau(A;R)-\mathcal{N}_\lambda^\varepsilon(A;R)\\ &= -\int_\tau^\varepsilon\int_{\mathbb{R}^{m-1}}\int_{\mathbb{R}^{m-1}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y;u)\mathbb{1}_A(x,y+v)R^{-m-1}\lambda^{-m+1}\\ &\quad (\sigma*\mathbb{k}_t)_{m!|u_1\cdots u_{m-1}|^{-1}}(v)\left(\prod_{k=1}^{m-1}\mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k)\right)\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}x\,\frac{\mathrm{d}t}{2\pi t}\end{aligned}$$

Triangles and simplices — Proof: uniform part

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, y; u)\, \mathbb{1}_A(x, y + v)\, (\sigma * \Bbbk_t)_a(v)\, \mathrm{d}v\, \mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}^2} \widehat{\mathbb{1}_A}(x, \xi)\, \overline{\widehat{f}(x, \xi; u)}\, \widehat{\sigma}(a\xi)\, \widehat{\Bbbk}(ta\xi)\, \mathrm{d}\xi \right| \\
&\lesssim t^{1/2} \left(\int_{\mathbb{R}^2} \left| \widehat{\mathbb{1}_A}(x, \xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \left(\int_{\mathbb{R}^2} \left| \widehat{f}(x, \xi; u) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&= t^{1/2} \left(\int_{\mathbb{R}^2} \mathbb{1}_A(x, y)^2\, \mathrm{d}y \right)^{1/2} \left(\int_{\mathbb{R}^2} f(x, y; u)^2\, \mathrm{d}y \right)^{1/2} \leqslant t^{1/2} R^2.
\end{aligned}
$$

$$
\left| \mathcal{N}_\lambda^\tau(A; R) - \mathcal{N}_\lambda^\varepsilon(A; R) \right| \lesssim \int_\tau^\varepsilon t^{-1/2}\, \mathrm{d}t \lesssim \varepsilon^{1/2}
$$

Let $\tau \to 0$ $\square$

Lemma

$R \in (0, \infty)$, $\varepsilon \in (0, 1]$, $A \subseteq [0, R]^{m+1}$

$$\implies \int_0^\infty \left(\mathcal{N}_\lambda^\varepsilon(A; R) - \mathcal{N}_\lambda^1(A; R)\right)^2 \frac{\mathrm{d}\lambda}{\lambda} \lesssim \theta^{-4(m-1)} \left(\log \frac{1}{\varepsilon}\right)^2$$

Estimated using basic Littlewood–Paley theory

Pigeonholing gives an appropriate parameter λ

$$\mathcal{N}_\lambda^\varepsilon(A;R) - \mathcal{N}_\lambda^1(A;R) = -\int_\varepsilon^1 \int_{e^{-1}t\lambda^{-m+1}}^{t\lambda^{-m+1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x,y;u)\mathbb{1}_A(x,y+v)$$
$$R^{-m-1}\lambda^{-m+1}(\sigma * \Bbbk_t)_a(v)\Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k)\Big)\, \mathrm{d}v\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}x\, \frac{\mathrm{d}s}{s}\, \frac{\mathrm{d}t}{2\pi t}$$

A few L–P tricks and Cauchy–Schwarz reduce:

$$\int_{\mathbb{R}} \big(\mathcal{N}_{e^\alpha}^\varepsilon(A;R) - \mathcal{N}_{e^\alpha}^1(A;R)\big)^2\, \mathrm{d}\alpha \lesssim \theta^{-4m+4} R^{-m-1}\Big(\log\frac{1}{\varepsilon}\Big)$$
$$\int_\varepsilon^1 \underbrace{\int_{\mathbb{R}} \int_{te^{-(m-1)\alpha-1}}^{te^{-(m-1)\alpha}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \big|\widehat{\mathbb{1}_A}(x,\xi)\big|^2 e^{-2\pi s^2|\xi|^2}|\xi|^2\, \mathrm{d}\xi\, \mathrm{d}x\, s\, \mathrm{d}s\, \mathrm{d}\alpha}_{\lesssim R^{m+1}}\ \frac{\mathrm{d}t}{t}$$

□

Triangles and simplices — More open problems

Erdős

A measurable set $A \subseteq \mathbb{R}^2$ of infinite area (and even an unbounded set of positive area) necessarily contains the vertices of a triangle of area 1.

Open problem (Erdős, 1983)

Must a measurable set $A \subseteq \mathbb{R}^2$ with infinite area contain the vertices of a right triangle of area 1?

Open problem (Erdős, 1978)

Is it true that there is an absolute constant C so that, if $A \subseteq \mathbb{R}^2$ has area $> C$, then A contains the vertices of a triangle of area 1?

Problem (Erdős and Graham, 1979)

The question is: Is this also true for rectangles?

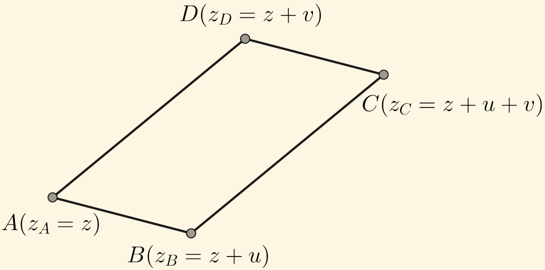
Ad: T. Bloom, *Erdős Problems*
www.erdosproblems.com

Theorem (K., 2024)

There exists a Jordan-measurable coloring of the plane in 25 colors such that no color-class contains the vertices of a rectangle of area 1

Rectangles — Proof of the theorem

Relax a rectangle of area 1 to a parallelogram $\mathcal{P}$ with $|AB| \cdot |AD| = 1$



Define a complex “invariant” quantity:

$$\mathscr{I}(\mathcal{P}) := z_A^2 - z_B^2 + z_C^2 - z_D^2 = 2uv$$

Rectangles — Proof of the theorem

On the one hand, for a parallelogram with $|AB| \cdot |AD| = 1$,

$$|\mathscr{I}(\mathcal{P})| = 2|u||v| = 2$$

For each pair $(j, k) \in \{0, 1, 2, 3, 4\}^2$ define a color-class $\mathscr{C}_{j,k}$ as

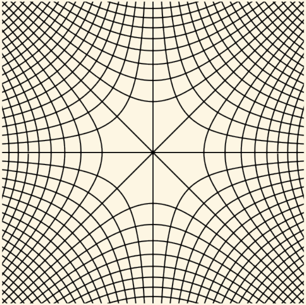
$$\mathscr{C}_{j,k} := \left\{ z \in \mathbb{C} \,:\, z^2 \in \frac{10}{3}\left(\mathbb{Z} + \mathbb{i}\mathbb{Z} + \frac{j + \mathbb{i}k}{5} + \left[0, \frac{1}{5}\right) + \mathbb{i}\left[0, \frac{1}{5}\right)\right)\right\}$$

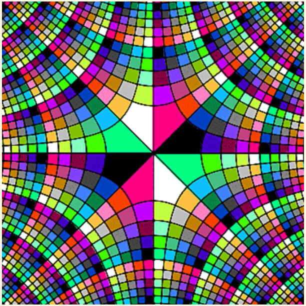
One the other hand, for a monochromatic parallelogram,

$$\mathscr{I}(\mathcal{P}) \in \frac{10}{3}\left(\mathbb{Z} + \mathbb{i}\mathbb{Z} + \left(-\frac{2}{5}, \frac{2}{5}\right) + \mathbb{i}\left(-\frac{2}{5}, \frac{2}{5}\right)\right),$$

which is never $= 2$ in the absolute value $\square$

The coloring $(\mathscr{C}_{j,k})$ of $\mathbb{R}^2$:



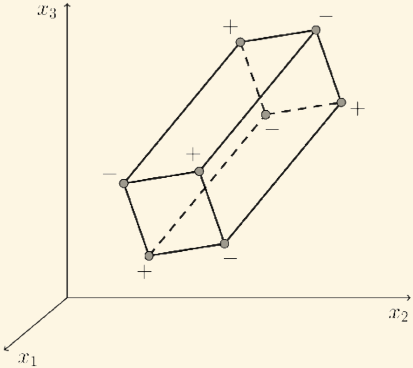


Rectangular boxes

A higher-dimensional generalization

Theorem (K., 2024)

For every $n \in \mathbb{N}$ $\exists$ a finite Jordan-measurable coloring of $\mathbb{R}^n$ s.t., $\forall m \leqslant n$, there is no m-dimensional rectangular box of m-volume equal to 1 with all 2^m vertices colored the same



Proof idea. A real-valued quantity invariant for slightly tilted boxes:

$$\mathscr{I}(\mathcal{R}) := \sum_{x=(x_1,\ldots,x_n) \text{ is a vertex of } \mathcal{R}} (-1)^{m-\text{parity}(x)}\, x_1 \cdots x_m$$

Rectangular boxes

$n \geqslant m+1$ & the coloring is measurable
$\implies$ all sufficiently large volumes are attained

Theorem (K., 2024)

(a) $A \subseteq \mathbb{R}^n$, $\overline{\delta}_n(A) > 0$
$\implies \exists V_0 = V_0(A) > 0 \;\; \forall V \geqslant V_0 \;\; \exists$ m-dimensional rectangular box of m-volume V with all 2^m vertices in A

(b) For every finite measurable coloring of $\mathbb{R}^n$ $\exists$ a color-class $\mathscr{C}$ $\exists V_0 > 0 \; \forall V \geqslant V_0 \; \exists$ an m-dimensional rectangular box of m-volume V with all vertices in $\mathscr{C}$

- Previously known for
  - $n \geqslant 5m$ (Durcik and K., 2018)
  - $n \geqslant 2m$ (Lyall and Magyar, 2019)
- Still open for $n = m$

Rectangular boxes — Proof of the theorem

Configuration-counting form: $\theta = m^{-1}2^{-2^m n}\delta^{2^m}, \quad \lambda > 0$

$$\mathcal{N}_\lambda^0(A;R) := \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} \Big(\prod_{(r_1,\dots,r_m)\in\{0,1\}^m} \mathbb{1}_A(x_1 + r_1u_1, \dots, x_{m-1} + r_{m-1}u_{m-1}, y + r_m v)\Big) \, d\sigma_{\lambda^m |u_1\cdots u_{m-1}|^{-1}}(v) \, \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k)\Big) \, du \, R^{-n} \, dy \, dx$$

Smoothed configuration-counting form: $\varepsilon \in (0,1]$

$$\mathcal{N}_\lambda^\varepsilon(A;R) := \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} \Big(\prod_{(r_1,\dots,r_m)\in\{0,1\}^m} \mathbb{1}_A(x_1 + r_1u_1, \dots, x_{m-1} + r_{m-1}u_{m-1}, y + r_m v)\Big) \, (\sigma * \mathrm{g}_\varepsilon)_{\lambda^m |u_1\cdots u_{m-1}|^{-1}}(v) \, dv \, \lambda^{-m+1} \Big(\prod_{k=1}^{m-1} \mathbb{1}_{[-\lambda,-\theta\lambda]\cup[\theta\lambda,\lambda]}(u_k)\Big) \, du \, R^{-n} \, dy \, dx$$

Lemma

$0 < \lambda \leqslant R,\;\; A \subseteq [0,R]^n,\;\; \delta = |A|/R^n$

$$\implies \mathcal{N}^1_\lambda(A;R) \gtrsim_\delta 1$$

Estimated by cutting $[0,R]^n$ into cubes of size

$$\lambda \times \cdots \times \lambda$$

and using inequalities for the so-called box-Gowers norms

Rectangular boxes — Proof: uniform part

Lemma

$\lambda, R \in (0, \infty), \;\; \varepsilon \in (0, 1], \;\; A \subseteq [0, R]^n$

$$\implies \left|\mathcal{N}^0_\lambda(A; R) - \mathcal{N}^\varepsilon_\lambda(A; R)\right| \lesssim \varepsilon^{1/2}$$

Very similar to the lemma for simplices

Rectangular boxes — Proof: error part

Lemma

$R \in (0, \infty)$, $\varepsilon \in (0, 1]$, $(\lambda_j)_{1 \leqslant j \leqslant J}$ satisfying $\lambda_{j+1} \geqslant 2\lambda_j$, $A \subseteq [0, R]^n$

$$\Longrightarrow \sum_{j=1}^{J} \left| \mathcal{N}^{\varepsilon}_{\lambda_j}(A; R) - \mathcal{N}^{1}_{\lambda_j}(A; R) \right| \lesssim_{\delta, \varepsilon} 1$$

Estimated using superpositions of

- *entangled singular integral forms* — K. (2010, 2011), Durcik (2014, 2015), Durcik, K., Škreb, Thiele (2016),
- recently a.k.a. *singular Brascamp–Lieb estimates* — Durcik, Thiele (2018, 2019), Durcik, Slavíková, Thiele (2021, 2023)

Pigeonholing gives an appropriate parameter λ_j, $1 \leqslant j \leqslant J$

Parallelograms and parallelotopes

Open problem (Erdős and Graham, 1979)

Or perhaps parallelograms?

Here we only give a partial answer

Theorem (K., 2024)

Suppose that we are given lines $\ell_1, \dots, \ell_m \subset \mathbb{R}^2$ and $\varepsilon > 0$

There exists a Jordan-measurable coloring of the plane s.t. there is no parallelogram of area 1 with monochromatic vertices that, additionally, has one side parallel to some line ℓ_i or it has all angles greater than ε

$\Longrightarrow$ Possible counterexamples are almost degenerate parallelograms and infinitely many directions should be considered

$n \geqslant 2, \ \varepsilon > 0$

Theorem (K., 2024)

There exists a Jordan-measurable set $A \subseteq \mathbb{R}^n$ of infinite volume such that every n-dimensional parallelotope with all 2^n vertices in A has volume less than ε

Taking $\varepsilon = 1$ we guarantee that parallelotopes with vertices in A cannot have volume 1

Case $n = 2$ previously claimed by Erdős and Mauldin (1983)

Proof idea. This result is much easier:

$$\left\{(x_1, x_2, \ldots, x_n) \in (0, \infty)^n \ : \ x_1 x_2 \cdots x_n \leqslant \frac{\varepsilon}{n!}\right\}$$

Harmonic analyst's point of view

How are **area** 1 **rectangles** (density results impossible)

different from **area** 1 **right-angled triangles** (density results possible in dimensions $n \geqslant 3$)?

$\varphi, \psi \in \mathcal{S}(\mathbb{R})$, $0 \notin \operatorname{supp}(\widehat{\varphi})$ or $0 \notin \operatorname{supp}(\widehat{\psi})$

$(p_1, p_2, p_3, p_4) \in [1, \infty]^4$, $\sum_{k=1}^{4} \frac{1}{p_k} = 1$, $0 < r < R$

Let $C_{r,R}$ be the best constant in:

$$\Big| \int_r^R \int_{\mathbb{R}^4} f_1(x,y) f_2(x,y') f_3(x',y) f_4(x',y') \varphi_t(x-x') \psi_{1/t}(y-y') \,\mathrm{d}y \,\mathrm{d}y' \,\mathrm{d}x \,\mathrm{d}x' \,\frac{\mathrm{d}t}{t} \Big| \leqslant C_{r,R} \prod_{k=1}^{4} \|f_k\|_{\mathrm{L}^{p_k}(\mathbb{R}^2)}$$

We claim: $C_{r,R} \sim \log(R/r)$ as $R/r \to \infty$

(Not any better than the trivial estimate obtained from Hölder)

Harmonic analyst's point of view — rectangles

$M > 0, \quad \mathrm{g}(x) := e^{-\pi x^2}$

$$f_1(x,y) := e^{2\pi \mathrm{i} xy}\, \mathrm{g}\Big(\frac{x}{M}\Big)\mathrm{g}\Big(\frac{y}{M}\Big)$$

$f_2 := \overline{f_1}, \quad f_3 := \overline{f_1}, \quad f_4 := f_1$

$$RHS \sim C_{r,R} M^2$$

$$\lim_{M\to\infty} \frac{1}{M^2} LHS = \lim_{M\to\infty} \frac{1}{M^2} \int_r^R \int_{\mathbb{R}^4} f_1(x,y) f_2(x,y') f_3(x',y) f_4(x',y')$$

$$\varphi_t(x-x')\psi_{1/t}(y-y')\,\mathrm{d}(x,x',y,y')\,\frac{\mathrm{d}t}{t}$$

[substitute $u = x - x', v = y - y'$]

$$= \lim_{M\to\infty} \frac{1}{4} \int_r^R \int_{\mathbb{R}^2} e^{2\pi \mathrm{i} uv} \varphi_t(u)\psi_{1/t}(v)\, \mathrm{g}\Big(\frac{u}{M}\Big)\mathrm{g}\Big(\frac{v}{M}\Big)\,\mathrm{d}(u,v)\,\frac{\mathrm{d}t}{t}$$

$$= \frac{1}{4}\Big(\log\frac{R}{r}\Big) \int_{\mathbb{R}} \widehat{\varphi}(-v)\psi(v)\,\mathrm{d}v$$

Harmonic analyst's point of view — triangles

Let $C'_{r,R}$ be the best constant in: $(p_4 = \infty)$

$$\Big| \int_r^R \int_{\mathbb{R}^4} f_1(x,y) f_2(x,y') f_3(x',y)\, \varphi_t(x-x')\, \psi_{1/t}(y-y')\, \mathrm{d}y\, \mathrm{d}y'\, \mathrm{d}x\, \mathrm{d}x'\, \frac{\mathrm{d}t}{t} \Big|$$

$$\leqslant C'_{r,R} \prod_{k=1}^{3} \|f_k\|_{\mathrm{L}^{p_k}(\mathbb{R}^2)}$$

A single Cauchy–Schwarz + a square function estimate:

$$C'_{r,R} = O\big((\log(R/r))^{1/2}\big)$$

It could be interesting to study boundedness/cancellation of "volume-preserving" or "time reversed" multilinear singular integral operators

¡Gracias!

Thank you for your attention!