Density theorems for anisotropic point configurations

and related joint work with P. Durcik, K. Falconer, L. Rimanić, and A. Yavicoli

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Density theorems

EUCLIDEAN DENSITY THEOREMS

There exist patterns in large but otherwise arbitrary structures.

The main idea behind Ramsey theory ($\subseteq$ combinatorics), but also widespread in other areas of mathematics.

Euclidean density theorems belong to:

- geometric measure theory (28A12, etc.),
- Ramsey theory (05D10),
- arithmetic combinatorics (11B25, 11B30, etc.).

Harmonic analysis seems to be the most powerful tool for attacking this type of problems.

Eucl. density theorems study “large” measurable sets

When is a measurable set A considered *large*?

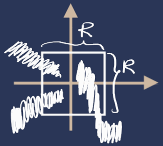
- For $A \subseteq [0,1]^d$ this means

$$|A| > 0$$

(the Lebesgue measure).

- For $A \subseteq \mathbb{R}^d$ this means

$$\overline{\delta}(A) := \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|A \cap (x+[0,R]^d)|}{R^d} > 0$$



(the upper Banach density).

Classical results

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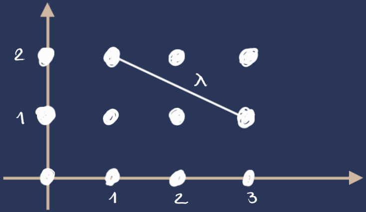
A question by Székely (1982):

Does for every measurable set $A \subseteq \mathbb{R}^2$ satisfying $\overline{\delta}(A) > 0$ there exist a number $\lambda_0 = \lambda_0(A)$ such that for each $\lambda \in [\lambda_0, \infty)$ there exist points $x, x' \in A$ satisfying $|x - x'| = \lambda$?



CLASSICAL RESULTS

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Yes.

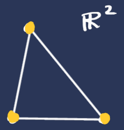
Furstenberg, Katznelson, and Weiss (1980s),
Falconer and Marstrand (1986),
Bourgain (1986).

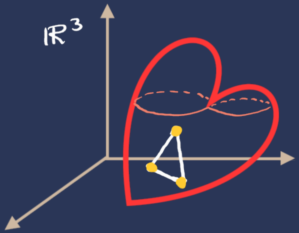
SIMPLICES

$\Delta =$ the set of vertices of a non-degenerate n-dimensional simplex

Bourgain (1986):

For every measurable set $A \subseteq \mathbb{R}^{n+1}$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \Delta)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of $\lambda\Delta$.





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Alternative proofs by Lyall and Magyar (2016, 2018, 2019), K. (2021).

Open question (Bourgain?):
When $n \geq 2$, does the same hold for subsets of $\mathbb{R}^n$?

DEGENERATE TRIANGLE COUNTEREXAMPLE

Bourgain (1986) also gave a counterexample for 3-*term arithmetic progressions*, i.e., isometric copies of dilates of $\{0, 1, 2\}$.

$$A := \{x \in \mathbb{R}^d : (\exists m \in \mathbb{Z})(m - \varepsilon < |x|^2 < m + \varepsilon)\}$$

- Take some $0 < \varepsilon < 1/8$. We have $\overline{\delta}(A) = 2\varepsilon > 0$.
- The parallelogram law:

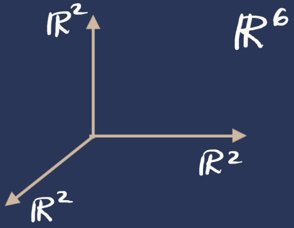
$$|x|^2 - 2|x + y|^2 + |x + 2y|^2 = |y|^2.$$

- $x, x + y, x + 2y \in A$
  dist$(|x|^2, \mathbb{Z})$, dist$(|x + y|^2, \mathbb{Z})$, dist$(|x + 2y|^2, \mathbb{Z}) < \varepsilon$
  $\implies$ dist$(2|y|^2, \mathbb{Z}) < 4\varepsilon < 1/2$
  $\implies$ not all large numbers are attained by $|y|$

BOXES (PRODUCT-TYPE CONFIGURATIONS)

$\square =$ the set of vertices of an n-dimensional rectangular box

Lyall and Magyar (2016, 2019):
For every measurable set $A \subseteq \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = (\mathbb{R}^2)^n$ satisfying $\overline{\delta}(A) > 0$ there is a number $\lambda_0 = \lambda_0(A, \square)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of $\lambda\square$ with sides parallel to the distinguished 2-dimensional coordinate planes.



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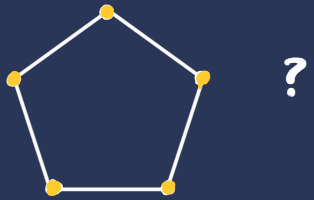
Alternative proofs by Durcik and K. (2018: in $(\mathbb{R}^5)^n$, 2020).

Generalizations

Open question (Graham? Furstenberg?):
Which point configurations P have the following property: for some (sufficiently large) dimension d and every measurable $A \subseteq \mathbb{R}^d$ with $\overline{\delta}(A) > 0$ there exists $\lambda_0 = \lambda_0(P, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ the set A contains an isometric copy of λP?



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The most general positive result is due to Lyall and Magyar (2019):
This **holds** for products of vertex-sets of nondegenerate simplices $\Delta_1 \times \cdots \times \Delta_m$.

The most general negative result is due to Graham (1993):
This **fails** for configurations that cannot be inscribed in a sphere.

Compact formulations — Back to simplices

$\Delta =$ the set of vertices of a non-degenerate n-dimensional simplex

Bourgain (1986):

Take $\delta \in (0, 1/2]$, $A \subseteq [0,1]^{n+1}$ measurable, $|A| \geq \delta$.

Then the set of "scales"

$$\{\lambda \in (0,1] : A \text{ contains an isometric copy of } \lambda\Delta\}$$

contains an interval of length at least $\left(\exp(\delta^{-C(\Delta,n)})\right)^{-1}$.

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Such formulation is **qualitatively weaker**, but it is **quantitative**.

One can try to "beat the current record" for dependencies on δ.

General approach

General scheme of the approach

Abstracted from:
Bourgain (1986) and Cook, Magyar, and Pramanik (2017)

$\mathcal{N}_\lambda^0$ = *configuration "counting" form*, identifies the configuration associated with the parameter $\lambda > 0$ (i.e., of "size" λ)

$\mathcal{N}_\lambda^\varepsilon$ = *smoothened counting form*; the picture is blurred up to scale $0 < \varepsilon \leq 1$

The largeness–smoothness multiscale approach:

- $\lambda =$ scale of largeness,
- $\varepsilon =$ scale of smoothness.

General scheme of the approach (continued)

Decompose:

$$\mathcal{N}_\lambda^0 = \mathcal{N}_\lambda^1 + \left(\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1\right) + \left(\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon\right).$$

$\mathcal{N}_\lambda^1 =$ *structured part*,

$\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1 =$ *error part*,

$\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon =$ *uniform part*.

GENERAL SCHEME OF THE APPROACH (CONTINUED)

For the structured part $\mathcal{N}_\lambda^1$ we need a lower bound

$$\mathcal{N}_\lambda^1 \geq c(\delta)$$

that is uniform in λ, but this should be a simpler/smoother problem.

For the uniform part $\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon$ we want

$$\lim_{\varepsilon \to 0} \left| \mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon \right| = 0$$

uniformly in λ; this usually leads to some oscillatory integrals.

For the error part $\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1$ one tries to prove

$$\sum_{j=1}^{J} \left| \mathcal{N}_{\lambda_j}^\varepsilon - \mathcal{N}_{\lambda_j}^1 \right| \leq C(\varepsilon) o(J)$$

for lacunary scales $\lambda_1 < \cdots < \lambda_J$; this usually leads to some multilinear singular integrals.

General scheme of the approach (continued)

We argue by contradiction. Take sufficiently many lacunary scales $\lambda_1 < \cdots < \lambda_J$ such that $\mathcal{N}^0_{\lambda_j} = 0$ for each j.

The structured part

$$\mathcal{N}^1_{\lambda_j} \geq c(\delta)$$

dominates the uniform part

$$\left|\mathcal{N}^0_{\lambda_j} - \mathcal{N}^\varepsilon_{\lambda_j}\right| \ll 1 \quad \text{(for sufficiently small } \varepsilon\text{)}$$

and the error part

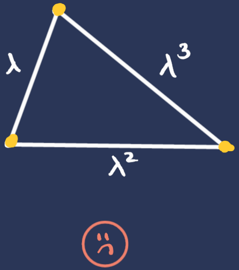
$$\left|\mathcal{N}^\varepsilon_{\lambda_j} - \mathcal{N}^1_{\lambda_j}\right| \ll C(\varepsilon) \quad \text{(for some } j \text{ by pigeonholing)}$$

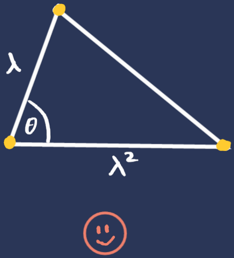
for at least one index j. This contradicts $\mathcal{N}^0_{\lambda_j} = 0$.

Anisotropic configurations

POLYNOMIAL GENERALIZATIONS?

- There are no triangles with sides λ, λ^2, and λ^3 for large λ.
- One can look for triangles with two sides of lengths λ, λ^2 and a fixed angle between them.





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We will be working with *anisotropic power-type dilations*

$$(x_1, \ldots, x_n) \mapsto (\lambda^{a_1} b_1 x_1, \ldots, \lambda^{a_n} b_n x_n).$$

Here $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n > 0$ are fixed parameters.

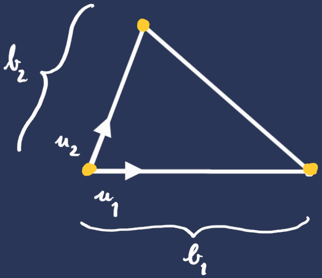
$a_1 = \cdots = a_n = 1$ is the (classical) “linear” case.

Open question:
Which families of dilations are also good?

ANISOTROPIC DILATES OF SIMPLICES

We are given linearly independent unit vectors

$$u_1, u_2, \ldots, u_n \in \mathbb{R}^n.$$



$\Delta = \{0, b_1 u_1, b_2 u_2\}$

ANISOTROPIC DILATES OF SIMPLICES

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K. (2021):

For every measurable set $A \subseteq \mathbb{R}^{n+1}$ satisfying $\overline{\delta}(A) > 0$ there is a positive number $\lambda_0 = \lambda_0(A, a_1, \ldots, a_n, b_1, \ldots, b_n, u_1, \ldots, u_n)$ such that for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of

$$\{\mathbf{0}, \lambda^{a_1} b_1 u_1, \lambda^{a_2} b_2 u_2, \ldots, \lambda^{a_n} b_n u_n\}.$$

Anisotropic dilates of boxes

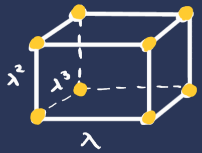
K. (2021):

For every measurable set $A \subseteq (\mathbb{R}^2)^n$ satisfying $\overline{\delta}(A) > 0$ there is a positive number $\lambda_0 = \lambda_0(A, a_1, \ldots, a_n, b_1, \ldots, b_n)$ such that for each $\lambda \in [\lambda_0, \infty)$ one can find $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}^2$ satisfying

$$\{(x_1 + r_1 y_1, x_2 + r_2 y_2, \ldots, x_n + r_n y_n) : (r_1, \ldots, r_n) \in \{0,1\}^n\} \subseteq A$$

and

$$|y_k| = \lambda^{a_k} b_k \quad \text{for } k = 1, 2, \ldots, n.$$



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$$\big\{(x_1 + r_1 y_1, x_2 + r_2 y_2, \dots, x_n + r_n y_n) : (r_1, \dots, r_n) \in \{0,1\}^n\big\} \subseteq A$$

and

$$|y_k| = \lambda^{a_k} b_k \quad \text{for } k = 1, 2, \dots, n.$$

In other words, for each $\lambda \in [\lambda_0, \infty)$ the set A contains an isometric copy of

$$\{0, \lambda^{a_1} b_1\} \times \{0, \lambda^{a_2} b_2\} \times \cdots \times \{0, \lambda^{a_n} b_n\} \subset \mathbb{R}^n$$

with sides parallel to the 2-dimensional coordinate planes.

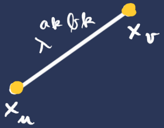
Anisotropic dilates of trees

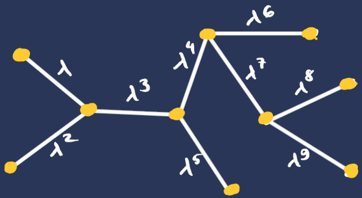
$\mathcal{T} = (V, E)$ a finite tree

K. (2021):

For every measurable set $A \subseteq \mathbb{R}^2$ satisfying $\overline{\delta}(A) > 0$ there is a positive number $\lambda_0 = \lambda_0(A, \mathcal{T}, a_1, \ldots, a_n, b_1, \ldots, b_n)$ such that for each $\lambda \in [\lambda_0, \infty)$ one can find a set of points $\{x_v : v \in V\} \subseteq A$ satisfying

$$|x_u - x_v| = \lambda^{a_k} b_k \quad \text{for each edge } k \in E \text{ joining vertices } u, v \in V.$$





ANISOTROPIC DILATES OF TREES (CONTINUED)

In other words, for each $\lambda \in [\lambda_0, \infty)$ the set A contains an embedding of the distance tree combinatorially isomorphic to $\mathcal{T}$ and having the numbers $\ell(k) = \lambda^{a_k} b_k$ as lengths of its edges.

This is not a **rigid** point configuration.

The corresponding isotropic result is due to Lyall and Magyar (2018) and it generalizes to nondegenerate distance graphs.

BACK TO THE GENERAL SCHEME

Durcik and K. (2020): $\mathcal{N}_\lambda^\varepsilon$ could be obtained by "heating up" $\mathcal{N}_\lambda^0$.

$$\mathrm{g} = \text{standard Gaussian}, \quad \mathbb{k} = \Delta \mathrm{g}$$

The present topic mainly benefits from the fact that the *heat equation*

$$\frac{\partial}{\partial t}\big(\mathrm{g}_t(x)\big) = \frac{1}{2\pi t}\mathbb{k}_t(x)$$

is unaffected by a power-type change of the time variable

$$\frac{\partial}{\partial t}\big(\mathrm{g}_{t^a b}(x)\big) = \frac{a}{2\pi t}\mathbb{k}_{t^a b}(x).$$

ANISOTROPIC SIMPLICES

For simplicity consider right-angled simplices, i.e., $u_k = \mathbb{e}_k$.

Pattern counting form:

$$\mathcal{N}_\lambda^0(f) := \int_{\mathbb{R}^{n+1}} \int_{\mathrm{SO}(n+1,\mathbb{R})} f(x) \Big(\prod_{k=1}^{n} f(x + \lambda^{a_k} b_k U \mathbb{e}_k) \Big) \,\mathrm{d}\mu(U)\,\mathrm{d}x.$$

Smoothened counting form:

$$\mathcal{N}_\lambda^\varepsilon(f) := \int_{\mathbb{R}^{n+1}} \int_{\mathrm{SO}(n+1,\mathbb{R})} f(x) \Big(\prod_{k=1}^{n} (f * \mathrm{g}_{(\varepsilon\lambda)^{a_k} b_k})(x + \lambda^{a_k} b_k U \mathbb{e}_k) \Big) \,\mathrm{d}\mu(U)\,\mathrm{d}x.$$

Anisotropic simplices (continued)

It is sufficient to show:

$$\mathcal{N}^1_\lambda(\mathbb{1}_B) \gtrsim \delta^{n+1} R^{n+1},$$

$$\sum_{j=1}^{J} \left|\mathcal{N}^\varepsilon_{\lambda_j}(\mathbb{1}_B) - \mathcal{N}^1_{\lambda_j}(\mathbb{1}_B)\right| \lesssim \varepsilon^{-C} J^{1/2} R^{n+1},$$

$$\left|\mathcal{N}^0_\lambda(\mathbb{1}_B) - \mathcal{N}^\varepsilon_\lambda(\mathbb{1}_B)\right| \lesssim \varepsilon^{c} R^{n+1}.$$

$\lambda > 0,\ J \in \mathbb{N},\ 0 < \lambda_1 < \cdots < \lambda_J$ satisfy $\lambda_{j+1} \geq 2\lambda_j$,
$R > 0$ is sufficiently large, $0 < \delta \leq 1$,
$B \subseteq [0,R]^{n+1}$ has measure $|B| \geq \delta R^{n+1}$.

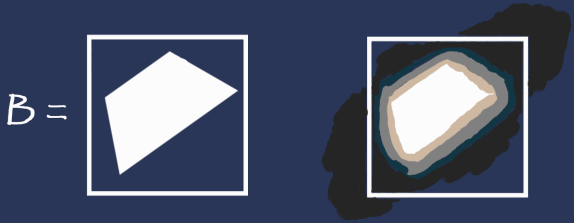
(We take $B := (A - x) \cap [0,R]^{n+1}$ for appropriate x, R.)

Anisotropic simplices — structured part

σ^H = the spherical measure inside a subspace H

$$\mathcal{N}_\lambda^\varepsilon(f) = \int_{(\mathbb{R}^{n+1})^{n+1}} f(x) \left(\prod_{k=1}^{n} (f * g_{(\varepsilon\lambda)^{a_k} b_k})(x + y_k) \right) \mathrm{d}\sigma_{\lambda^{a_n} b_n}^{\{y_1,\dots,y_{n-1}\}^\perp}(y_n)$$

$$\mathrm{d}\sigma_{\lambda^{a_{n-1}} b_{n-1}}^{\{y_1,\dots,y_{n-2}\}^\perp}(y_{n-1}) \cdots \mathrm{d}\sigma_{\lambda^{a_2} b_2}^{\{y_1\}^\perp}(y_2)\, \mathrm{d}\sigma_{\lambda^{a_1} b_1}^{\mathbb{R}^{n+1}}(y_1)\, \mathrm{d}x$$



ANISOTROPIC SIMPLICES — STRUCTURED PART

σ^H = the spherical measure inside a subspace H

$$\mathcal{N}_\lambda^\varepsilon(f) = \int_{(\mathbb{R}^{n+1})^{n+1}} f(x) \Big(\prod_{k=1}^{n} (f * \mathrm{g}_{(\varepsilon\lambda)^{a_k} b_k})(x+y_k) \Big) \, \mathrm{d}\sigma_{\lambda^{a_n} b_n}^{\{y_1,\ldots,y_{n-1}\}^\perp}(y_n)$$
$$\mathrm{d}\sigma_{\lambda^{a_{n-1}} b_{n-1}}^{\{y_1,\ldots,y_{n-2}\}^\perp}(y_{n-1}) \cdots \mathrm{d}\sigma_{\lambda^{a_2} b_2}^{\{y_1\}^\perp}(y_2) \, \mathrm{d}\sigma_{\lambda^{a_1} b_1}^{\mathbb{R}^{n+1}}(y_1) \, \mathrm{d}x$$

$$\sigma^H * \mathrm{g} \geq \Big(\min_{\mathrm{B}(0,2)} \mathrm{g} \Big) \mathbb{1}_{\mathrm{B}(0,1)} \gtrsim \varphi := |\mathrm{B}(0,1)|^{-1} \mathbb{1}_{\mathrm{B}(0,1)}$$

Bourgain's lemma (1988):

$$\fint_{[0,R]^d} f(x) \Big(\prod_{k=1}^{n} (f * \varphi_{t_k})(x) \Big) \, \mathrm{d}x \gtrsim \Big(\fint_{[0,R]^d} f(x) \, \mathrm{d}x \Big)^{n+1}$$

ANISOTROPIC SIMPLICES — ERROR PART

$$\mathcal{N}_\lambda^{\alpha}(f) - \mathcal{N}_\lambda^{\beta}(f) = \sum_{m=1}^{n} \mathcal{L}_\lambda^{\alpha,\beta,m}(f)$$

$$\mathcal{L}_\lambda^{\alpha,\beta,m}(f) := -\frac{a_m}{2\pi} \int_{\alpha}^{\beta} \int_{\mathbb{R}^{n+1}} \int_{\mathrm{SO}(n+1,\mathbb{R})} f(x)\, (f * \mathbb{k}_{(t\lambda)^{a_m} b_m})(x + \lambda^{a_m} b_m U \mathbb{e}_m) \times \Big(\prod_{\substack{1\le k\le n \\ k\neq m}} (f * \mathbb{g}_{(t\lambda)^{a_k} b_k})(x + \lambda^{a_k} b_k U \mathbb{e}_k) \Big)\, \mathrm{d}\mu(U)\, \mathrm{d}x\, \frac{\mathrm{d}t}{t}$$

These look like certain paraproducts.

Anisotropic simplices — error part (continued)

From $\sum_{j=1}^{J} \left|\mathcal{N}_{\lambda_j}^{\varepsilon}(\mathbb{1}_B) - \mathcal{N}_{\lambda_j}^{1}(\mathbb{1}_B)\right|$ we are lead to study

$$\Lambda_K(f_0,\ldots,f_n) := \int_{(\mathbb{R}^d)^{n+1}} K(x_1 - x_0,\ldots,x_n - x_0) \Big(\prod_{k=0}^{n} f_k(x_k)\,\mathrm{d}x_k\Big).$$

Multilinear C–Z operators: Coifman and Meyer (1970s), Grafakos and Torres (2002).

Here K is a C–Z kernel, but with respect to the quasinorm associated with our anisotropic dilation structure.

Anisotropic simplices — uniform part

$$\left|\mathcal{L}_{\lambda}^{0,\varepsilon,n}(f)\right| \lesssim \|f\|_{\mathrm{L}^2(\mathbb{R}^{n+1})} \int_0^{\varepsilon} \left(\int_{\mathbb{R}^{n+1}} \left|\widehat{f}(\xi)\right|^2 \left|\widehat{\Bbbk}(t^{a_n}\lambda^{a_n} b_n \xi)\right|^2 \mathcal{I}(\lambda^{a_n} b_n \xi) \mathrm{d}\xi \right)^{1/2} \frac{\mathrm{d}t}{t}$$

$$\mathcal{I}(\zeta) := \int_{(\mathbb{R}^{n+1})^{n-1}} \left|\widehat{\sigma}^{\{y_1,\ldots,y_{n-1}\}^{\perp}}(\zeta)\right|^2 \mathrm{d}\sigma^{\{y_1,\ldots,y_{n-2}\}^{\perp}}(y_{n-1}) \cdots \mathrm{d}\sigma^{\mathbb{R}^{n+1}}(y_1)$$

$$\left|\widehat{\sigma}^{\{y_1,\ldots,y_{n-1}\}^{\perp}}(\zeta)\right| \lesssim \operatorname{dist}\left(\zeta, \operatorname{span}(\{y_1,\ldots,y_{n-1}\})\right)^{-1/2}$$

$$\left|\mathcal{L}_{\lambda}^{0,\varepsilon,n}(f)\right| \lesssim \|f\|_{\mathrm{L}^2(\mathbb{R}^{n+1})}^2 \int_0^{\varepsilon} t^c \frac{\mathrm{d}t}{t}$$

Anisotropic boxes

Pattern counting form ($\sigma =$ circle measure in $\mathbb{R}^2$):

$$\mathcal{N}^0_\lambda(f) := \int_{(\mathbb{R}^2)^{2n}} \Big(\prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} f(x_1+r_1y_1,\ldots,x_n+r_ny_n) \Big) \Big(\prod_{k=1}^{n} \mathrm{d}x_k \, \mathrm{d}\sigma_{\lambda^{a_k b_k}}(y_k) \Big)$$

Smoothened counting form:

$$\mathcal{N}^\varepsilon_\lambda(f) := \int_{(\mathbb{R}^2)^{2n}} \Big(\cdots \Big) \Big(\prod_{k=1}^{n} (\sigma * \mathrm{g}_{\varepsilon^{a_k}})_{\lambda^{a_k b_k}}(y_k) \, \mathrm{d}x_k \, \mathrm{d}y_k \Big)$$

$$= \int_{(\mathbb{R}^2)^{2n}} \mathcal{F}(\mathbf{x}) \Big(\prod_{k=1}^{n} (\sigma * \mathrm{g}_{\varepsilon^{a_k}})_{\lambda^{a_k b_k}}(x_k^0 - x_k^1) \Big) \, \mathrm{d}\mathbf{x}$$

$$\mathcal{F}(\mathbf{x}) := \prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} f(x_1^{r_1},\ldots,x_n^{r_n}), \quad \mathrm{d}\mathbf{x} := \mathrm{d}x_1^0 \, \mathrm{d}x_1^1 \, \mathrm{d}x_2^0 \, \mathrm{d}x_2^1 \cdots \mathrm{d}x_n^0 \, \mathrm{d}x_n^1$$

Anisotropic boxes (continued)

It is sufficient to show:

$$\mathcal{N}^1_\lambda(\mathbb{1}_B) \gtrsim \delta^{2^n} R^{2n},$$

$$\sum_{j=1}^{J} \left|\mathcal{N}^\varepsilon_{\lambda_j}(\mathbb{1}_B) - \mathcal{N}^1_{\lambda_j}(\mathbb{1}_B)\right| \lesssim \varepsilon^{-C} R^{2n},$$

$$\left|\mathcal{N}^0_\lambda(\mathbb{1}_B) - \mathcal{N}^\varepsilon_\lambda(\mathbb{1}_B)\right| \lesssim \varepsilon^{c} R^{2n}.$$

$B \subseteq ([0,R]^2)^n$ has measure $|B| \geq \delta R^{2n}$.

ANISOTROPIC BOXES — STRUCTURED PART

Partition "most" of the cube $([0,R]^2)^n$ into rectangular boxes $Q_1 \times \cdots \times Q_n$, where

$$Q_k = [l\lambda^{a_k} b_k, (l+1)\lambda^{a_k} b_k) \times [l'\lambda^{a_k} b_k, (l'+1)\lambda^{a_k} b_k).$$

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We only need the *box–Gowers–Cauchy–Schwarz inequality*:

$$⨍_{Q_1\times Q_1\times\cdots\times Q_n\times Q_n} \mathcal{F}(\mathbf{x})\,\mathrm{d}\mathbf{x} \geq \Big(⨍_{Q_1\times\ldots\times Q_n} f \Big)^{2^n}.$$

Anisotropic boxes — error part

$$\mathcal{N}_\lambda^\alpha(f) - \mathcal{N}_\lambda^\beta(f) = \sum_{m=1}^{n} \mathcal{L}_\lambda^{\alpha,\beta,m}(f)$$

$$\mathcal{L}_\lambda^{\alpha,\beta,m}(f) := -\frac{a_m}{2\pi} \int_\alpha^\beta \int_{(\mathbb{R}^2)^{2n}} \mathcal{F}(\mathbf{x})\,(\sigma * \Bbbk_{t^{a_m}})_{\lambda^{a_m} b_m}(x_m^0 - x_m^1)$$
$$\times \Big(\prod_{\substack{1\le k\le n \\ k\neq m}} (\sigma * \mathbb{g}_{t^{a_k}})_{\lambda^{a_k} b_k}(x_k^0 - x_k^1) \Big)\, \mathrm{d}\mathbf{x}\, \frac{\mathrm{d}t}{t}$$

These look like certain “entangled” paraproducts.

ANISOTROPIC BOXES — ERROR PART (CONTINUED)

From $\sum_{j=1}^{J} \left|\mathcal{N}_{\lambda_j}^{\varepsilon}(\mathbb{1}_B) - \mathcal{N}_{\lambda_j}^{1}(\mathbb{1}_B)\right|$ we are lead to study

$$\Theta_K\big((f_{r_1,\ldots,r_n})_{(r_1,\ldots,r_n)\in\{0,1\}^n}\big) := \int_{(\mathbb{R}^d)^{2n}} \prod_{(r_1,\ldots,r_n)\in\{0,1\}^n} f_{r_1,\ldots,r_n}(x_1+r_1y_1,\ldots,x_n+r_ny_n)\Big) K(y_1,\ldots,y_n)\Big(\prod_{k=1}^{n} \mathrm{d}x_k\,\mathrm{d}y_k\Big)$$

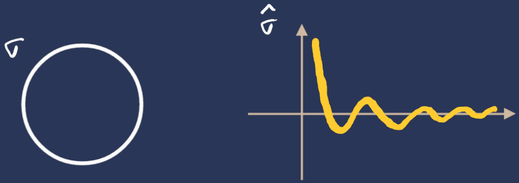
Entangled multilinear singular integral forms with cubical structure: K. (2010), Durcik (2014), Durcik and Thiele (2018: entangled Brascamp–Lieb)

ANISOTROPIC BOXES — UNIFORM PART

Exactly the same as for the simplices

Again one only needs some decay of $\widehat{\sigma}$

(coming from circle curvature)



Other configurations

Rectangular boxes — quantitative strengthening

Fix $b_1, \ldots, b_n > 0$ (box sidelengths).

Durcik and K. (2020), interesting already for isotropic boxes:
For $0 < \delta \leq 1/2$ and measurable $A \subseteq ([0,1]^2)^n$ with $|A| \geq \delta$ there exists an interval $I = I(A, b_1, \ldots, b_n) \subseteq (0,1]$ of length at least

$$\left(\exp(\delta^{-C(n)})\right)^{-1}$$

s. t. for every $\lambda \in I$ one can find $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}^2$ satisfying

$$(x_1 + r_1 y_1, x_2 + r_2 y_2, \ldots, x_n + r_n y_n) \in A \ \text{ for } (r_1, \ldots, r_n) \in \{0,1\}^n;$$

$$|y_i| = \lambda b_i \ \text{ for } i = 1, \ldots, n.$$

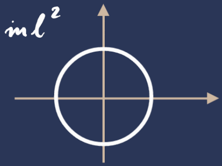
This improves the bound of Lyall and Magyar (2019) of the form
$\left(\exp(\exp(\cdots\exp(C(n)\delta^{-3\cdot 2^n})\cdots))\right)^{-1}$ (a tower of height n).

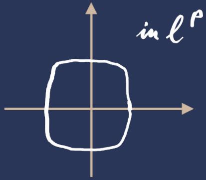
ARITHMETIC PROGRESSIONS

Bourgain's counterexample applies.

Cook, Magyar, and Pramanik (2015) decided to measure gap lengths in the ℓ^p-norm for $p \neq 1, 2, \infty$.

Cook, Magyar, and Pramanik (2015):
If $p \neq 1, 2, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(p, d, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ one can find $x, y \in \mathbb{R}^d$ satisfying $x, x+y, x+2y \in A$ and $\|y\|_{\ell^p} = \lambda$.





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Open question (Cook, Magyar, and Pramanik):
Is it possible to lower the dimensional threshold all the way to $d = 2$ or $d = 3$?

ARITHMETIC PROGRESSIONS (CONTINUED)

Open question (Durcik, K., and Rimanić):
Prove or disprove: if $n \geq 4$, $p \neq 1, 2, \ldots, n-1, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(n, p, d, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ one can find $x, y \in \mathbb{R}^d$ satisfying $x, x+y, \ldots, x+(n-1)y \in A$ and $\|y\|_{\ell^p} = \lambda$.

It is necessary to assume $p \neq 1, 2, \ldots, n-1, \infty$.

In fact, we have the following weaker but quantitative "compact" result.

Arithmetic progressions — compact formulation

Durcik and K. (2020):

Take $n \geq 3$, $p \neq 1, 2, \ldots, n-1, \infty$, $d \geq d_{\text{min}}(n,p)$, $\delta \in (0, 1/2]$, $A \subseteq [0,1]^d$ measurable, $|A| \geq \delta$. Then the set of ℓ^p-norms of the gaps of n-term APs in the set A contains an interval of length at least

$$\begin{cases} \big(\exp(\exp(\delta^{-C(n,p,d)}))\big)^{-1} & \text{when } 3 \leq n \leq 4, \\ \big(\exp(\exp(\exp(\delta^{-C(n,p,d)})))\big)^{-1} & \text{when } n \geq 5. \end{cases}$$

One can take $d_{\text{min}}(n,p) = 2^{n+3}(n+p)$ (certainly not sharp).

These “weird” bounds in terms of δ come from the best known bounds in Szemerédi’s theorem (with an additional “exp”).

Arithmetic progressions — compact formulation

The error part uses bounds for (what is essentially) the multilinear Hilbert transform,

$$\int_{\mathbb{R}} \int_{[-R,-r]\cup[r,R]} \prod_{k=0}^{n-1} f_k(x+ky) \frac{\mathrm{d}y}{y} \, \mathrm{d}x.$$

- When $n \geq 4$, no L^p-bounds uniform in r, R are known.
- Tao (2016) showed a bound of the form $o(J)$, where $J \sim \log(R/r)$ is the "number of scales" involved.
- Reproved and generalized by Zorin-Kranich (2016), still with $o(J)$.
- Durcik, K., and Thiele (2016) showed a bound $O(J^{1-\varepsilon})$.

Other arithmetic configurations

Allowed symmetries play a major role.

Note a difference between:

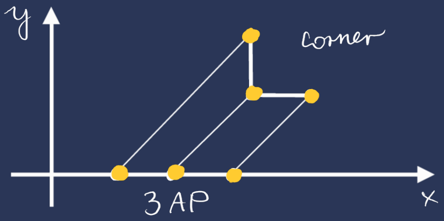
- the so-called *corners*: (x, y), $(x + s, y)$, $(x, y + s)$ (harder),
- isosceles right triangles: (x, y), $(x + s, y)$, $(x, y + t)$ with $\|s\|_{\ell^2} = \|t\|_{\ell^2}$ (easier).

CORNERS

Durcik, K., and Rimanić (2016):

If $p \neq 1, 2, \infty$, d sufficiently large, $A \subseteq \mathbb{R}^d \times \mathbb{R}^d$ measurable, $\overline{\delta}(A) > 0$, then $\exists \lambda_0 = \lambda_0(p, d, A) \in (0, \infty)$ such that for every $\lambda \geq \lambda_0$ one can find $x, y, s \in \mathbb{R}^d$ satisfying $(x, y), (x+s, y), (x, y+s) \in A$ and $\|s\|_{\ell^p} = \lambda$.

Generalizes the result of Cook, Magyar, and Pramanik (2015) via the skew projection $(x, y) \mapsto y - x$.



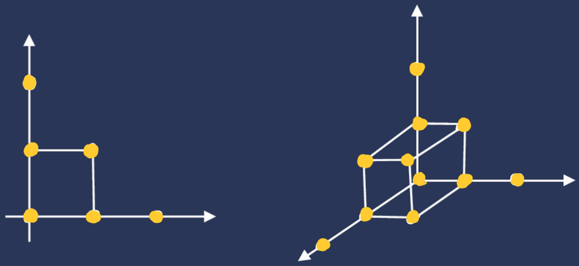
AP-extended boxes

Consider the configuration in $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$ consisting of:

$$(x_1 + k_1 s_1,\, x_2 + k_2 s_2,\, \ldots,\, x_n + k_n s_n), \quad k_1, k_2, \ldots, k_n \in \{0, 1\},$$

$$(x_1 + 2s_1, x_2, \ldots, x_n),\ (x_1, x_2 + 2s_2, \ldots, x_n),\ \ldots,\ (x_1, x_2, \ldots, x_n + 2s_n).$$

Fix $b_1, \ldots, b_n > 0$ and $p \neq 1, 2, \infty$.



AP-EXTENDED BOXES

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Fix $b_1, \ldots, b_n > 0$ and $p \neq 1, 2, \infty$.

Durcik and K. (2018):
There exists a dimensional threshold d_{min} such that for any $d_1, d_2, \ldots,$ $d_n \geq d_{\text{min}}$ and any measurable set A with $\overline{\delta}(A) > 0$ one can find $\lambda_0 > 0$ with the property that for any $\lambda \geq \lambda_0$ the set A contains the above 3AP-extended box with $\|s_i\|_{\ell^p} = \lambda b_i$, $i = 1, 2, \ldots, n$.

Corner-extended boxes

Consider the config. in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n} \times \mathbb{R}^{d_n}$ consisting of:

$$(x_1 + k_1 s_1,\, \ldots,\, x_n + k_n s_n,\, y_1, y_2, \ldots, y_n), \quad k_1, k_2, \ldots, k_n \in \{0, 1\},$$

$$(x_1, x_2, \ldots, x_n,\, y_1 + s_1, y_2, \ldots, y_n), \ldots, (x_1, x_2, \ldots, x_n,\, y_1, y_2, \ldots, y_n + s_n).$$

Fix $b_1, \ldots, b_n > 0$ and $p \neq 1, 2, \infty$.

CORNER-EXTENDED BOXES

Consider the config. in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n} \times \mathbb{R}^{d_n}$ consisting of:

$$(x_1 + k_1 s_1, \ldots, x_n + k_n s_n, y_1, y_2, \ldots, y_n), \quad k_1, k_2, \ldots, k_n \in \{0, 1\},$$

$$(x_1, x_2, \ldots, x_n, y_1 + s_1, y_2, \ldots, y_n), \ldots, (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n + s_n).$$

Fix $b_1, \ldots, b_n > 0$ and $p \neq 1, 2, \infty$.

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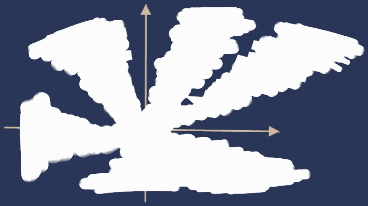
There exists a dimensional threshold d_{min} such that for any $d_1, \ldots, d_n \geq d_{\text{min}}$ and any measurable set A with $\overline{\delta}(A) > 0$ one can find $\lambda_0 > 0$ with the property that for any $\lambda \geq \lambda_0$ the set A contains the above corner-extended box with $\|s_i\|_{\ell^p} = \lambda b_i$, $i = 1, 2, \ldots, n$.

VERY DENSE SETS

Falconer, K., and Yavicoli (2020):
If $d \geq 2$ and $A \subseteq \mathbb{R}^d$ is measurable with $\overline{\delta}(A) > 1 - \frac{1}{n-1}$, then for **every** n-point configuration P there exists $\lambda_0 > 0$ s. t. for every $\lambda \geq \lambda_0$ the set A contains an isometric copy of λP.

The result would be trivial for $\overline{\delta}(A) > 1 - \frac{1}{n}$ and rotations would not even be needed there.



VERY DENSE SETS — LOWER BOUND

What can one say about the lower bound for such density threshold (depending on the # of points n)?

Let us return to arithmetic progressions!

Falconer, K., and Yavicoli (2020):
For all $n, d \geq 2$ there exists a measurable set $A \subseteq \mathbb{R}^d$ of density at least

$$1 - \frac{10 \log n}{n^{1/5}}$$

s.t. there are arbitrarily large values of λ for which A contains no congruent copy of $\lambda\{0, 1, \ldots, n-1\}$.

Conclusion

- *The largeness–smoothness multiscale approach* is quite flexible.
- It also gives superior quantitative bounds.
- Its applicability largely depends on the current state of the art on estimates for multilinear singular and oscillatory integrals.

Thank you for your attention!