Bilinear and trilinear embeddings for complex elliptic operators

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Semigroups

STRONGLY CONTINUOUS SEMIGROUPS

$X =$ a Banach space, $\mathsf{B}(X) =$ bounded linear operators on X

Function

$$[0,\infty\rangle \to \mathsf{B}(X), \quad t \mapsto T_t$$

is a *strongly continuous one-parameter operator semigroup* if:

- $T_0 = I$;
- for all $s, t \in [0,\infty\rangle$ one has $T_{s+t} = T_s T_t$;
- for all $f \in X$ the function $[0,\infty\rangle \to X$, $t \mapsto T_t f$ is continuous.

Infinitesimal generator of the semigroup is

$$Lf := \lim_{t \to 0+} \frac{T_t f - f}{t}.$$

Its domain $\mathcal{D}(L) \subseteq X$ is dense in X, while its graph is a closed subset of $X \times X$.

Strongly continuous semigroups

Conversely, L uniquely determines the semigroup and we write

$$T_t = \exp(tL) = e^{tL}.$$

For each $f \in \mathcal{D}(L)$ the function

$$u(t) := T_t f$$

represents a classical solution of the differential equation (in X)

$$u'(t) = Lu(t)$$

(a “proof” is $\frac{\mathrm{d}}{\mathrm{d}t} e^{tL} = L e^{tL}$) with the initial condition

$$u(0) = f.$$

For each $f \in X$ the formula gives a “mild” solution of the problem,

$$u(t) = f + L \int_0^t u(s)\,\mathrm{d}s.$$

Examples of classical questions: Generation

Under which conditions does operator L really generate a strongly continuous semigroup?

Hille–Yosida–Feller–Miyadera–Phillips theorem:
if and only if

- $\mathcal{D}(L)$ is dense in X,
- the graph of L is a closed subset of $X \times X$,
- there exist $\omega \in [0, \infty\rangle$, $M \in [1, \infty\rangle$ such that for $\lambda \in \langle\omega, \infty\rangle$, $n \in \mathbb{N}$

one has

$$\|(\lambda I - L)^{-n}\|_{X \to X} \leqslant \frac{M}{(\lambda - \omega)^n}.$$

Moreover, the semigroup then satisfies the bound

$$\|T_t\|_{X \to X} \leqslant M e^{\omega t}.$$

EXAMPLES OF CLASSICAL QUESTIONS: CONTRACTIVITY

Under which conditions is the semigroup contractive, i.e.,

$$\|T_t f\|_X \leqslant \|f\|_X$$

for all $t \in [0, \infty\rangle$ and $f \in X$?

Hille–Yosida theorem:
if and only if for every $\lambda > 0$ one has $\|(\lambda I - L)^{-1}\|_{X \to X} \leqslant \frac{1}{\lambda}$.

Lumer–Phillips theorem:
if and only if for every $f \in \mathcal{D}(L) \subseteq X$ there exists $g \in X^*$ such that $g(f) = \|f\|_X^2 = \|g\|_{X^*}^2$ and $\operatorname{Re} g(Lf) \leqslant 0$.

Contractivity can be investigated in different norms $\|\cdot\|$, but for the same semigroup. For instance, when $X = \mathrm{L}^2(\mathbb{R}^d)$ we could consider $\|\cdot\|_{\mathrm{L}^p(\mathbb{R}^d)}$ for some $1 < p < \infty$.

Elliptic operators

MATRIX FUNCTIONS WITH COMPLEX L^∞ COEFFICIENTS

$A\colon \mathbb{R}^d \to \mathrm{M}_d(\mathbb{C})$ a matrix function

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,d} \\ \vdots & & \vdots \\ a_{d,1} & \cdots & a_{d,d} \end{bmatrix}, \quad a_{i,j} \in \mathrm{L}^\infty(\mathbb{R}^d)$$

(For simplicity we are not working on domains $\Omega \subseteq \mathbb{R}^d$.)

We say that A is *uniformly elliptic (accretive)* if:

$$\Lambda(A) := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \max_{\substack{\zeta,\eta\in\mathbb{C}^d \\ |\zeta|=|\eta|=1}} \left|\langle A(x)\zeta, \eta\rangle_{\mathbb{C}^d}\right| < \infty,$$

$$\lambda(A) := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} \min_{\substack{\xi\in\mathbb{C}^d \\ |\xi|=1}} \operatorname{Re}\left\langle A(x)\xi, \xi\right\rangle_{\mathbb{C}^d} > 0.$$

Elliptic operators in divergence form

Elliptic operator in divergence form (complex and non-smooth) associated with A is formally defined:

$$L_A f := -\,\mathrm{div}(A\nabla f),$$

i.e.,

$$(L_A f)(x) := -\sum_{i,j=1}^{d} \partial_i\big(a_{i,j}(x)\partial_j f(x)\big).$$

(One might also add first-order derivatives.)

This is well defined in the classical sense only when the coefficients of A are smooth.

If $A \equiv I$, then $L_I = -\,\mathrm{div}\,\nabla = -\Delta$.

Elliptic operators in divergence form

In general, L_A is defined via duality:

$$\langle L_A f, g\rangle_{\mathrm{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle A(x)\nabla f(x), \nabla g(x)\rangle_{\mathbb{C}^d}\,\mathrm{d}x$$

(as if we formally applied integration by parts) and its domain $\mathcal{D}(L_A) \subseteq \mathrm{L}^2(\mathbb{R}^d)$ is the set of all $f \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ for which the RHS, as a function of $g \in \mathrm{W}^{1,2}(\mathbb{R}^d)$, extends to a bounded antilinear functional on $\mathrm{L}^2(\mathbb{R}^d)$.

We will only work with $f, g \in \mathrm{C}_c^\infty(\mathbb{R}^d)$.

Consider the strongly continuous operator semigroup $(T_t^A)_{t\geqslant 0}$ on $\mathrm{L}^2(\mathbb{R}^d)$ generated by $-L_A$:

$$T_t^A := \exp(-tL_A).$$

Elliptic operators in divergence form

Why is this concept important or interesting?

For every $f \in \mathcal{D}(L_A)$ the function

$$u\colon [0,\infty\rangle \times \mathbb{R}^d \to \mathbb{C},$$

$$u(t,x) := (T_t^A f)(x)$$

is a classical solution of the evolution PDE

$$\frac{\partial}{\partial t} u(t,x) = -L_A(x) u(t,x)$$

with the initial condition

$$u(0,x) = f(x).$$

For $A \equiv I$ we obtain the heat equation:

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x).$$

AN INTERESTING QUESTION: CONTRACTIVITY

Under which condition on A is the semigroup contractive on $\mathrm{L}^p(\mathbb{R}^d)$ for some $1 < p < \infty$, i.e.,

$$\|T_t^A f\|_{\mathrm{L}^p(\mathbb{R}^d)} \leqslant \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

for $t \in [0, \infty\rangle$ and $f \in \mathrm{L}^p(\mathbb{R}^d)$?

Under the additional requirement that $\operatorname{Im} A$ is symmetric, Cialdea and Maz'ya (2005) showed that the contractivity on $\mathrm{L}^p(\mathbb{R}^d)$ is equivalent with

$$|p-2|\left|\langle \operatorname{Im} A(x)\xi, \xi\rangle_{\mathbb{R}^d}\right| \leqslant 2(p-1)^{1/2}\langle \operatorname{Re} A(x)\xi, \xi\rangle_{\mathbb{R}^d}$$

for a.e. $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$.

A general characterization is an open problem.

Carbonaro and Dragičević (2016) showed:
$\Delta_p(A) \geqslant 0$ is sufficient, $\Delta_p\left(\frac{A+A^*}{2}\right) \geqslant 0$ is necessary.

p-ELLIPTICITY

Carbonaro and Dragičević (2016) defined A to be p-*elliptic* for $1 < p < \infty$ if $\Lambda(A) < \infty$ and

$$\Delta_p(A) := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} \min_{\substack{\xi\in\mathbb{C}^d\\|\xi|=1}} \operatorname{Re}\left\langle A(x)\xi,\ \xi + \left|1-\frac{2}{p}\right|\bar{\xi}\right\rangle_{\mathbb{C}^d} > 0.$$

For $2 \leqslant p_1 \leqslant p_2 < \infty$ we have

$$\lambda(A) = \Delta_2(A) \geqslant \Delta_{p_1}(A) \geqslant \Delta_{p_2}(A)$$

and inclusions

$$\left\{\begin{matrix}\text{elliptic}\\\text{matrices}\end{matrix}\right\} = \left\{\begin{matrix}\text{2-elliptic}\\\text{matrices}\end{matrix}\right\} \supseteq \left\{\begin{matrix}p_1\text{-elliptic}\\\text{matrices}\end{matrix}\right\} \supseteq \left\{\begin{matrix}p_2\text{-elliptic}\\\text{matrices}\end{matrix}\right\}$$

$$\supseteq \left\{\begin{matrix}\text{matrices that are } p\text{-elliptic}\\\text{for every } 1<p<\infty\end{matrix}\right\} = \left\{\begin{matrix}\text{real elliptic}\\\text{matrices}\end{matrix}\right\}.$$

Bilinear embeddings

Bilinear embeddings

Let $(T_t)_{t\geqslant 0}$ and $(\widetilde{T}_t)_{t\geqslant 0}$ be operator semigroups on some space of functions on $\mathbb{R}^d$. Let $\|\cdot\|$ and $\|\cdot\|^*$ be mutually dual norms.

Bi-sub-linear estimates of the form

$$\int_0^\infty \int_{\mathbb{R}^d} \left|\nabla T_t f(x)\right| \left|\nabla \widetilde{T}_t g(x)\right| \mathrm{d}x\,\mathrm{d}t \leqslant C\|f\|\|g\|^*$$

are called *bilinear embeddings*.

Some history of bilinear embeddings:

- Estimates for the Ahlfors–Beurling operator and iterated Riesz transforms (Petermichl–Volberg, 2002; Nazarov–Volberg, 2003):

$$\int_{\mathbb{R}^2} (R_1^2 f)(x)\, g(x)\, \mathrm{d}x = -2\int_0^\infty \int_{\mathbb{R}^2} \left(\partial_{x_1} T_t f(x)\right)\left(\partial_{x_1} T_t g(x)\right) \mathrm{d}x\,\mathrm{d}t,$$

where $(T_t f)_{t\geqslant 0}$ is the heat extension of f.

More history of bilinear embeddings

- Dimension-free Littlewood–Paley estimates (Dragičević–Volberg, 2006)
- Dimension-free estimates for Schrödinger operators (Dragičević–Volberg, 2011, 2012)
- Dimension-free estimates for Riesz transforms associated with a Riemannian manifold (Carbonaro–Dragičević, 2011)
- Functional calculus for generators of symmetric contraction semigroups (Carbonaro–Dragičević, 2013)
- Bilinear embedding for divergence-form operators with complex coefficients (Carbonaro–Dragičević, 2016)

BILINEAR EMBEDDING FOR COMPLEX ELLIPTIC OPERATORS

Theorem [Carbonaro–Dragičević, 2016]. If $A, B\colon \mathbb{R}^d \to \mathrm{M}_d(\mathbb{C})$ are p-elliptic, then

$$\int_0^\infty \int_{\mathbb{R}^d} |(\nabla T_t^A f)(x)|\,|(\nabla T_t^B g)(x)|\,\mathrm{d}x\,\mathrm{d}t \leqslant 20\, C_p(A,B)\, \|f\|_{\mathrm{L}^p} \|g\|_{\mathrm{L}^q}.$$

Here

$$C_p(A,B) := \frac{\max\{\Lambda(A),\Lambda(B)\}}{\min\{\Delta_p(A),\Delta_p(B)\}\min\{\lambda(A),\lambda(B)\}}.$$

BILINEAR EMBEDDING FOR COMPLEX ELLIPTIC OPERATORS

They use certain convexity properties of the Bellman function constructed by Nazarov and Treil (1996):

$$\mathfrak{X}(u,v) := \frac{|u|^p}{p} + \frac{|v|^q}{q} + \delta \begin{cases} \frac{2}{p}|u|^p + \left(\frac{2}{q}-1\right)|v|^q & \text{for } |u|^p \geqslant |v|^q, \\ |u|^2|v|^{2-q} & \text{for } |u|^p < |v|^q, \end{cases}$$

for an appropriate $\delta > 0$.

It is crucially used that $\mathfrak{X}$ is made of powers.

Orlicz spaces

ORLICZ SPACES

They generalize L^p spaces.

We want to have function spaces "close to" L^1 or L^∞, or a "finer scale" of function spaces between L^1 and L^∞.

$\Phi\colon [0,\infty\rangle \to [0,\infty\rangle$ is a *Young function* if

$$\Phi \text{ is convex}, \quad \Phi(0)=0, \quad \lim_{s\to 0+}\frac{\Phi(s)}{s}=0, \quad \lim_{s\to\infty}\frac{\Phi(s)}{s}=\infty.$$

Conjugated Young function is $\Psi\colon [0,\infty\rangle \to [0,\infty\rangle$,

$$\Psi(t) := \sup_{s\in\langle 0,\infty\rangle}\big(st-\Phi(s)\big) = \int_0^t (\Phi')^{-1}(r)\,\mathrm{d}r.$$

Young's inequality:

$$st \leqslant \Phi(s)+\Psi(t) \quad \text{for all } s,t\in[0,\infty\rangle.$$

ORLICZ SPACES

Luxemburg norm $\|\cdot\|_\Phi$ is defined for (a.e. classes of) measurable functions $f\colon \mathbb{R}^d \to \mathbb{C}$ as

$$\|f\|_\Phi := \inf\Big\{\alpha \in \langle 0,\infty\rangle : \int_{\mathbb{R}^d} \Phi\Big(\frac{|f(x)|}{\alpha}\Big)\,\mathrm{d}x \leqslant 1\Big\}.$$

That way we arrive at the *Orlicz space* $\mathrm{L}^\Phi(\mathbb{R}^d)$.

The doubling condition

$$\Phi(2s) \leqslant K\Phi(s) \quad \text{for every } s \in [0,\infty\rangle$$

guarantees

$$\|\cdot\|_\Phi^* \sim \|\cdot\|_\Psi$$

and

$$\mathrm{L}^\Phi(\mathbb{R}^d)^* \cong \mathrm{L}^\Psi(\mathbb{R}^d).$$

Additional assumptions

We additionally assume that Φ and Ψ are "like powers":

- Φ and Ψ are mutually conjugate Young functions,
- Φ and Ψ are C^1 on $[0,\infty\rangle$ and C^2 on $\langle 0,\infty\rangle$,
- $\Phi''(s), \Psi''(s) > 0$ for every $s \in \langle 0,\infty\rangle$,
- Φ' is strictly convex on $\langle 0,\infty\rangle$ and $\displaystyle\lim_{s\to 0+}\frac{\Phi'(s)}{s} = 0$,
- $\displaystyle\sup_{s\in(0,\infty)}\frac{s\Phi'(s)}{\Phi(s)} < \infty$,
- $\displaystyle 1 < \inf_{s\in(0,\infty)}\frac{s\Phi''(s)}{\Phi'(s)} \leqslant \sup_{s\in(0,\infty)}\frac{s\Phi''(s)}{\Phi'(s)} < \infty.$

ADDITIONAL ASSUMPTIONS

Consequently:

Φ' i Ψ' are mutually inverse increasing bijections of $[0, \infty\rangle$.

The last three conditions are equivalent with:

- Ψ' is strictly concave on $\langle 0, \infty\rangle$ and $\displaystyle\lim_{s\to 0+} \frac{\Psi'(s)}{s} = \infty$,
- $\displaystyle\inf_{s\in(0,\infty)} \frac{s\Psi'(s)}{\Psi(s)} > 1$,
- $\displaystyle 0 < \inf_{s\in(0,\infty)} \frac{s\Psi''(s)}{\Psi'(s)} \leqslant \sup_{s\in(0,\infty)} \frac{s\Psi''(s)}{\Psi'(s)} < 1$.

Consequently:

$$\|\cdot\|_{\Phi}^{*} \sim \|\cdot\|_{\Psi}.$$

EXAMPLES

Example 1. Lebesgue spaces L^p.

$$\Phi(s) = \frac{s^p}{p}, \quad \Psi(s) = \frac{s^q}{q} \quad \text{for } p \in \langle 2, \infty \rangle,\ q \in \langle 1, 2 \rangle,\ \frac{1}{p} + \frac{1}{q} = 1.$$

$$\|\cdot\|_\Phi \sim \|\cdot\|_{\mathsf{L}^p}, \qquad \|\cdot\|_\Psi \sim \|\cdot\|_{\mathsf{L}^q}.$$

Example 2. Zygmund spaces $\mathsf{L}^r \log \mathsf{L}$.

$$\Phi(s) = s^r \log(s+e) \quad \text{for } r \in \langle 2, \infty \rangle.$$

EXAMPLES

Example 3. Superposition of powers with exponents from $\langle 2,\infty\rangle$.

$$\Phi(s) = s^p + s^r \quad \text{for } 2 < r < p < \infty.$$

$$\Phi(s) = \int s^t \, \mathrm{d}\mu(t)$$

for a positive Borel measure μ with compact support in $\langle 2,\infty\rangle$.

Example 4. Superposition of powers with exponents from $\langle 1,2\rangle$.

$$\Psi(s) = s^q + s^r \quad \text{for } 1 < q < r < 2.$$

$$\Psi(s) = \int s^t \, \mathrm{d}\mu(t)$$

for a positive Borel measure μ with compact support in $\langle 1,2\rangle$.

Embeddings meet Orlicz

What would be Φ-ellipticity?

For a Young function Φ one could define:

$$\Delta_\Phi(A) := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} \inf_{\substack{\xi\in\mathbb{C}^d,\, |\xi|=1 \\ s\in(0,\infty)}} \operatorname{Re}\left\langle A(x)\xi,\ \xi + \frac{s\Phi''(s)-\Phi'(s)}{s\Phi''(s)+\Phi'(s)}\,\bar{\xi}\right\rangle_{\mathbb{C}^d}.$$

In the special case $\Phi(s) = s^p/p$ this simplifies as $\Delta_p(A)$.

However,

$$\Delta_\Phi(A) = \Delta_p(A)$$

for a unique $p \in [2,\infty]$ s.t.

$$\sup_{s\in(0,\infty)} \left|\frac{s\Phi''(s)-\Phi'(s)}{s\Phi''(s)+\Phi'(s)}\right| = 1 - \frac{2}{p}, \quad \text{i.e.,}\ \ p = \sup_{s\in(0,\infty)} \frac{s\Phi''(s)}{\Phi'(s)} + 1.$$

Relation to recent literature

Cialdea and Maz'ya (2021) introduced the notion of certain Φ-*dissipativity*, as an attempt to characterize L^Φ-contractivity.

Under the additional assumption that $\operatorname{Im} A$ is symmetric they showed that Φ-dissipativity is equivalent with

$$\left|\Phi''(s)-\frac{\Phi'(s)}{s}\right|\left|\langle \operatorname{Im} A(x)\xi,\xi\rangle_{\mathbb{R}^d}\right| \leqslant 2\left(\frac{\Phi'(s)\Phi''(s)}{s}\right)^{1/2}\langle \operatorname{Re} A(x)\xi,\xi\rangle_{\mathbb{R}^d}$$

for a.e. $x\in\mathbb{R}^d$, every $s\in(0,\infty)$, and every $\xi\in\mathbb{R}^d$.

In the special case $\Phi(s)=s^p/p$ this becomes the previously mentioned characterization of L^p-contractivity.

A bilinear embedding in Orlicz spaces

We recall all aditional assumptions on Φ, Ψ, and we set

$$p = \sup_{s\in(0,\infty)} \frac{s\Phi''(s)}{\Phi'(s)} + 1 = \sup_{s\in(0,\infty)} \frac{\Psi'(s)}{s\Psi''(s)} + 1.$$

Theorem [K.–Škreb, 2021]. If $A, B\colon \mathbb{R}^d \to \mathrm{M}_d(\mathbb{C})$ are p-elliptic, then

$$\int_0^\infty \int_{\mathbb{R}^d} |(\nabla T_t^A f)(x)|\,|(\nabla T_t^B g)(x)|\,\mathrm{d}x\,\mathrm{d}t \leqslant 40\, C_p(A,B)\, D(\Phi,\Psi)\, \|f\|_\Phi \|g\|_\Psi .$$

Here $C_p(A,B)$ is as before, while

$$D(\Phi,\Psi) := \max\left\{1, \frac{M}{\tilde m}\right\}\left(\frac{\tilde m}{\tilde M}\frac{\tilde M - 1}{\tilde m - 1}\right)^{1/2}$$

with

$$M := \sup_{s\in(0,\infty)} \frac{s\Phi'(s)}{\Phi(s)}, \quad \tilde m := \inf_{s\in(0,\infty)} \frac{s\Phi''(s)}{\Phi'(s)}, \quad \tilde M := \sup_{s\in(0,\infty)} \frac{s\Phi''(s)}{\Phi'(s)}.$$

LIMITATION OF THE COMPLEX CASE

Take

$$A \equiv e^{i\phi} I, \quad B \equiv e^{-i\phi} I$$

for some $\phi \in (-\pi/2, \pi/2)$, $\phi \neq 0$.

Orlicz-space bilinear embedding implies boundedness:

$$\sup_{t \in (0,\infty)} \| \exp(t e^{i\phi} \Delta) \|_{\mathsf{L}^\Phi \to \mathsf{L}^\Phi} \lesssim_{\Phi,\phi} 1.$$

However, a dimensionless bound

$$\sup_{t \in (0,\infty)} \| \exp(t e^{i\phi} \Delta) \|_{\mathsf{L}^p \to \mathsf{L}^p} \lesssim_{p,\phi} 1$$

can only hold when $|1 - 2/p| \leqslant \cos \phi$.

SKETCH OF THE PROOF

A *generalized Hessian* of a function $\mathfrak{X}\colon \mathbb{C}^2 \to \mathbb{R}$ w.r.t. matrices $A, B \in \mathrm{M}_d(\mathbb{C})$, denoted by

$$H_{\mathfrak{X}}^{A,B}[(u,v);(\zeta,\eta)],$$

is a dot product of

$$\left(\mathrm{Hess}(\mathfrak{X};(u,v)) \otimes I_d\right) \begin{bmatrix} \mathrm{Re}\,\zeta \\ \mathrm{Im}\,\zeta \\ \mathrm{Re}\,\eta \\ \mathrm{Im}\,\eta \end{bmatrix} \in (\mathbb{R}^d)^4$$

and

$$\begin{bmatrix} \mathrm{Re}\,A & -\mathrm{Im}\,A & \mathbf{0} & \mathbf{0} \\ \mathrm{Im}\,A & \mathrm{Re}\,A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{Re}\,B & -\mathrm{Im}\,B \\ \mathbf{0} & \mathbf{0} & \mathrm{Im}\,B & \mathrm{Re}\,B \end{bmatrix} \begin{bmatrix} \mathrm{Re}\,\zeta \\ \mathrm{Im}\,\zeta \\ \mathrm{Re}\,\eta \\ \mathrm{Im}\,\eta \end{bmatrix} \in (\mathbb{R}^d)^4.$$

Sketch of the proof

A function $\mathfrak{X}\colon \mathbb{C}^2 \to [0,\infty)$ will be determined later.
For $R > 0$ denote $\psi_R(x) := \psi(x/R)$ and define

$$\mathcal{E}_R(t) := \int_{\mathbb{R}^d} \psi_R(x)\, \mathfrak{X}\big((T_t^A f)(x), (T_t^B g)(x)\big)\, \mathrm{d}x.$$

On the one hand,

$$\begin{aligned}\mathcal{E}_R(0) - \mathcal{E}_R(\tau) \leqslant \mathcal{E}_R(0) &= \int_{\mathbb{R}^d} \psi_R(x)\, \mathfrak{X}\big(f(x), g(x)\big)\, \mathrm{d}x \\ &\lesssim_{\Phi,\Psi} \int_{\mathbb{R}^d} \psi_R(x)\, \big(\Phi(|f(x)|) + \Psi(|g(x)|)\big)\, \mathrm{d}x.\end{aligned}$$

One the other hand,

$$-\int_0^\tau \mathcal{E}_R'(t)\, \mathrm{d}t = -\int_0^\tau \int_{\mathbb{R}^d} \psi_R(x)\, \frac{\partial}{\partial t} \mathfrak{X}\big((T_t^A f)(x), (T_t^B g)(x)\big)\, \mathrm{d}x\, \mathrm{d}t.$$

Sketch of the proof

Continued:

$$-\int_0^\tau \mathcal{E}_R'(t)\,\mathrm{d}t$$
$$= \int_0^\tau \int_{\mathbb{R}^d} \psi_R(x)\, H_{\mathfrak{X}}^{A(x),B(x)}\left[\left((T_t^A f)(x), (T_t^B g)(x)\right); \left((\nabla T_t^A f)(x), (\nabla T_t^B g)(x)\right)\right] \mathrm{d}x\,\mathrm{d}t + \mathcal{R}_{R,\tau}$$
$$\gtrsim_{A,B,\Phi,\Psi} \int_0^\tau \int_{\mathbb{R}^d} \psi_R(x)\, \left|(\nabla T_t^A f)(x)\right| \left|(\nabla T_t^B g)(x)\right| \mathrm{d}x\,\mathrm{d}t + \mathcal{R}_{R,\tau},$$

where

$$\mathcal{R}_{R,\tau} = 2\,\mathrm{Re} \int_0^\tau \int_{\mathbb{R}^d} \Big((\partial_{\bar{u}}\mathfrak{X})\left((T_t^A f)(x), (T_t^B g)(x)\right) \left\langle (\nabla \psi_R)(x), A(x)(\nabla T_t^A f)(x) \right\rangle_{\mathbb{C}^d}$$
$$+ (\partial_{\bar{v}}\mathfrak{X})\left((T_t^A f)(x), (T_t^B g)(x)\right) \left\langle (\nabla \psi_R)(x), B(x)(\nabla T_t^B g)(x) \right\rangle_{\mathbb{C}^d} \Big)\, \mathrm{d}x\,\mathrm{d}t.$$

We want

$$\lim_{R\to\infty} \mathcal{R}_{R,\tau} = 0.$$

Sketch of the proof

Therefore

$$\int_0^\tau \int_{\mathbb{R}^d} \psi_R(x) \left|(\nabla T_t^A f)(x)\right| \left|(\nabla T_t^B g)(x)\right| \mathrm{d}x\, \mathrm{d}t + \mathcal{R}_{R,\tau}$$
$$\lesssim_{A,B,\Phi,\Psi} \int_{\mathbb{R}^d} \psi_R(x) \left(\Phi(|f(x)|) + \Psi(|g(x)|)\right) \mathrm{d}x.$$

Letting $R \to \infty$ we get

$$\int_0^\tau \int_{\mathbb{R}^d} \left|(\nabla T_t^A f)(x)\right| \left|(\nabla T_t^B g)(x)\right| \mathrm{d}x\, \mathrm{d}t \lesssim_{A,B,\Phi,\Psi} \int_{\mathbb{R}^d} \Phi(|f(x)|)\, \mathrm{d}x + \int_{\mathbb{R}^d} \Psi(|g(x)|)\, \mathrm{d}x.$$

Finally, letting $\tau \to \infty$ and "homogenizing"

$$f \to \alpha f, \quad g \to g/\alpha$$

we obtain

$$\int_0^\infty \int_{\mathbb{R}^d} \left|(\nabla T_t^A f)(x)\right| \left|(\nabla T_t^B g)(x)\right| \mathrm{d}x\, \mathrm{d}t \lesssim_{A,B,\Phi,\Psi} \|f\|_\Phi \|g\|_\Psi.$$

PROPERTIES OF THE DESIRED BELLMAN FUNCTION

The proof is complete if there exists a function $\mathfrak{X}\colon \mathbb{C}^2 \to [0,\infty\rangle$ with the following properties (the so-called *Bellman function*):

- $\mathfrak{X}$ is C^1 on $\mathbb{C}^2 \equiv \mathbb{R}^4$ and "piecewise" C^2 with locally integrable second derivatives;
- $\mathfrak{X}(u,v) \lesssim_{\Phi,\Psi} \Phi(|u|) + \Psi(|v|)$;
- $H_{\mathfrak{X}}^{A(x),B(x)}[(u,v);(\zeta,\eta)] \gtrsim_{\Phi,\Psi} |\zeta||\eta|$;
- $|\partial_{\bar{u}}\mathfrak{X}(u,v)| \leqslant \max\{\Phi'(|u|),|v|\}$,
  $|\partial_{\bar{v}}\mathfrak{X}(u,v)| \leqslant \Psi'(|v|)$,
  where $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$.

EXISTENCE OF THE DESIRED BELLMAN FUNCTION

We define $\mathfrak{X}$ by the formula

$$\mathfrak{X}(u,v) := \begin{cases} (1+\delta)\big(\Phi(|u|)+\Psi(|v|)\big)+\delta|u|^2\displaystyle\int_0^{|u|}\frac{\Phi'(s)\,\mathrm{d}s}{s^2}; & |v|\leqslant\Phi'(|u|),\\ \Phi(|u|)+\Psi(|v|)+\delta|u|^2\displaystyle\int_0^{|v|}\frac{\mathrm{d}s}{\Psi'(s)}; & |v|>\Phi'(|u|),\end{cases}$$

for an appropriate $\delta > 0$.

A good choice is

$$\delta := \frac{\tilde m-1}{\tilde m}\min\left\{\frac{\Delta_p(A)}{8\Lambda(A)},\frac{\Delta_p(B)}{4\Lambda(B)},\frac{\lambda(A)\Delta_p(B)}{100\max\{\Lambda(A)^2,\Lambda(B)^2\}}\right\}.$$

The verification is tedious.

A trilinear embedding

TRILINEAR EMBEDDING

What if we have three semigroups generated by three elliptic div-form operators?

We would like to have a paraproduct-type estimate involving

$$T_t^A f, \quad \nabla T_t^B g, \quad \nabla T_t^C h.$$

What conditions on A, B, C are needed to guarantee an estimate on $\mathrm{L}^p(\mathbb{R}^d) \times \mathrm{L}^q(\mathbb{R}^d) \times \mathrm{L}^r(\mathbb{R}^d)$?

TRILINEAR EMBEDDING

Take $p, q, r \in (1, \infty)$ such that $1/p + 1/q + 1/r = 1$.

Theorem [Carbonaro–Dragičević–K.–Škreb, 2020].
Suppose that $A, B, C \colon \mathbb{R}^d \to \mathrm{M}_d(\mathbb{C})$ are matrix functions such that
A is p-elliptic,
B is q-elliptic and $(1 + q/r)$-elliptic,
C is r-elliptic and $(1 + r/q)$-elliptic.
Then we have

$$\int_0^\infty \int_{\mathbb{R}^d} \left|(T_t^A f)(x)\right| \left|(\nabla T_t^B g)(x)\right| \left|(\nabla T_t^C h)(x)\right| \mathrm{d}x\, \mathrm{d}t \leqslant \mathcal{C} \left\|f\right\|_{\mathrm{L}^p(\mathbb{R}^d)} \left\|g\right\|_{\mathrm{L}^q(\mathbb{R}^d)} \left\|h\right\|_{\mathrm{L}^r(\mathbb{R}^d)} .$$

The embedding constant $\mathcal{C}$ only depends on p, q, r and the implied ellipticity constants of A, B, C.

PROPERTIES OF THE DESIRED BELLMAN FUNCTION

For the proof we need a function $\mathfrak{X}\colon \mathbb{C}^3 \to [0,\infty\rangle$ with the following properties:

- $\mathfrak{X}$ is C[1] on $\mathbb{C}^3 \equiv \mathbb{R}^6$ and "piecewise" C[2] with locally integrable second derivatives;
- $\mathfrak{X}(u,v,w) \lesssim_{p,q,r} |u|^p + |v|^q + |w|^r$;
- $H_{\mathfrak{X}}^{A(x),B(x),C(x)}[(u,v,w);(\zeta,\eta,\xi)] \gtrsim_{p,q,r} |u||\eta||\xi|$;
- $|\partial_{\bar{u}}\mathfrak{X}(u,v,w)| \lesssim |u|^{p-1}$,
  $|\partial_{\bar{v}}\mathfrak{X}(u,v,w)| \lesssim \max\{|u|^p,|v|^q,|w|^r\}^{1-1/q}$,
  $|\partial_{\bar{w}}\mathfrak{X}(u,v,w)| \lesssim \max\{|u|^p,|v|^q,|w|^r\}^{1-1/r}$.

Construction of the desired Bellman function

WLOG assume $q > r$ and use the ansatz:

$$\mathfrak{X}(u,v,w) = |u|^p\,\gamma\Big(\underbrace{\frac{|v|^q}{|u|^p}}_{t},\underbrace{\frac{|w|^r}{|u|^p}}_{s}\Big).$$

$$\gamma(t,s) = \begin{cases} a_1 + b_1 t + c_1 s; & 1 \leqslant s \leqslant t, \\ a_2 + b_2 t + c_2 s^{\frac{1}{p'}}; & s \leqslant 1 \leqslant t, \\ a_3 + b_3 t^{\frac{1}{p'}} + c_3 s^{\frac{1}{p'}}; & s \leqslant t \leqslant 1, \\ a_4 + b_4 t^{\frac{2}{q}} s^{\frac{1}{r}-\frac{1}{q}} + c_4 s^{\frac{1}{p'}}; & t \leqslant s \leqslant 1, \\ a_5 + b_5 t^{\frac{2}{q}} + c_5 t^{\frac{2}{q}} s^{1-\frac{2}{q}} + d_5 s; & t \leqslant 1 \leqslant s, \\ a_6 + b_6 t + c_6 t^{\frac{2}{q}} s^{1-\frac{2}{q}} + d_6 s; & 1 \leqslant t \leqslant s. \end{cases}$$

Adjust the coefficients so that γ is C^1 on $(0,\infty)^2$.

CONSTRUCTION OF THE DESIRED BELLMAN FUNCTION

$$\gamma(t,s)=\begin{cases} a+bt+cs; & 1\leqslant s\leqslant t,\\ \frac{a(p-1)-c}{p-1}+bt+\frac{cp}{p-1}s^{\frac{1}{p'}}; & s\leqslant 1\leqslant t,\\ \frac{a(p-1)-(b+c)}{p-1}+\frac{bp}{p-1}t^{\frac{1}{p'}}+\frac{cp}{p-1}s^{\frac{1}{p'}}; & s\leqslant t\leqslant 1,\\ \frac{a(p-1)-(b+c)}{p-1}+\frac{bq}{2}t^{\frac{2}{q}}s^{\frac{1}{r}-\frac{1}{q}}+\frac{2cpr-bp(q-r)}{2r(p-1)}s^{\frac{1}{p'}}; & t\leqslant s\leqslant 1,\\ \frac{2ar(p-1)-b(q+r)}{2r(p-1)}+\frac{bq^2}{2p(q-2)}t^{\frac{2}{q}}+\frac{bq(q-r)}{2r(q-2)}t^{\frac{2}{q}}s^{1-\frac{2}{q}} & \\ \qquad +\frac{2cr-b(q-r)}{2r}s; & t\leqslant 1\leqslant s,\\ a+\frac{bq}{p(q-2)}t+\frac{bq(q-r)}{2r(q-2)}t^{\frac{2}{q}}s^{1-\frac{2}{q}}+\frac{2cr-b(q-r)}{2r}s; & 1\leqslant t\leqslant s. \end{cases}$$

Choose a,b,c appropriately.

Again similar to the function constructed by Nazarov and Treil (1996), used in the L^p bilinear embedding.

Square functions

SQUARE FUNCTION ESTIMATES

“Vertical” (Littlewood–Paley–Stein) square function, defined as

$$(\mathcal{G}^A f)(x) := \left(\int_0^\infty \left|(\nabla T_t^A f)(x)\right|^2 \, \mathrm{d}t \right)^{1/2},$$

is only bounded in a very restrictive range of exponents $(p_-(A), p_+(A))$.

Even for real elliptic A this range could be only $(1, 2+\varepsilon)$.

(Auscher, 2007)

Consequently, there are no easy (maximal–square–square) shortcuts to trilinear embeddings.

Square function estimates

"Conical" square function, defined as

$$(\mathcal{C}^A f)(x) := \left(\iint_{\{|x-y|<\sqrt{t}\}} \left| \nabla (T_t^A f)(y) \right|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{d/2}} \right)^{1/2},$$

is bounded in the full range of exponents $(1,\infty)$ for real elliptic A.

(Auscher–Hofmann–Martell, 2012)

The trilinear embedding reproves and sharpens their result.

AN APPLICATION OF THE TRILINEAR EMBEDDING

Consider the *modified square function*, defined as

$$(\widetilde{\mathcal{G}}^A f)(x) := \left(\int_0^\infty T_t^I \left(\left| \nabla T_t^A f \right|^2 \right)(x) \, \mathrm{d}t \right)^{1/2}.$$

From the formula for the heat kernel on $\mathbb{R}^d$:

$$\mathcal{C}^A f \leqslant e^{1/8} (4\pi)^{d/4} \widetilde{\mathcal{G}}^A f.$$

For $p \in (2, \infty)$, from the trilinear embedding applied to matrix functions I, A, A and exponents $p/(p-2), p, p$ we obtain the bound

$$\left\| \widetilde{\mathcal{G}}^A f \right\|_{\mathrm{L}^p(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

under the condition that A is p-elliptic. (A “dimensionless” estimate.)

Thank you for your attention!