Variants of the Christ–Kiselev lemma and an application to the maximal Fourier restriction

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Block-diagonal truncation

$(\mathbb{X}, \mathcal{X}, \mu)$, $(\mathbb{Y}, \mathcal{Y}, \nu)$ σ-finite measure spaces, $N \in \mathbb{N}$
$(A_n)_{n=1}^N$ an $\mathcal{X}$-measurable partition of $\mathbb{X}$
$(B_n)_{n=1}^N$ a $\mathcal{Y}$-measurable partition of $\mathbb{Y}$
$1 \leqslant p \leqslant q < \infty$

Suppose that $T\colon \mathrm{L}^p(\mathbb{Y}, \mathcal{Y}, \nu) \to \mathrm{L}^q(\mathbb{X}, \mathcal{X}, \mu)$ is a bounded linear operator

A warm-up result

If

$$\widetilde{T} f := \sum_{n=1}^{N} \mathbb{1}_{A_n} T(f \mathbb{1}_{B_n}),$$

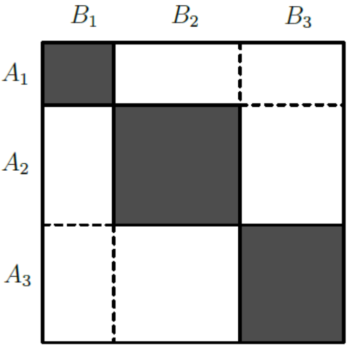
then

$$\|\widetilde{T}\|_{\mathrm{L}^p \to \mathrm{L}^q} \leqslant \|T\|_{\mathrm{L}^p \to \mathrm{L}^q}.$$

Block-diagonal truncation

$$(Tf)(x) := \int_{\mathbb{Y}} K(x,y)f(y)d\nu(y)$$

$$\widetilde{K}(x,y) := \sum_{n=1}^{N} K(x,y)\mathbb{1}_{A_n}(x)\mathbb{1}_{B_n}(y)$$



Block-diagonal truncation

Proof.

Denote $C = \|T\|_{\mathrm{L}^p\to\mathrm{L}^q}$

$$\|\widetilde{T}f\|_{\mathrm{L}^q} = \Big(\sum_{n=1}^{N}\|\mathbb{1}_{A_n}\widetilde{T}f\|_{\mathrm{L}^q}^q\Big)^{1/q} = \Big(\sum_{n=1}^{N}\|\mathbb{1}_{A_n}T(f\mathbb{1}_{B_n})\|_{\mathrm{L}^q}^q\Big)^{1/q}$$

$$\leqslant \Big(\sum_{n=1}^{N}\|T(f\mathbb{1}_{B_n})\|_{\mathrm{L}^q}^q\Big)^{1/q} \leqslant C\Big(\sum_{n=1}^{N}\|f\mathbb{1}_{B_n}\|_{\mathrm{L}^p}^q\Big)^{1/q}$$

$$\leqslant C\Big(\sum_{n=1}^{N}\|f\mathbb{1}_{B_n}\|_{\mathrm{L}^p}^p\Big)^{1/p} = C\|f\|_{\mathrm{L}^p}$$

□

Block-triangular truncation

Now assume $1 \leqslant \boxed{p < q} < \infty$

Suppose that $T\colon \mathrm{L}^p(\mathbb{Y}, \mathcal{Y}, \nu) \to \mathrm{L}^q(\mathbb{X}, \mathcal{X}, \mu)$ is a bounded linear operator

Theorem (*Christ* and *Kiselev*, 2000)

If

$$\widetilde{T}f := \sum_{\substack{m,n \\ 1 \leqslant n \leqslant m \leqslant N}} \mathbb{1}_{A_m} T(f \mathbb{1}_{B_n}),$$

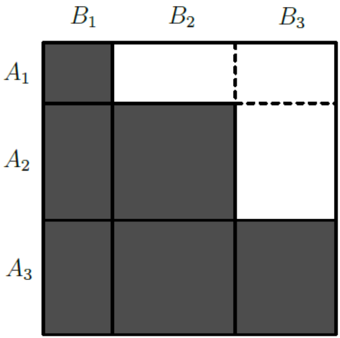
then

$$\|\widetilde{T}\|_{\mathrm{L}^p \to \mathrm{L}^q} \leqslant C_{p,q} \|T\|_{\mathrm{L}^p \to \mathrm{L}^q}.$$

Block-triangular truncation

$$(Tf)(x) := \int_{\mathbb{Y}} K(x,y) f(y) d\nu(y)$$

$$\widetilde{K}(x,y) := \sum_{1 \leqslant n \leqslant m \leqslant N} K(x,y) \mathbb{1}_{A_m}(x) \mathbb{1}_{B_n}(y)$$



Block-triangular truncation

Proof (Tao, 2006).

We are showing

$$\|\widetilde{T}\|_{\mathrm{L}^p\to\mathrm{L}^q} \leqslant \underbrace{\big(1-2^{1/q-1/p}\big)^{-1}}_{C_{p,q}} \|T\|_{\mathrm{L}^p\to\mathrm{L}^q}$$

by the induction on N

- Normalize $\|T\|_{\mathrm{L}^p\to\mathrm{L}^q}=1$, $\|f\|_{\mathrm{L}^p}=1$.
- Choose the unique $k\in\{1,2,\ldots,N\}$ such that

$$\|f\mathbb{1}_{B_1\cup\cdots\cup B_{k-1}}\|_{\mathrm{L}^p}^p \leqslant \tfrac{1}{2} < \|f\mathbb{1}_{B_1\cup\cdots\cup B_k}\|_{\mathrm{L}^p}^p$$

$$\Longrightarrow \|f\mathbb{1}_{B_{k+1}\cup\cdots\cup B_N}\|_{\mathrm{L}^p}^p < \tfrac{1}{2}.$$

Block-triangular truncation

Proof (Tao, 2006).

- The induction hypothesis gives:

$$\big\|\mathbb{1}_{A_1\cup\cdots\cup A_{k-1}}\widetilde{T}(f\mathbb{1}_{B_1\cup\cdots\cup B_{k-1}})\big\|_{\mathrm{L}^q} \leqslant C_{p,q}\|f\mathbb{1}_{B_1\cup\cdots\cup B_{k-1}}\|_{\mathrm{L}^p} \leqslant C_{p,q}2^{-1/p}$$

$$\big\|\mathbb{1}_{A_{k+1}\cup\cdots\cup A_N}\widetilde{T}(f\mathbb{1}_{B_{k+1}\cup\cdots\cup B_N})\big\|_{\mathrm{L}^q} \leqslant C_{p,q}\|f\mathbb{1}_{B_{k+1}\cup\cdots\cup B_N}\|_{\mathrm{L}^p} \leqslant C_{p,q}2^{-1/p}$$

- Also clearly:

$$\big\|\mathbb{1}_{A_k\cup\cdots\cup A_N}\widetilde{T}(f\mathbb{1}_{B_1\cup\cdots\cup B_k})\big\|_{\mathrm{L}^q} = \big\|\mathbb{1}_{A_k\cup\cdots\cup A_N}T(f\mathbb{1}_{B_1\cup\cdots\cup B_k})\big\|_{\mathrm{L}^q} \leqslant 1$$

- Combine these to obtain:

$$\|\widetilde{T}f\|_{\mathrm{L}^q} \leqslant C_{p,q}2^{1/q-1/p} + 1 = C_{p,q}$$

□

A cheap maximal estimate

$(J, \preceq)$ a countable totally ordered set, $(E_j)_{j\in J}$ an increasing collection in $\mathcal{Y}$
$1 \leqslant \boxed{p < q} \leqslant \infty$
Suppose that $T\colon \mathrm{L}^p(\mathbb{Y}, \mathcal{Y}, \nu) \to \mathrm{L}^q(\mathbb{X}, \mathcal{X}, \mu)$ is a bounded linear operator

Theorem (*Christ* and *Kiselev*, 2000)

If

$$T_\star f := \sup_{j\in J} |T(f\mathbb{1}_{E_j})|,$$

then $T_\star$ *is also bounded with*

$$\|T_\star\|_{\mathrm{L}^p\to\mathrm{L}^q} \leqslant C_{p,q}\|T\|_{\mathrm{L}^p\to\mathrm{L}^q}.$$

Idea of proof. Assume $J = \{1, 2, \ldots, N\}$.
Linearize $(T_\star f)(x) = |T(f\mathbb{1}_{E_{j(x)}})(x)|$
and take $A_m := \{x \in \mathbb{X} : j(x) = m\}$, $B_n := E_n \setminus E_{n-1}$.

A cheap variational estimate

$(E_n)_{n\in\mathbb{Z}}$ an increasing collection in $\mathcal{Y}$
$1 \leqslant \boxed{p < q} \leqslant \infty, \ \ p < \varrho < \infty$

Suppose that $T\colon \mathrm{L}^p(\mathbb{Y},\mathcal{Y},\nu) \to \mathrm{L}^q(\mathbb{X},\mathcal{X},\mu)$ is a bounded linear operator

Theorem (*Oberlin*, *Seeger*, *Tao*, *Thiele*, and *Wright*, 2009)

$$\Bigg\| \sup_{\substack{k\in\mathbb{N} \\ n_0,n_1,\ldots,n_k\in\mathbb{Z} \\ n_0<n_1<\cdots<n_k}} \Big(\sum_{j=1}^{k} \big| T(f\mathbb{1}_{E_{n_j}}) - T(f\mathbb{1}_{E_{n_{j-1}}}) \big|^{\varrho} \Big)^{1/\varrho} \Bigg\|_{\mathrm{L}^q(\mathbb{X},\mathcal{X},\mu)} \leqslant C_{p,q,\varrho} \|T\|_{\mathrm{L}^p\to\mathrm{L}^q} \|f\|_{\mathrm{L}^p(\mathbb{Y},\mathcal{Y},\nu)}$$

Idea of proof. Reduce to $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_N$ and induct on N.

The Fourier transform

$\mathcal{F}\colon \mathrm{L}^1(\mathbb{R}^d) \to \mathrm{C}_0(\mathbb{R}^d), \quad \mathcal{F}\colon f \mapsto \widehat{f}$

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\,\mathrm{d}x$$

$\mathcal{F}$ extends to a unitary operator $\mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d)$

$\mathcal{F}$ extends to a linear contraction $\mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^{p'}(\mathbb{R}^d)$ for $1 < p < 2$,
where $\frac{1}{p} + \frac{1}{p'} = 1$, i.e. $p' = \frac{p}{p-1}$

Restriction of the Fourier transform

$S =$ a (hyper)surface in $\mathbb{R}^d$ (e.g. a paraboloid, a cone, a sphere)

Is it possible to give a meaning to $\widehat{f}|_S$ when $f \in \mathrm{L}^p(\mathbb{R})$? (*Stein*, late 1960s)

- $p = 1 \rightsquigarrow$ YES, because $\widehat{f}$ is continuous
- $p = 2 \rightsquigarrow$ NO, since $\widehat{f}$ is an arbitrary L^2 function
- What can be said for $1 < p < 2$?
  A question depending on S and p

A priori estimate

σ = a measure on S, e.g. (appropriately weighted) surface measure

We want an estimate for the restriction operator

$$\mathcal{R}f := \widehat{f}|_S$$

of the form

$$\|\widehat{f}\|_{\mathrm{L}^q(S,\sigma)} \leqslant C\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

for some $1 \leqslant q \leqslant \infty$ and all functions $f \in \mathcal{S}(\mathbb{R}^d)$

How to "compute" $\widehat{f}|_S$?

$\chi \in \mathcal{S}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \chi = 1$, $\chi_\varepsilon(x) := \varepsilon^{-d}\chi(\varepsilon^{-1}x)$

$$\lim_{\varepsilon\to 0^+} \big(\widehat{f} * \chi_\varepsilon\big)\big|_S \text{ exists in the norm of } \mathrm{L}^q(S,\sigma)$$

Adjoint formulation

Equivalently, we want an estimate for the extension operator

$$(\mathcal{E}g)(x) = \int_S e^{2\pi i x\cdot\xi} g(\xi)\, \mathrm{d}\sigma(\xi)$$

$\mathcal{R}$ and $\mathcal{E}$ are mutually adjoint:

$$\langle \mathcal{R}f, g\rangle_{\mathrm{L}^2(S,\sigma)} = \langle f, \mathcal{E}g\rangle_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\mathcal{R}\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^q(S,\sigma) \quad \Longleftrightarrow \quad \mathcal{E}\colon \mathrm{L}^{q'}(S,\sigma) \to \mathrm{L}^{p'}(\mathbb{R}^d)$$

T^*T method

For $\boxed{q=2}$ one can use the T^*T trick: $\|T^*T\| = \|T\|^2$

$$(\mathcal{E}\mathcal{R}f)(x) = \int_{\mathbb{R}^d} f(y)\Big(\int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\xi}\,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}y$$
$$\Longrightarrow\ \mathcal{E}\mathcal{R}f = f * \check{\sigma}$$

$\mathcal{E}\mathcal{R}\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^{p'}(\mathbb{R}^d)$ (?)

Take the sphere $\mathbb{S}^{d-1}$ with its surface measure σ
$\Longrightarrow\ |\check{\sigma}(x)| \leqslant C(1+|x|)^{-(d-1)/2}$

Young's inequality for convolution (*Fefferman/Stein*, 1970): $p < \frac{4d}{3d+1}$

Applied on dyadic annuli (*Tomas/Stein*, 1975): $p \leqslant \frac{2(d+1)}{d+3}$ $\rightsquigarrow$ optimal
(note $p < q = 2$)

Restriction conjecture in $d = 2$

Essentially solved: $p < \frac{4}{3}$, $q \leqslant \frac{p'}{3}$

- For $S = \mathbb{S}^1 \rightsquigarrow$ *Zygmund*, 1974
- For compact C^2 curves S with curvature $\kappa \geqslant 0$

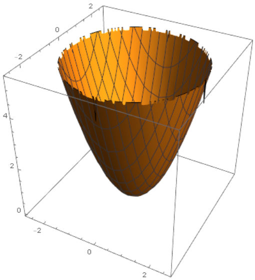
  $\mathrm{d}\sigma =$ arclength measure weighted by $\kappa^{1/3}$

  $\rightsquigarrow$ *Carleson and Sjölin*, 1972; *Sjölin*, 1974

Restriction conjecture in $d \geqslant 3$

Largely open, even for the three classical hypersurfaces

- *paraboloid* $\{\xi = (\eta, -2\pi k|\eta|^2) : \eta \in \mathbb{R}^{d-1}\}$ with $d\sigma(\xi) = d\eta$



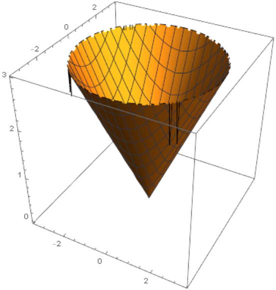
$q = 2 \rightsquigarrow$ Strichartz estimates for the Schrödinger equation

$$\begin{cases} i\partial_t u + k\Delta u = 0 & \text{in } \mathbb{R}^{d-1} \\ u(\cdot, 0) = u_0 \end{cases}$$

Conjecture: $p < \frac{2d}{d+1}$, $q = \frac{(d-1)p'}{d+1}$ (note $p < q$)

Restriction conjecture in $d \geqslant 3$

- *cone* $\{\xi = (\eta, \pm k|\eta|) : \eta \in \mathbb{R}^{d-1}\}$ with $\mathrm{d}\sigma(\xi) = \mathrm{d}\eta/|\eta|$



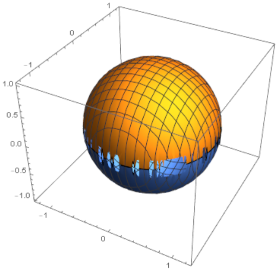
$q = 2 \rightsquigarrow$ Strichartz estimates for the wave equation

$$\begin{cases} \partial_t^2 u - k^2 \Delta u = 0 \quad \text{in } \mathbb{R}^{d-1} \\ u(\cdot, 0) = u_0, \ \partial_t u(\cdot, 0) = u_1 \end{cases}$$

Conjecture: $p < \frac{2(d-1)}{d}$, $q = \frac{(d-2)p'}{d}$ (note $p < q$)

Restriction conjecture in $d \geqslant 3$

- *sphere* $\{\xi \in \mathbb{R}^d : |\xi| = k/2\pi\}$ with surface measure σ



$q = 2 \rightsquigarrow$ The Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^d$$

Conjecture: $p < \frac{2d}{d+1}$, $q \leqslant \frac{(d-1)p'}{d+1}$ (note $p < q$ at the endpoint)

Maximal Fourier restriction

Theorem (*Müller*, *Ricci*, and *Wright*, 2016)

$d = 2$, $S = \mathrm{C}^2$ *curve with* $\kappa \geqslant 0$, $\mathrm{d}\sigma = \kappa^{1/3}\,\mathrm{d}l$,
$\chi \in \mathcal{S}(\mathbb{R}^d)$, $p < \frac{4}{3}$, $q \leqslant \frac{p'}{3}$

$$\Big\| \sup_{t>0} \big|\widehat{f} * \chi_t\big| \Big\|_{\mathrm{L}^q(S,\sigma)} \leqslant C \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

For $f \in \mathrm{L}^p(\mathbb{R}^d)$ the restriction $\widehat{f}\,|_S$ makes sense pointwise, i.e.

$$\lim_{\varepsilon \to 0^+} \big(\widehat{f} * \chi_\varepsilon\big)(\xi) \quad \text{exists for } \sigma\text{-a.e. } \xi \in S$$

(pointwise convergence on $\mathcal{S}(\mathbb{R}^d)$ + maximal estimate)

Higher dimensions

Theorem (*Vitturi*, 2017)

$d \geqslant 3$, $S = \mathbb{S}^{d-1}$, σ = surface measure,
$\chi \in \mathcal{S}(\mathbb{R}^d)$, $p \leqslant \frac{4}{3}$, $q \leqslant \frac{(d-1)p'}{d+1}$
(strict subset of the Tomas–Stein range, i.e. $q = 2$, when $d \geqslant 4$)

$$\Big\| \sup_{t>0} \big|\widehat{f} * \chi_t\big| \Big\|_{\mathrm{L}^q(\mathbb{S}^{d-1},\sigma)} \leqslant C \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

Idea: inserting the maximal function inside the non-oscillatory restriction estimate

Variational Fourier restriction

Can we do it "quantitatively"? Variational (semi)norms

Theorem (*K.* and *Oliveira e Silva*, 2018)

σ = surface measure on $\mathbb{S}^2 \subset \mathbb{R}^3$, $\chi \in \mathcal{S}(\mathbb{R}^3)$ or $\chi = \mathbb{1}_{\mathrm{B}(0,1)}$, $\varrho > 2$

$$\Big\| \sup_{0<t_0<t_1<\cdots<t_m} \Big(\sum_{j=1}^{m} \big| \widehat{f} * \chi_{t_j} - \widehat{f} * \chi_{t_{j-1}} \big|^{\varrho} \Big)^{1/\varrho} \Big\|_{\mathrm{L}^2(\mathbb{S}^2,\sigma)} \leqslant C \|f\|_{\mathrm{L}^{4/3}(\mathbb{R}^3)}$$

If $f \in \mathrm{L}^{4/3}(\mathbb{R}^3)$, then for σ-a.e. $\xi \in \mathbb{S}^2$:

$$\sup_{0<t_0<t_1<\cdots<t_m} \Big(\sum_{j=1}^{m} \big| (\widehat{f} * \chi_{t_j})(\xi) - (\widehat{f} * \chi_{t_{j-1}})(\xi) \big|^{\varrho} \Big)^{1/\varrho} < \infty$$

$\Longrightarrow$ $\big((\widehat{f} * \chi_{\varepsilon})(\xi) \big)_{\varepsilon>0}$ makes $O(\delta^{-\varrho})$ jumps of size $\geqslant \delta$

Abstract principle

Theorem (*K.*, 2018)

$S \subseteq \mathbb{R}^d$ a measurable set, σ = a measure on S,
μ = a complex measure on $\mathbb{R}^d$, $\mu_t(E) := \mu(t^{-1}E)$, $\widehat{\mu} \in \mathrm{C}^\infty$, $\eta > 0$

$$|\nabla\widehat{\mu}(x)| \leqslant D(1+|x|)^{-1-\eta}$$

Suppose that for some $1 \leqslant p \leqslant 2$, $\boxed{p < q} < \infty$ the a priori F. r. estimate holds. Then we have the maximal F. r. estimate:

$$\Big\| \sup_{t\in(0,\infty)} \big|\widehat{f} * \mu_t\big| \Big\|_{\mathrm{L}^q(S,\sigma)} \leqslant C\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

and for $p < \varrho < \infty$ we have the variational F. r. estimate:

$$\Big\| \sup_{0<t_0<t_1<\cdots<t_m} \Big(\sum_{j=1}^{m} \big|\widehat{f} * \mu_{t_j} - \widehat{f} * \mu_{t_{j-1}}\big|^{\varrho} \Big)^{1/\varrho} \Big\|_{\mathrm{L}^q(S,\sigma)} \leqslant C\|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

Consequences

- Covers the full Tomas–Stein range for the sphere $\mathbb{S}^{d-1}$
- Covers any known range for the paraboloid or the cone
- One can take $\mathrm{d}\mu(x) = \chi(x)\,\mathrm{d}x$, $\chi \in \mathcal{S}(\mathbb{R}^d)$ or $\chi = \mathbb{1}_{\mathrm{B}(0,1)}$
- One can take μ to be the surface measure on $\mathbb{S}^{d-1}$ in dimensions $d \geqslant 4 \rightsquigarrow$ spherical averages of $\widehat{f}$:

$$\frac{1}{\mu(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} \widehat{f}(\xi + \varepsilon\zeta)\,\mathrm{d}\mu(\zeta)$$

The proof

The idea of proof

- "Imagine" that $\check{\mu} = \mathbb{1}_E$, where $0 \in E \in \mathbb{R}^d$ is a convex set

  $\implies \check{\mu}(tx) = \mathbb{1}_{E_t}(x)$, where $E_t = t^{-1}E$ for $t \in \mathbb{Q}^+$

  $t < t' \implies E_t \supseteq E_{t'}$
- $\widehat{f} * \mu_t = (f(x)\check{\mu}(tx))\widehat{\ }$

  $\implies \sup_{t\in\mathbb{Q}^+} |\widehat{f} * \mu_t| = \mathcal{F}_* f$
- maximal and variational Christ–Kiselev lemmae apply
- $\mathcal{F}_*$ was already known to be bounded (in 1D) by the Menshov–Paley–Zygmund theorem

The proof

The actual proof (handling overlaps)

- Begin by proving the variational estimate
- Represent $\check{\mu}$ as a superposition of "nice" cutoffs
- Split into long variations (over $\{2^k : k \in \mathbb{Z}\}$) and short variations (over $[2^k, 2^{k+1}]$) following the approach of *Jones*, *Seeger*, and *Wright*, 2008
- For long variations use the variational Christ–Kiselev lemma by *Oberlin*, *Seeger*, *Tao*, *Thiele*, and *Wright*, 2009
- Short variations are trivial by an off-diagonal square function estimate

Subsequent research

Theorem (*Ramos*, 2019)

μ = a measure on $\mathbb{R}^2$, $p < \frac{4}{3}$, $q \leqslant \frac{p'}{3}$

Maximal F. r. for $\mathbb{S}^1$ holds as soon as $\mathrm{M}_\mu g := \sup_{t>0} |g * \mu_t|$ is bounded on $\mathrm{L}^r(\mathbb{R}^2)$ for $r > 2$.

μ = a measure on $\mathbb{R}^3$, $p \leqslant \frac{4}{3}$, $q \leqslant 2$

Maximal F. r. for $\mathbb{S}^2$ holds as soon as $\mathrm{M}_\mu g := \sup_{t>0} |g * \mu_t|$ is bounded on $\mathrm{L}^2(\mathbb{R}^3)$.

$\Longleftarrow$ spherical averages in $d = 2, 3$ (*Bourgain*, 1986; *Stein*, 1976)

Thank you for your attention!