Boundedness of the Twisted Paraproduct

Vjekoslav Kovač, UCLA

Mathematical Institute, University of Bonn

February 3, 2011

Ordinary paraproduct

Dyadic version

$$T_{\mathrm{d}}(f,g) := \sum_{k\in\mathbb{Z}} (\mathbb{E}_k f)(\Delta_k g)$$

$\mathbb{E}_k f := \sum_{|I|=2^{-k}} \left(\frac{1}{|I|}\int_I f\right) \mathbf{1}_I, \quad \Delta_k g := \mathbb{E}_{k+1} g - \mathbb{E}_k g$

Continuous version

$$T_{\mathrm{c}}(f,g) := \sum_{k\in\mathbb{Z}} (\mathrm{P}_{\varphi_k} f)(\mathrm{P}_{\psi_k} g)$$

$\mathrm{P}_{\varphi_k} f := f * \varphi_k, \quad \mathrm{P}_{\psi_k} g := g * \psi_k$

φ, ψ Schwartz, $\quad \mathrm{supp}(\hat{\psi}) \subseteq \{\xi \in \mathbb{R} : \frac{1}{2} \le |\xi| \le 2\}$

$\varphi_k(x) := 2^k \varphi(2^k x), \quad \psi_k(x) := 2^k \psi(2^k x)$

Twisted paraproduct

Dyadic version

$$T_{\mathrm{d}}(F,G) := \sum_{k\in\mathbb{Z}} (\mathbb{E}_k^{(1)} F)(\Delta_k^{(2)} G)$$

$\mathbb{E}_k^{(1)}$ martingale averages in the 1st variable
$\Delta_k^{(2)}$ martingale differences in the 2nd variable

Continuous version

$$T_{\mathrm{c}}(F,G) := \sum_{k\in\mathbb{Z}} (\mathrm{P}_{\varphi_k}^{(1)} F)(\mathrm{P}_{\psi_k}^{(2)} G)$$

$\mathrm{P}_{\varphi_k}^{(1)}$, $\mathrm{P}_{\psi_k}^{(2)}$ Littlewood-Paley projections in the 1st and the 2nd variable respectively
$(\mathrm{P}_{\varphi_k}^{(1)} F)(x,y) := \int_{\mathbb{R}} F(x-t,y)\varphi_k(t)dt$
$(\mathrm{P}_{\psi_k}^{(2)} G)(x,y) := \int_{\mathbb{R}} G(x,y-t)\psi_k(t)dt$

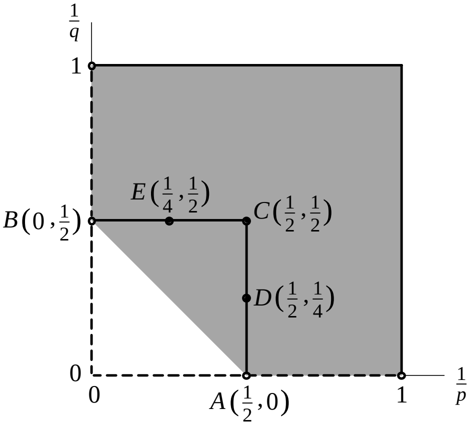
We are interested in

- strong-type estimates
  $\|T(F,G)\|_{\mathrm{L}^{pq/(p+q)}(\mathbb{R}^2)} \lesssim_{p,q} \|F\|_{\mathrm{L}^p(\mathbb{R}^2)} \|G\|_{\mathrm{L}^q(\mathbb{R}^2)}$
- weak-type estimates
  $\alpha \left|\{|T(F,G)| > \alpha\}\right|^{(p+q)/pq} \lesssim_{p,q} \|F\|_{\mathrm{L}^p(\mathbb{R}^2)} \|G\|_{\mathrm{L}^q(\mathbb{R}^2)}$

Theorem (K., 2010)

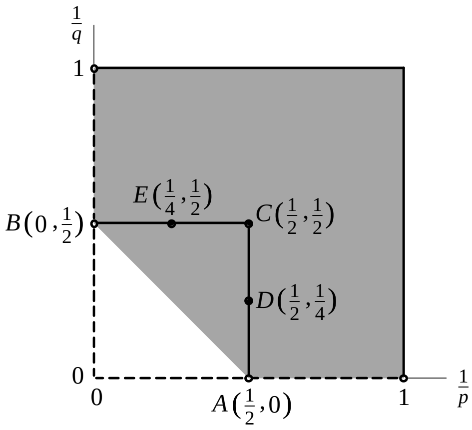
- *Operators T_{d} and T_{c} satisfy the strong bound if* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$.
- *Additionally, operators T_{d} and T_{c} satisfy the weak bound when* $p = 1,\ 1 \le q < \infty$ *or* $q = 1,\ 1 \le p < \infty$.
- *The weak estimate fails for* $p = \infty$ *or* $q = \infty$.

Range of exponents



- the shaded region – the strong estimate
- two solid sides of the square – the weak estimate
- two dashed sides of the square – no estimates
- the white region – unresolved

Proof outline



Dyadic version T_{d}

- $\triangle ABC$ – direct proof, K. (2010)
- the rest of the shaded region – conditional proof, Bernicot (2010)
- dashed segments – counterexamples, K. (2010)
- points D, E – the direct proof simplifies

Continuous version T_{c}

- transition using the Jones-Seeger-Wright square function

Some motivation

1D bilinear Hilbert transform

$$T_{\alpha,\beta}(f,g)(x) := \text{p.v.} \int_{\mathbb{R}} f(x-\alpha t)\, g(x-\beta t)\, \frac{dt}{t}$$

Bounds proved by Lacey and Thiele (1997–1999).

2D bilinear Hilbert transform

$$T_{A,B}(F,G)(x,y) :=$$
$$\text{p.v.} \int_{\mathbb{R}^2} F\big((x,y)-A(s,t)\big)\, G\big((x,y)-B(s,t)\big)\, K(s,t)\, ds\, dt$$

$|\partial^\alpha \hat{K}(\xi,\eta)| \lesssim_\alpha (\xi^2+\eta^2)^{-|\alpha|/2}$

Introduced by Demeter and Thiele (2008). They proved bounds in the range $2 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, and for almost all cases depending on matrices A and B.

Some motivation

Essentially the only unresolved case was (denoted "Case 6"):

$$T(F,G)(x,y) := \mathrm{p.v.} \int_{\mathbb{R}^2} F(x-s,y)\, G(x,y-t)\, K(s,t)\, ds\, dt$$

Using "cone decomposition" it reduces to bilinear multipliers with symbols

$$\hat{K}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \hat{\varphi}(2^{-k}\xi)\, \hat{\psi}(2^{-k}\eta)$$

which are twisted paraproducts.

Bounding the dyadic version

$\varphi_I^{\mathrm{d}} := |I|^{-1/2}\mathbf{1}_I$ the Haar scaling function

$\psi_I^{\mathrm{d}} := |I|^{-1/2}(\mathbf{1}_{I_{\mathrm{left}}} - \mathbf{1}_{I_{\mathrm{right}}})$ the Haar wavelet

$\mathcal{C}$ = dyadic squares in $\mathbb{R}^2$

$$T_{\mathrm{d}}(F,G)(x,y) = \sum_{I\times J\in\mathcal{C}} \int_{\mathbb{R}^2} F(u,y)G(x,v)\varphi_I^{\mathrm{d}}(u)\varphi_I^{\mathrm{d}}(x)\psi_J^{\mathrm{d}}(v)\psi_J^{\mathrm{d}}(y)dudv$$

$$\Lambda_{\mathrm{d}}(F,G,H) = \int_{\mathbb{R}^2} T_{\mathrm{d}}(F,G)(x,y)H(x,y)\,dx\,dy$$

$$= \sum_{I\times J\in\mathcal{C}} \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y)\,\varphi_I^{\mathrm{d}}(u)\varphi_I^{\mathrm{d}}(x)\psi_J^{\mathrm{d}}(v)\psi_J^{\mathrm{d}}(y)\,du\,dx\,dv\,dy$$

A *finite convex tree* is a collection $\mathcal{T}$ of dyadic squares such that

- There exists $Q_{\mathcal{T}} \in \mathcal{T}$, called the *root* of $\mathcal{T}$, satisfying $Q \subseteq Q_{\mathcal{T}}$ for every $Q \in \mathcal{T}$.
- Whenever $Q_1 \subseteq Q_2 \subseteq Q_3$ and $Q_1, Q_3 \in \mathcal{T}$, then also $Q_2 \in \mathcal{T}$.

A *leaf* of $\mathcal{T}$ is a square that is not contained in $\mathcal{T}$, but its parent is.
$\mathcal{L}(\mathcal{T}) =$ the family of leaves of $\mathcal{T}$

Squares in $\mathcal{L}(\mathcal{T})$ partition $Q_{\mathcal{T}}$.

Local forms

For any finite convex tree $\mathcal{T}$ we define:

$$\Lambda_{\mathcal{T}}(F,G,H) := \sum_{I\times J\in\mathcal{T}} \int_{\mathbb{R}^4} F(u,y)G(x,v)H(x,y) \varphi_I^{\mathrm{d}}(u)\varphi_I^{\mathrm{d}}(x)\psi_J^{\mathrm{d}}(v)\psi_J^{\mathrm{d}}(y)\, du\, dx\, dv\, dy$$

$$\Theta_{\mathcal{T}}^{(2)}(F_1,F_2,F_3,F_4) := \sum_{I\times J\in\mathcal{T}} \int_{\mathbb{R}^4} F_1(u,v)F_2(x,v)F_3(u,y)F_4(x,y) \varphi_I^{\mathrm{d}}(u)\varphi_I^{\mathrm{d}}(x)\psi_J^{\mathrm{d}}(v)\psi_J^{\mathrm{d}}(y)\, du\, dv\, dx\, dy$$

$$\Theta_{\mathcal{T}}^{(1)}(F_1,F_2,F_3,F_4) := \sum_{I\times J\in\mathcal{T}} \int_{\mathbb{R}^4} F_1(u,v)F_2(x,v)F_3(u,y)F_4(x,y) \psi_I^{\mathrm{d}}(u)\psi_I^{\mathrm{d}}(x)\Big(\varphi_{J_{\mathrm{left}}}^{\mathrm{d}}(v)\varphi_{J_{\mathrm{left}}}^{\mathrm{d}}(y)+\varphi_{J_{\mathrm{right}}}^{\mathrm{d}}(v)\varphi_{J_{\mathrm{right}}}^{\mathrm{d}}(y)\Big)\, du\, dv\, dx\, dy$$

Observe that $\Lambda_{\mathcal{T}}(F,G,H) = \Theta_{\mathcal{T}}^{(2)}(\mathbf{1},G,F,H)$.

A notion from additive combinatorics

For $Q = I \times J \in \mathcal{C}$ we define:

Gowers box inner-product and norm

$$[F_1, F_2, F_3, F_4]_{\square(Q)} :=$$
$$\frac{1}{|Q|^2} \int_I \int_I \int_J \int_J F_1(u,v) F_2(x,v) F_3(u,y) F_4(x,y)\, du\, dx\, dv\, dy$$
$$\|F\|_{\square(Q)} := [F, F, F, F]_{\square(Q)}^{1/4}$$

The box Cauchy-Schwarz inequality:
$$\big|[F_1, F_2, F_3, F_4]_{\square(Q)}\big| \le \|F_1\|_{\square(Q)} \|F_2\|_{\square(Q)} \|F_3\|_{\square(Q)} \|F_4\|_{\square(Q)}$$
A special case of Brascamp-Lieb inequalities:
$$\|F\|_{\square(Q)} \le \big(\tfrac{1}{|Q|} \int_Q |F|^2\big)^{1/2}$$

For any finite collection of squares $\mathcal{F}$ we define:

$$\Xi_{\mathcal{F}}(F_1, F_2, F_3, F_4) := \sum_{Q \in \mathcal{F}} |Q| \left[F_1, F_2, F_3, F_4\right]_{\Box(Q)}$$

$$= \sum_{I \times J \in \mathcal{F}} \int_{\mathbb{R}^4} F_1(u, v) F_2(x, v) F_3(u, y) F_4(x, y) \varphi_I^{\mathrm{d}}(u) \varphi_I^{\mathrm{d}}(x) \varphi_J^{\mathrm{d}}(v) \varphi_J^{\mathrm{d}}(y)\, du\, dv\, dx\, dy$$

A single-scale estimate:

$$\left|\Xi_{\mathcal{L}(\mathcal{T})}(F_1, F_2, F_3, F_4)\right| \ \leq \ |Q_{\mathcal{T}}| \prod_{j=1}^{4} \max_{Q \in \mathcal{L}(\mathcal{T})} \|F_j\|_{\Box(Q)}$$

Telescoping identity over trees

Lemma (Telescoping identity)

For any finite convex tree $\mathcal{T}$ with root $Q_{\mathcal{T}}$ we have

$$\begin{aligned}&\Theta^{(1)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)+\Theta^{(2)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)\\ &=\ \Xi_{\mathcal{L}(\mathcal{T})}(F_1,F_2,F_3,F_4)-\Xi_{\{Q_{\mathcal{T}}\}}(F_1,F_2,F_3,F_4)\end{aligned}$$

Telescoping identity over trees

Lemma (Telescoping identity)

For any finite convex tree $\mathcal{T}$ with root $Q_{\mathcal{T}}$ we have

$$\begin{aligned} &\Theta^{(1)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)+\Theta^{(2)}_{\mathcal{T}}(F_1,F_2,F_3,F_4) \\ &= \ \Xi_{\mathcal{L}(\mathcal{T})}(F_1,F_2,F_3,F_4)-\Xi_{\{Q_{\mathcal{T}}\}}(F_1,F_2,F_3,F_4) \end{aligned}$$

Proof, part 1.

When $\mathcal{T}$ consists of only one square, the identity reduces to

$$\begin{aligned} &\sum_{j\in\{\text{left},\text{right}\}} \psi^{\mathrm{d}}_I(u)\psi^{\mathrm{d}}_I(x)\varphi^{\mathrm{d}}_{J_j}(v)\varphi^{\mathrm{d}}_{J_j}(y) \ + \ \varphi^{\mathrm{d}}_I(u)\varphi^{\mathrm{d}}_I(x)\psi^{\mathrm{d}}_J(v)\psi^{\mathrm{d}}_J(y) \\ &= \sum_{i,j\in\{\text{left},\text{right}\}} \varphi^{\mathrm{d}}_{I_i}(u)\varphi^{\mathrm{d}}_{I_i}(x)\varphi^{\mathrm{d}}_{J_j}(v)\varphi^{\mathrm{d}}_{J_j}(y) \ - \ \varphi^{\mathrm{d}}_I(u)\varphi^{\mathrm{d}}_I(x)\varphi^{\mathrm{d}}_J(v)\varphi^{\mathrm{d}}_J(y) \end{aligned}$$

Telescoping identity over trees

Lemma (Telescoping identity)

For any finite convex tree $\mathcal{T}$ with root $Q_{\mathcal{T}}$ we have

$$\begin{aligned}&\Theta^{(1)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)+\Theta^{(2)}_{\mathcal{T}}(F_1,F_2,F_3,F_4)\\ &=\ \Xi_{\mathcal{L}(\mathcal{T})}(F_1,F_2,F_3,F_4)-\Xi_{\{Q_{\mathcal{T}}\}}(F_1,F_2,F_3,F_4)\end{aligned}$$

Proof, part 2.

For a general finite convex tree $\mathcal{T}$ the following sum

$$\sum_{Q\in\mathcal{T}}\Bigg(\sum_{\widetilde{Q}\ \text{is a child of}\ Q}\Xi_{\{\widetilde{Q}\}}\ -\ \Xi_{\{Q\}}\Bigg)$$

telescopes into the right hand side. □

Reduction inequalities

Lemma (Reduction inequalities)

$$\left|\Theta_{\mathcal{T}}^{(1)}(F_1, F_2, F_3, F_4)\right| \;\leq\; \Theta_{\mathcal{T}}^{(1)}(F_1, F_1, F_3, F_3)^{\frac{1}{2}}\;\Theta_{\mathcal{T}}^{(1)}(F_2, F_2, F_4, F_4)^{\frac{1}{2}}$$

$$\left|\Theta_{\mathcal{T}}^{(2)}(F_1, F_2, F_3, F_4)\right| \;\leq\; \Theta_{\mathcal{T}}^{(2)}(F_1, F_2, F_1, F_2)^{\frac{1}{2}}\;\Theta_{\mathcal{T}}^{(2)}(F_3, F_4, F_3, F_4)^{\frac{1}{2}}$$

Reduction inequalities

Lemma (Reduction inequalities)

$$\left|\Theta^{(1)}_{\mathcal{T}}(F_1, F_2, F_3, F_4)\right| \;\le\; \Theta^{(1)}_{\mathcal{T}}(F_1, F_1, F_3, F_3)^{\frac{1}{2}}\; \Theta^{(1)}_{\mathcal{T}}(F_2, F_2, F_4, F_4)^{\frac{1}{2}}$$

$$\left|\Theta^{(2)}_{\mathcal{T}}(F_1, F_2, F_3, F_4)\right| \;\le\; \Theta^{(2)}_{\mathcal{T}}(F_1, F_2, F_1, F_2)^{\frac{1}{2}}\; \Theta^{(2)}_{\mathcal{T}}(F_3, F_4, F_3, F_4)^{\frac{1}{2}}$$

Proof.

Rewrite $\Theta^{(2)}_{\mathcal{T}}(F_1, F_2, F_3, F_4)$ as

$$\sum_{I\times J\in\mathcal{T}} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} F_1(u,v)F_2(x,v)\psi^{\mathrm{d}}_J(v)\,dv \Big) \cdot \Big(\int_{\mathbb{R}} F_3(u,y)F_4(x,y)\psi^{\mathrm{d}}_J(y)\,dy \Big)\, \varphi^{\mathrm{d}}_I(u)\varphi^{\mathrm{d}}_I(x)\, du\, dx$$

and apply the Cauchy-Schwarz inequality, first over $(u,x) \in I \times I$, and then over $I \times J \in \mathcal{T}$. $\square$

Single tree estimate

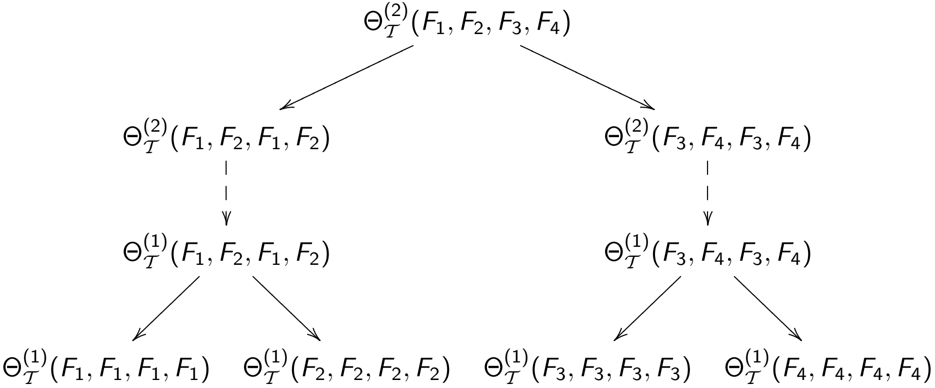
For any finite convex tree $\mathcal{T}$ we have:

Proposition (Single tree estimate)

$$\left|\Theta^{(2)}_{\mathcal{T}}(F_1, F_2, F_3, F_4)\right| \lesssim |Q_{\mathcal{T}}| \prod_{j=1}^{4} \max_{Q\in\mathcal{L}(\mathcal{T})} \|F_j\|_{\Box(Q)}$$

$$\left|\Lambda_{\mathcal{T}}(F, G, H)\right| \lesssim |Q_{\mathcal{T}}| \Big(\max_{Q\in\mathcal{L}(\mathcal{T})} \|F\|_{\Box(Q)} \Big) \Big(\max_{Q\in\mathcal{L}(\mathcal{T})} \|G\|_{\Box(Q)} \Big) \Big(\max_{Q\in\mathcal{L}(\mathcal{T})} \|H\|_{\Box(Q)} \Big)$$

Proof of the single tree estimate



How do we control $\Theta^{(1)}_{\mathcal{T}}(F_j, F_j, F_j, F_j)$?

$$\underbrace{\Theta^{(1)}_{\mathcal{T}}(F_j, F_j, F_j, F_j)}_{\geq 0} + \underbrace{\Theta^{(2)}_{\mathcal{T}}(F_j, F_j, F_j, F_j)}_{\geq 0} + \underbrace{\Xi_{\{Q_{\mathcal{T}}\}}(F_j, F_j, F_j, F_j)}_{\geq 0 \text{ if } F_j \geq 0}$$
$$= \Xi_{\mathcal{L}(\mathcal{T})}(F_j, F_j, F_j, F_j)$$

$\square$

Estimates in the local L^2 range

Proposition

$$|\Lambda_{\mathrm{d}}(F,G,H)| \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}\|H\|_{\mathrm{L}^r}$$

for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $2<p,q,r<\infty$

Estimates in the local L^2 range

Proposition

$$|\Lambda_{\mathrm{d}}(F, G, H)| \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^q} \|H\|_{\mathrm{L}^r}$$

for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $2 < p, q, r < \infty$

Proof, part 1.

$\mathcal{P}_k^F := \{ Q \ : \ 2^k \le \sup_{Q' \supseteq Q} \|F\|_{\square(Q')} < 2^{k+1} \}$
$\mathcal{M}_k^F =$ maximal squares in $\mathcal{P}_k^F$
$\mathcal{P}_k^G$, $\mathcal{M}_k^G$, $\mathcal{P}_k^H$, $\mathcal{M}_k^H$ defined analogously
$\mathcal{P}_{k_1,k_2,k_3} := \mathcal{P}_{k_1}^F \cap \mathcal{P}_{k_2}^G \cap \mathcal{P}_{k_3}^H$
$\mathcal{M}_{k_1,k_2,k_3} =$ maximal squares in $\mathcal{P}_{k_1,k_2,k_3}$

Estimates in the local L^2 range

Proposition

$$|\Lambda_{\mathrm{d}}(F,G,H)| \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}\|H\|_{\mathrm{L}^r}$$

for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,\ 2<p,q,r<\infty$

Proof, part 2.

For each $Q\in\mathcal{M}_{k_1,k_2,k_3}$

$$\mathcal{T}_Q := \{\widetilde{Q}\in\mathcal{P}_{k_1,k_2,k_3} \,:\, \widetilde{Q}\subseteq Q\}$$

is a convex tree with root Q. We apply the single tree estimate to each of the trees $\mathcal{T}_Q$.

It remains to show

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}} 2^{k_1+k_2+k_3} \sum_{Q\in\mathcal{M}_{k_1,k_2,k_3}} |Q| \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}\|H\|_{\mathrm{L}^r}$$

Estimates in the local L^2 range

Proposition

$$|\Lambda_{\mathrm{d}}(F,G,H)| \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}\|H\|_{\mathrm{L}^r}$$

for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,\ 2<p,q,r<\infty$

Proof, part 3.

i.e.
$$\sum_{k_1,k_2,k_3\in\mathbb{Z}} 2^{k_1+k_2+k_3}\min\Big(\sum_{Q\in\mathcal{M}^F_{k_1}}|Q|,\ \sum_{Q\in\mathcal{M}^G_{k_2}}|Q|,\ \sum_{Q\in\mathcal{M}^H_{k_3}}|Q|\Big) \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}\|H\|_{\mathrm{L}^r}$$

i.e.
$$\sum_{k\in\mathbb{Z}} 2^{pk}\sum_{Q\in\mathcal{M}^F_k}|Q| \lesssim_p \|F\|^p_{\mathrm{L}^p},$$

$$\sum_{k\in\mathbb{Z}} 2^{qk}\sum_{Q\in\mathcal{M}^G_k}|Q| \lesssim_q \|G\|^q_{\mathrm{L}^q}, \quad \sum_{k\in\mathbb{Z}} 2^{rk}\sum_{Q\in\mathcal{M}^H_k}|Q| \lesssim_r \|H\|^r_{\mathrm{L}^r}$$

Estimates in the local L^2 range

Proposition

$$|\Lambda_{\mathrm{d}}(F,G,H)| \lesssim_{p,q,r} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}\|H\|_{\mathrm{L}^r}$$

for $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,\ 2<p,q,r<\infty$

Proof, part 4.

$$\mathrm{M}_2 F := \sup_{Q\in\mathcal{C}} \Big(\frac{1}{|Q|}\int_Q |F|^2\Big)^{1/2}\mathbf{1}_Q$$

For each $Q\in\mathcal{M}_k^F$ we have $\big(\frac{1}{|Q|}\int_Q |F|^2\big)^{1/2} \geq \|F\|_{\Box(Q)} \geq 2^k$ and so by disjointness

$$\sum_{Q\in\mathcal{M}_k^F} |Q| \leq |\{\mathrm{M}_2 F \geq 2^k\}|$$

$$\sum_{k\in\mathbb{Z}} 2^{pk}|\{\mathrm{M}_2 F\geq 2^k\}| \sim_p \|\mathrm{M}_2 F\|_{\mathrm{L}^p}^p \lesssim_p \|F\|_{\mathrm{L}^p}^p$$

Extending the range of exponents

Proposition (Bernicot, 2010)

If for some (p, q), $1 < p, q < \infty$

$$\|T_{\mathrm{d}}(F, G)\|_{\mathrm{L}^{pq/(p+q)}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^q}$$

then

$$\|T_{\mathrm{d}}(F, G)\|_{\mathrm{L}^{p/(p+1),\infty}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^1}$$

$$\|T_{\mathrm{d}}(F, G)\|_{\mathrm{L}^{q/(q+1),\infty}} \lesssim_{p,q} \|F\|_{\mathrm{L}^1} \|G\|_{\mathrm{L}^q}$$

Extending the range of exponents

Proposition (Bernicot, 2010)

If for some (p, q), $1 < p, q < \infty$

$$\| T_{\mathrm{d}}(F, G)\|_{\mathrm{L}^{pq/(p+q)}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^q}$$

then

$$\| T_{\mathrm{d}}(F, G)\|_{\mathrm{L}^{p/(p+1),\infty}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p}\|G\|_{\mathrm{L}^1}$$

$$\| T_{\mathrm{d}}(F, G)\|_{\mathrm{L}^{q/(q+1),\infty}} \lesssim_{p,q} \|F\|_{\mathrm{L}^1}\|G\|_{\mathrm{L}^q}$$

Proof.

Perform one-dimensional Calderón-Zygmund decomposition in each fiber $F(\cdot, y)$ or $G(x, \cdot)$. □

Even simpler in the dyadic setting: there is no contribution of "the bad part" outside of "the bad set".

Bounding the continuous version

Let us assume that $\psi_k = \phi_{k+1} - \phi_k$ for some ϕ Schwartz, $\int_{\mathbb{R}} \phi = 1$. The general case is obtained by composing with a bounded Fourier multiplier in the second variable.

Proposition (Jones, Seeger, Wright, 2008)

If φ is Schwartz and $\int_{\mathbb{R}} \varphi = 1$, then the square function

$$\mathcal{S}_{\mathrm{JSW}} f := \Big(\sum_{k \in \mathbb{Z}} \big| \mathrm{P}_{\varphi_k} f - \mathbb{E}_k f \big|^2 \Big)^{1/2}$$

satisfies

$$\|\mathcal{S}_{\mathrm{JSW}} f\|_{\mathrm{L}^p(\mathbb{R})} \lesssim_p \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

for $1 < p < \infty$.

Bounding the continuous version

Proposition

$$\left\| T_{\mathrm{c}}(F, G) - T_{\mathrm{d}}(F, G) \right\|_{\mathrm{L}^{pq/(p+q)}} \lesssim_{p,q} \|F\|_{\mathrm{L}^{p}} \|G\|_{\mathrm{L}^{q}}$$

Bounding the continuous version

Proposition

$$\big\| T_{\mathrm{c}}(F,G) - T_{\mathrm{d}}(F,G) \big\|_{\mathrm{L}^{pq/(p+q)}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^q}$$

Proof, part 1.

$$T_{\mathrm{aux}}(F,G) := \sum_{k\in\mathbb{Z}} (\mathbb{E}_k^{(1)} F)(\mathrm{P}_{\psi_k}^{(2)} G)$$

$$\big| T_{\mathrm{c}}(F,G) - T_{\mathrm{aux}}(F,G) \big| \le \underbrace{\Big(\sum_{k\in\mathbb{Z}} \big| \mathrm{P}_{\varphi_k}^{(1)} F - \mathbb{E}_k^{(1)} F \big|^2 \Big)^{1/2}}_{\mathcal{S}_{\mathrm{JSW}}^{(1)} F} \underbrace{\Big(\sum_{k\in\mathbb{Z}} \big| \mathrm{P}_{\psi_k}^{(2)} G \big|^2 \Big)^{1/2}}_{\mathcal{S}^{(2)} G}$$

Bounding the continuous version

Proposition

$$\big\| T_{\mathrm{c}}(F,G) - T_{\mathrm{d}}(F,G) \big\|_{\mathrm{L}^{pq/(p+q)}} \lesssim_{p,q} \|F\|_{\mathrm{L}^p} \|G\|_{\mathrm{L}^q}$$

Proof, part 2.

$$T_{\mathrm{aux}}(F,G) = FG - \sum_{k\in\mathbb{Z}} (\Delta_k^{(1)} F)(\mathrm{P}_{\phi_{k+1}}^{(2)} G)$$

$$T_{\mathrm{d}}(F,G) = FG - \sum_{k\in\mathbb{Z}} (\Delta_k^{(1)} F)(\mathbb{E}_{k+1}^{(2)} G)$$

$$\big| T_{\mathrm{aux}}(F,G) - T_{\mathrm{d}}(F,G) \big| \leq \underbrace{\Big(\sum_{k\in\mathbb{Z}} \big|\Delta_k^{(1)} F\big|^2 \Big)^{1/2}}_{\mathcal{S}_{\mathrm{d}}^{(1)} F} \underbrace{\Big(\sum_{k\in\mathbb{Z}} \big|\mathrm{P}_{\phi_k}^{(2)} G - \mathbb{E}_k^{(2)} G\big|^2 \Big)^{1/2}}_{\mathcal{S}_{\mathrm{JSW}}^{(2)} G}$$

Endpoint counterexample

We disprove

$$\mathrm{L}^{\infty}(\mathbb{R}^2) \times \mathrm{L}^{q}(\mathbb{R}^2) \to \mathrm{L}^{q,\infty}(\mathbb{R}^2)$$

for $1 \leq q < \infty$.

$$G(x,y) := \mathbf{1}_{[0,2^{-n})}(x) \sum_{k=1}^{n} R_k(y)$$

$$R_k := k\text{-th Rademacher function} = \sum_{J \subseteq [0,1),\, |J| = 2^{-k+1}} (\mathbf{1}_{J_{\mathrm{left}}} - \mathbf{1}_{J_{\mathrm{right}}})$$

$\|G\|_{\mathrm{L}^q} \sim_q 2^{-n/q} n^{1/2}$ by Khintchine's inequality

$(\Delta_k^{(2)} G)(x,y) = \mathbf{1}_{[0,2^{-n})}(x) R_{k+1}(y)$ for $k = 0, 1, \ldots, n-1$

Endpoint counterexample

We choose F so that there is no cancellation between different scales in $T_{\mathrm{d}}(F,G)$.
$(\mathbb{E}_k^{(1)}F)(x,y) = R_{k+1}(y)$ for $x \in [0,2^{-n})$, $k=0,1,\ldots,n-1$

$$F(x,y) := \begin{cases} 2R_j(y) - R_{j+1}(y), & \text{for } x \in [2^{-j},2^{-j+1}),\ j=1,\ldots,n-1 \\ R_n(y), & \text{for } x \in [0,2^{-n+1}) \end{cases}$$

Then $\quad T_{\mathrm{d}}(F,G) = n\,\mathbf{1}_{[0,2^{-n})\times[0,1)}$

$$\frac{\|T_{\mathrm{d}}(F,G)\|_{\mathrm{L}^{q,\infty}}}{\|F\|_{\mathrm{L}^\infty}\|G\|_{\mathrm{L}^q}} \gtrsim_q \frac{2^{-n/q}n}{2^{-n/q}n^{1/2}} = n^{1/2}$$

A byproduct of the proof

Theorem (K., 2010)

$$\Theta^{(2)}(F_1, F_2, F_3, F_4) := \sum_{I \times J \in \mathcal{C}} \int_{\mathbb{R}^4} F_1(u, v) F_2(x, v) F_3(u, y) F_4(x, y) \varphi_I^{\mathrm{d}}(u) \varphi_I^{\mathrm{d}}(x) \psi_J^{\mathrm{d}}(v) \psi_J^{\mathrm{d}}(y) \, du \, dv \, dx \, dy$$

$$\left|\Theta^{(2)}(F_1, F_2, F_3, F_4)\right| \lesssim_{p_1, p_2, p_3, p_4} \prod_{j=1}^{4} \|F_j\|_{\mathrm{L}^{p_j}},$$

for $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 1, \ \ 2 < p_1, p_2, p_3, p_4 < \infty$

Higher-dimensional generalizations are straightforward.

Subsequent research

The next step in the Demeter-Thiele program:

More singular 2D bilinear Hilbert transform

$$\mathrm{p.v.} \int_{\mathbb{R}} F(x-t, y)\, G(x, y-t)\, \frac{dt}{t}$$

A reason for caution:

Bi-parameter bilinear Hilbert transform

$$\mathrm{p.v.} \int_{\mathbb{R}^2} F(x-s, y-t)\, G(x+s, y+t)\, \frac{ds}{s}\frac{dt}{t}$$

satisfies no L^p estimates – Muscalu, Pipher, Tao, Thiele (2004)