Bellman functions and L^p estimates for paraproducts

Vjekoslav Kovač (University of Zagreb)

Joint work with *Kristina Ana Škreb* (University of Zagreb)

Harmonic analysis, complex analysis, spectral theory and all that

Będlewo, August 2, 2016

Dyadic paraproduct

Dyadic paraproduct

As a bilinear operator:

$$\Pi(f,g) := \sum_{I\in\mathcal{D}} \epsilon_I \left\langle f, \frac{\mathbb{1}_I}{|I|^{1/2}} \right\rangle \frac{\mathbb{1}_I}{|I|^{1/2}} \left\langle g, \frac{\mathbb{h}_I}{|I|^{1/2}} \right\rangle \frac{\mathbb{h}_I}{|I|^{1/2}}$$
$$= \sum_{k\in\mathbb{Z}} (E_k f)(\Delta_k g)$$

$\mathcal{D}$ = dyadic intervals $\subset \mathbb{R}$, $\quad \mathbb{h}_I = \mathbb{1}_{I_{\text{left}}} - \mathbb{1}_{I_{\text{right}}}, \quad |\epsilon_I| \leq 1$

Dyadic paraproduct

As a bilinear operator:

$$\Pi(f,g) := \sum_{I\in\mathcal{D}} \epsilon_I \left\langle f, \frac{\mathbb{1}_I}{|I|^{1/2}} \right\rangle \frac{\mathbb{1}_I}{|I|^{1/2}} \left\langle g, \frac{\mathbb{h}_I}{|I|^{1/2}} \right\rangle \frac{\mathbb{h}_I}{|I|^{1/2}}$$
$$= \sum_{k\in\mathbb{Z}} (E_k f)\,(\Delta_k g)$$

$\mathcal{D} =$ dyadic intervals $\subset \mathbb{R}$, $\quad \mathbb{h}_I = \mathbb{1}_{I_{\text{left}}} - \mathbb{1}_{I_{\text{right}}}$, $\quad |\epsilon_I| \le 1$

- $f \mapsto \Pi(f,g)$ is usually called the (linear) paraproduct
- $g \mapsto \Pi(f,g)$ is the martingale transform or the Haar multiplier

Dyadic paraproduct

As a bilinear operator:

$$\Pi(f,g) := \sum_{I\in\mathcal{D}} \epsilon_I \left\langle f, \frac{\mathbb{1}_I}{|I|^{1/2}} \right\rangle \frac{\mathbb{1}_I}{|I|^{1/2}} \left\langle g, \frac{\mathbb{h}_I}{|I|^{1/2}} \right\rangle \frac{\mathbb{h}_I}{|I|^{1/2}}$$
$$= \sum_{k\in\mathbb{Z}} (E_k f)\,(\Delta_k g)$$

$\mathcal{D} =$ dyadic intervals $\subset \mathbb{R}$, $\quad \mathbb{h}_I = \mathbb{1}_{I_{\text{left}}} - \mathbb{1}_{I_{\text{right}}}$, $\quad |\epsilon_I| \le 1$

- $f \mapsto \Pi(f,g)$ is usually called the (linear) paraproduct
- $g \mapsto \Pi(f,g)$ is the martingale transform or the Haar multiplier

As a trilinear form:

$$\Lambda(f,g,h) := \sum_{I\in\mathcal{D}} \epsilon_I |I|^{-2} \langle f, \mathbb{1}_I\rangle\langle g, \mathbb{h}_I\rangle\langle h, \mathbb{h}_I\rangle$$
$$= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} (E_k f)\,(\Delta_k g)\,(\Delta_k h)$$

Dyadic paraproduct

L^p estimates (in the Banach range only):

$$|\Lambda(f,g,h)| \leq C_{p,q,r}\|f\|_{\mathrm{L}^p}\|g\|_{\mathrm{L}^q}\|h\|_{\mathrm{L}^r}$$

hold whenever $1 < p,q,r \leq \infty,\ \ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

Dyadic paraproduct

L^p estimates (in the Banach range only):

$$|\Lambda(f,g,h)| \leq C_{p,q,r}\|f\|_{\mathrm{L}^p}\|g\|_{\mathrm{L}^q}\|h\|_{\mathrm{L}^r}$$

hold whenever $1 < p,q,r \leq \infty,\ \ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

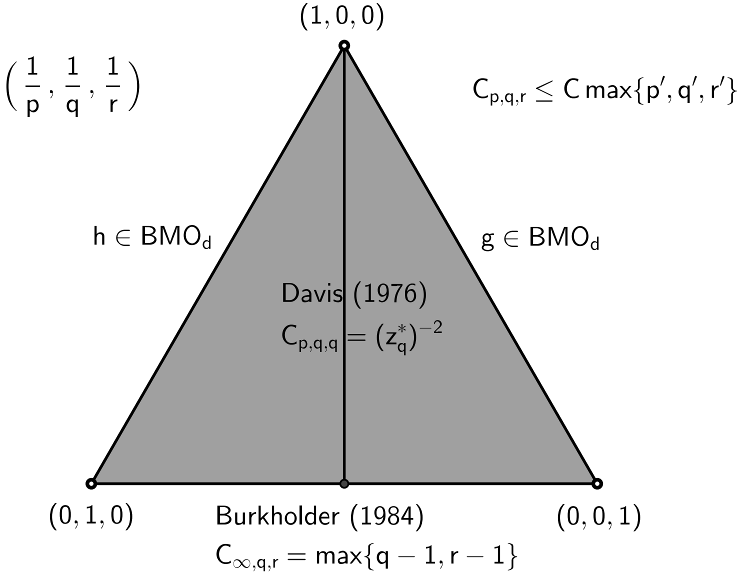
The easiest proof (when $q,r<\infty$):

$$\begin{aligned}|\Lambda(f,g,h)| &\leq \int_{\mathbb{R}} (\mathrm{M}f)\,(\mathrm{S}g)\,(\mathrm{S}h)\\ &\leq \|\mathrm{M}f\|_{\mathrm{L}^p}\|\mathrm{S}g\|_{\mathrm{L}^q}\|\mathrm{S}h\|_{\mathrm{L}^r}\end{aligned}$$

where

- $\mathrm{M}f := \sup_{I\in\mathcal{D}} |I|^{-1}\langle f,\mathbb{1}_I\rangle\mathbb{1}_I$ is the dyadic maximal function
- $\mathrm{S}f := \Big(\sum_{I\in\mathcal{D}} |I|^{-2}|\langle f,\mathbb{h}_I\rangle|^2\mathbb{1}_I\Big)^{1/2}$ is the dyadic square function

Dyadic paraproduct



The Bellman function restatement

The Bellman function restatement

Assume that $f, g, h \geq 0$
In terms of the averages:

$$\Phi_I(f,g,h) := \frac{1}{|I|} \sum_{J \subseteq I} |J| \, \langle f \rangle_J \;\; \frac{1}{2} \left| \langle g \rangle_{J_{\mathrm{left}}} - \langle g \rangle_{J_{\mathrm{right}}} \right| \;\; \frac{1}{2} \left| \langle h \rangle_{J_{\mathrm{left}}} - \langle h \rangle_{J_{\mathrm{right}}} \right|$$

The Bellman function restatement

Assume that $f, g, h \geq 0$
In terms of the averages:

$$\Phi_I(f,g,h) := \frac{1}{|I|}\sum_{J\subseteq I} |J| \left\langle f \right\rangle_J \ \frac{1}{2}\left|\left\langle g \right\rangle_{J_{\text{left}}} - \left\langle g \right\rangle_{J_{\text{right}}}\right| \ \frac{1}{2}\left|\left\langle h \right\rangle_{J_{\text{left}}} - \left\langle h \right\rangle_{J_{\text{right}}}\right|$$

The estimate:

$$\Phi_I(f,g,h) \leq C_{p,q,r}\left(\tfrac{1}{p}\left\langle f^p \right\rangle_I + \tfrac{1}{q}\left\langle g^q \right\rangle_I + \tfrac{1}{r}\left\langle h^r \right\rangle_I \right)$$

The Bellman function restatement

Assume that $f, g, h \geq 0$
In terms of the averages:

$$\Phi_I(f,g,h) := \frac{1}{|I|} \sum_{J \subseteq I} |J| \left\langle f \right\rangle_J \; \frac{1}{2} \left| \left\langle g \right\rangle_{J_{\text{left}}} - \left\langle g \right\rangle_{J_{\text{right}}} \right| \; \frac{1}{2} \left| \left\langle h \right\rangle_{J_{\text{left}}} - \left\langle h \right\rangle_{J_{\text{right}}} \right|$$

The estimate:

$$\Phi_I(f,g,h) \leq C_{p,q,r} \left(\tfrac{1}{p} \left\langle f^p \right\rangle_I + \tfrac{1}{q} \left\langle g^q \right\rangle_I + \tfrac{1}{r} \left\langle h^r \right\rangle_I \right)$$

The abstract Bellman function:

$$\mathcal{B}(u,v,w,U,V,W) := \sup_{f,g,h} \Phi_I(f,g,h),$$

where the supremum is taken over all $f, g, h \geq 0$ s.t.

$$\left\langle f \right\rangle_I = u, \; \left\langle g \right\rangle_I = v, \; \left\langle h \right\rangle_I = w, \; \left\langle f^p \right\rangle_I = U, \; \left\langle g^q \right\rangle_I = V, \; \left\langle h^r \right\rangle_I = W$$

The Bellman function restatement

($\mathcal{B}$1) *Domain*:

$$u, v, w, U, V, W \geq 0, \quad u^p \leq U, \quad v^q \leq V, \quad w^r \leq W$$

The Bellman function restatement

($\mathcal{B}1$) *Domain*:

$$u, v, w, U, V, W \geq 0, \quad u^p \leq U, \quad v^q \leq V, \quad w^r \leq W$$

($\mathcal{B}2$) *Range*:

$$0 \leq \mathcal{B}(u, v, w, U, V, W) \leq C_{p,q,r}\left(\tfrac{1}{p}U + \tfrac{1}{q}V + \tfrac{1}{r}W\right)$$

The Bellman function restatement

($\mathcal{B}$1) *Domain*:

$$u, v, w, U, V, W \geq 0, \quad u^p \leq U, \quad v^q \leq V, \quad w^r \leq W$$

($\mathcal{B}$2) *Range*:

$$0 \leq \mathcal{B}(u, v, w, U, V, W) \leq C_{p,q,r}\big(\tfrac{1}{p}U + \tfrac{1}{q}V + \tfrac{1}{r}W \big)$$

($\mathcal{B}$3) *The main inequality*:

$$\begin{aligned} \mathcal{B}(u, v, w, U, V, W) \geq\ & \frac{1}{2}\mathcal{B}(u_1, v_1, w_1, U_1, V_1, W_1) \\ & + \frac{1}{2}\mathcal{B}(u_2, v_2, w_2, U_2, V_2, W_2) \\ & + \frac{1}{2}(u_1 + u_2)\ \frac{1}{2}|v_1 - v_2|\ \frac{1}{2}|w_1 - w_2| \end{aligned}$$

whenever $u = \frac{1}{2}(u_1 + u_2)$, $v = \frac{1}{2}(v_1 + v_2)$, etc. and all 6-tuples belong to the domain

The Bellman function restatement

Substitute $\triangle u = \frac{1}{2}(u_1 - u_2)$, etc.

Assume that $\mathcal{B}$ is C^1 and "piecewise" C^2:

($\mathcal{B}3'$) *Infinitesimal version*:

$$-\frac{1}{2}(\underbrace{d^2\mathcal{B}}_{\text{quadratic form}})\underbrace{(u,v,w,U,V,W)}_{\text{at a point}}\underbrace{(\triangle u,\triangle v,\triangle w,\triangle U,\triangle V,\triangle W)}_{\text{on a vector}} \geq u|\triangle v||\triangle w|$$

The Bellman function restatement

Substitute $\triangle u = \frac{1}{2}(u_1 - u_2)$, etc.

Assume that $\mathcal{B}$ is C^1 and "piecewise" C^2:

$(\mathcal{B}3')$ *Infinitesimal version*:

$$-\frac{1}{2}(\underbrace{d^2\mathcal{B}}_{\text{quadratic form}})\underbrace{(u, v, w, U, V, W)}_{\text{at a point}}\underbrace{(\triangle u, \triangle v, \triangle w, \triangle U, \triangle V, \triangle W)}_{\text{on a vector}} \geq u|\triangle v||\triangle w|$$

$$(\mathcal{B}3) \overset{\text{Taylor's formula}}{\Longrightarrow} (\mathcal{B}3')$$

$$(\mathcal{B}3') \overset{\text{convexity of the domain}}{\Longrightarrow} (\mathcal{B}3)$$

Sufficiency of ($\mathcal{B}1$)–($\mathcal{B}3$)

The Bellman function restatement

Sufficiency of $(\mathcal{B}1)$–$(\mathcal{B}3)$

Apply $(\mathcal{B}3)$ n times:

$$|I|\,\mathcal{B}\big(\langle f\rangle_I,\langle g\rangle_I,\langle h\rangle_I,\langle f^p\rangle_I,\langle g^q\rangle_I,\langle h^r\rangle_I\big)$$

$$\geq \sum_{\substack{J\subseteq I\\ |J|=2^{-n}|I|}} |J|\,\underbrace{\mathcal{B}\big(\langle f\rangle_J,\langle g\rangle_J,\langle h\rangle_J,\langle f^p\rangle_J,\langle g^q\rangle_J,\langle h^r\rangle_J\big)}_{\geq 0}$$

$$+\sum_{\substack{J\subseteq I\\ |J|>2^{-n}|I|}} \langle f\rangle_J\;\frac{1}{2}\big|\langle g\rangle_{J_{\text{left}}}-\langle g\rangle_{J_{\text{right}}}\big|\;\frac{1}{2}\big|\langle h\rangle_{J_{\text{left}}}-\langle h\rangle_{J_{\text{right}}}\big|$$

The Bellman function restatement

Sufficiency of $(\mathcal{B}1)$–$(\mathcal{B}3)$

Apply $(\mathcal{B}3)$ n times:

$$
\begin{aligned}
&|I|\,\mathcal{B}\big(\langle f\rangle_I, \langle g\rangle_I, \langle h\rangle_I, \langle f^p\rangle_I, \langle g^q\rangle_I, \langle h^r\rangle_I\big) \\
&\geq \sum_{\substack{J\subseteq I \\ |J|=2^{-n}|I|}} |J|\, \underbrace{\mathcal{B}\big(\langle f\rangle_J, \langle g\rangle_J, \langle h\rangle_J, \langle f^p\rangle_J, \langle g^q\rangle_J, \langle h^r\rangle_J\big)}_{\geq 0} \\
&\quad + \sum_{\substack{J\subseteq I \\ |J|>2^{-n}|I|}} \langle f\rangle_J\ \frac{1}{2}\big|\langle g\rangle_{J_{\text{left}}} - \langle g\rangle_{J_{\text{right}}}\big|\ \frac{1}{2}\big|\langle h\rangle_{J_{\text{left}}} - \langle h\rangle_{J_{\text{right}}}\big|
\end{aligned}
$$

Use $(\mathcal{B}2)$ and let $n \to \infty$

Martingale paraproduct

Martingale paraproduct

Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be continuous-time martingales w.r.t. the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$

The martingale paraproduct $\rightsquigarrow$ Bañuelos & Bennett (1988):

$$\mathcal{P}(X, Y) := \int_0^\infty X_s dY_s$$

Established L^p, H^p, BMO estimates

Martingale paraproduct

Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be continuous-time martingales w.r.t. the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$

The martingale paraproduct $\rightsquigarrow$ Bañuelos & Bennett (1988):

$$\mathcal{P}(X, Y) := \int_0^\infty X_s dY_s$$

Established L^p, H^p, BMO estimates

Consider the process

$$(X \cdot Y)_t := \int_0^t X_s dY_s$$

Take Z and define $Z_s := \mathbb{E}(Z|\mathcal{F}_s)$

Martingale paraproduct

Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be continuous-time martingales w.r.t. the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$

The martingale paraproduct $\rightsquigarrow$ Bañuelos & Bennett (1988):

$$\mathcal{P}(X, Y) := \int_0^\infty X_s dY_s$$

Established L^p, H^p, BMO estimates

Consider the process

$$(X \cdot Y)_t := \int_0^t X_s dY_s$$

Take Z and define $Z_s := \mathbb{E}(Z|\mathcal{F}_s)$

As a trilinear form:

$$\Lambda_t(X, Y, Z) := \mathbb{E}\big((X \cdot Y)_t Z\big) = \mathbb{E}\big((X \cdot Y)_t (Z_t - Z_0)\big)$$

Martingale paraproduct

Using Itō's isometry:

$$\Lambda_t(X, Y, Z) = \mathbb{E}\Big(\Big(\int_0^t X_s dY_s\Big)\Big(\int_0^t 1 dZ_s\Big)\Big)$$
$$= \mathbb{E}\int_0^t X_s d\langle Y, Z\rangle_s$$

$\langle Y, Z\rangle_t$ is the predictable quadratic covariation process
$\rightsquigarrow$ coincides with the quadratic covariation $[Y, Z]_t$

Martingale paraproduct

Using Itō's isometry:

$$\Lambda_t(X,Y,Z) = \mathbb{E}\Big(\Big(\int_0^t X_s dY_s\Big)\Big(\int_0^t 1 dZ_s\Big)\Big)$$
$$= \mathbb{E}\int_0^t X_s d\langle Y,Z\rangle_s$$

$\langle Y,Z\rangle_t$ is the predictable quadratic covariation process
$\rightsquigarrow$ coincides with the quadratic covariation $[Y,Z]_t$

Let us show the estimate

$$|\Lambda_t(X,Y,Z)| \le C_{p,q,r}\|X_t\|_{\mathrm{L}^p}\|Y_t\|_{\mathrm{L}^q}\|Z_t\|_{\mathrm{L}^r}$$

for $1 < p,q,r < \infty$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$

A simple proof uses Doob's inequality and the Burkholder-Gundy inequality

Martingale paraproduct

Assume $X_s, Y_s, Z_s \geq 0$ (or split into $+$ and $-$ parts)

Also introduce martingales (for a fixed t and for $s \in [0, t]$):

$$U_s := \mathbb{E}(X_t^p | \mathcal{F}_s), \quad V_s := \mathbb{E}(Y_t^q | \mathcal{F}_s), \quad W_s := \mathbb{E}(Z_t^r | \mathcal{F}_s)$$

Martingale paraproduct

Assume $X_s, Y_s, Z_s \geq 0$ (or split into $+$ and $-$ parts)
Also introduce martingales (for a fixed t and for $s \in [0, t]$):

$$U_s := \mathbb{E}(X_t^p | \mathcal{F}_s), \quad V_s := \mathbb{E}(Y_t^q | \mathcal{F}_s), \quad W_s := \mathbb{E}(Z_t^r | \mathcal{F}_s)$$

We use Itō's formula (assuming for simplicity that $\mathcal{B} \in \mathrm{C}^2$):

$$\mathcal{B}(\mathbf{X}_t) - \mathcal{B}(\mathbf{X}_0) = \underbrace{\sum_i \int_0^t \partial_i \mathcal{B}(\mathbf{X}_s) dX_s^i}_{\text{the martingale part}} + \frac{1}{2} \sum_{i,j} \int_0^t \partial_i \partial_j \mathcal{B}(\mathbf{X}_s) d\langle X^i, X^j \rangle_s$$

$$\Longrightarrow \mathbb{E}\big(\mathcal{B}(\mathbf{X}_t) - \mathcal{B}(\mathbf{X}_0)\big) = \mathbb{E}\Big(\int_0^t \frac{1}{2} \sum_{i,j} \partial_i \partial_j \mathcal{B}(\mathbf{X}_s) d\langle X^i, X^j \rangle_s \Big)$$

for $\mathbf{X}_s = (X_s, Y_s, Z_s, U_s, V_s, W_s)$

Martingale paraproduct

Use the martingale representation theorem:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{A}_s dB_t,$$

where B_t is the 1D Brownian motion and $\mathbf{A}_t$ is predictable

Martingale paraproduct

Use the martingale representation theorem:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{A}_s dB_t,$$

where B_t is the 1D Brownian motion and $\mathbf{A}_t$ is predictable

Since $d\langle X^i, X^j\rangle_s = A_s^i A_s^j ds$:

$$\mathbb{E}\int_0^t \Big(-\frac{1}{2} \underbrace{\sum_{i,j} \partial_i \partial_j \mathcal{B}(\mathbf{X}_s) A_s^i A_s^j}_{(d^2\mathcal{B})(\mathbf{X}_s)(\mathbf{A}_s)} \Big) ds = \mathbb{E}\mathcal{B}(\mathbf{X}_0) - \underbrace{\mathbb{E}\mathcal{B}(\mathbf{X}_t)}_{\geq 0}$$

Martingale paraproduct

Use the martingale representation theorem:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{A}_s dB_t,$$

where B_t is the 1D Brownian motion and $\mathbf{A}_t$ is predictable

Since $d\langle X^i, X^j\rangle_s = A_s^i A_s^j ds$:

$$\mathbb{E}\int_0^t \Big(-\frac{1}{2} \underbrace{\sum_{i,j} \partial_i \partial_j \mathcal{B}(\mathbf{X}_s) A_s^i A_s^j}_{(d^2\mathcal{B})(\mathbf{X}_s)(\mathbf{A}_s)} \Big) ds = \mathbb{E}\mathcal{B}(\mathbf{X}_0) - \underbrace{\mathbb{E}\mathcal{B}(\mathbf{X}_t)}_{\geq 0}$$

Using $(\mathcal{B}3')$ and $(\mathcal{B}2)$:

$$\pm\mathbb{E}\int_0^t \underbrace{X_s^1 A_s^2 A_s^3 ds}_{X_s d\langle Y, Z\rangle_s} \leq \mathbb{E}\mathcal{B}(\mathbf{X}_0) \leq C_{p,q,r}\Big(\frac{1}{p} \underbrace{\mathbb{E}U_0}_{\|X_t\|_{\mathrm{L}^p}^p} + \frac{1}{q} \underbrace{\mathbb{E}V_0}_{\|Y_t\|_{\mathrm{L}^q}^q} + \frac{1}{r} \underbrace{\mathbb{E}W_0}_{\|Z_t\|_{\mathrm{L}^r}^r} \Big)$$

Martingale paraproduct

For general martingales:

- Consider

$$\sum_{k=1}^{\infty} X_{k-1}(Y_k - Y_{k-1}) \quad \text{or} \quad \int_0^{\infty} X_{s-}\, dY_s$$

Martingale paraproduct

For general martingales:

- Consider

$$\sum_{k=1}^{\infty} X_{k-1}(Y_k - Y_{k-1}) \quad \text{or} \quad \int_0^{\infty} X_{s-}\, dY_s$$

- $\mathcal{B}$ also works but not so directly

Martingale paraproduct

For general martingales:

- Consider

$$\sum_{k=1}^{\infty} X_{k-1}(Y_k - Y_{k-1}) \quad \text{or} \quad \int_0^{\infty} X_{s-}\, dY_s$$

- $\mathcal{B}$ also works but not so directly
- The optimal constant $C_{p,q,r}$ increases
  - at least when $1 < p < \infty$, $q = r$
  - unlike with Burkholder's martingale transform

Heat flow paraproduct

Heat flow paraproduct

$k(x,t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ the heat kernel on the line

Let u, v, w be the heat extensions of f, g, h:

$$u(x,t) := \int_{\mathbb{R}} f(y) k(x-y,t) dy, \text{ etc.}$$

$$\Lambda(f,g,h) := \int_{\mathbb{R}} \int_0^\infty u(x,t)\, \partial_x v(x,t)\, \partial_x w(x,t)\, dt\, dx$$

Heat flow paraproduct

$k(x,t) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ the heat kernel on the line

Let u, v, w be the heat extensions of f, g, h:

$$u(x,t) := \int_{\mathbb{R}} f(y) k(x-y,t) dy, \text{ etc.}$$

$$\Lambda(f,g,h) := \int_{\mathbb{R}} \int_0^\infty u(x,t)\, \partial_x v(x,t)\, \partial_x w(x,t)\, dt\, dx$$

We get a more familiar expression if we substitute

$$\varphi_s(x) := k(x,s^2), \quad \psi_s(x) := -2^{1/2} s\, \partial_x k(x,s^2)$$

$$\Lambda(f,g,h) = \int_{\mathbb{R}} \int_0^\infty (f * \varphi_s)(x)\, (g * \psi_s)(x)\, (h * \psi_s)(x)\, \frac{ds}{s}\, dx$$

Heat flow paraproduct

Imitating the "heating" technique from Petermichl & Volberg (2002) or Nazarov & Volberg (2003)

Heat flow paraproduct

Imitating the "heating" technique from Petermichl & Volberg (2002) or Nazarov & Volberg (2003)

Assume $f, g, h \geq 0$ and assume for simplicity that $\mathcal{B} \in \mathrm{C}^2$
Let also U, V, W be the heat extensions of f^p, g^q, h^r and define

$$b(x,t) := \mathcal{B}\big(u(x,t), v(x,t), w(x,t), U(x,t), V(x,t), W(x,t)\big)$$

$$\implies \big(\partial_t - \tfrac{1}{2}\partial_x^2\big) b(x,t) = (\nabla\mathcal{B})(u,v,\ldots) \cdot \underbrace{\big(\partial_t - \tfrac{1}{2}\partial_x^2\big)(u,v,\ldots)}_{=0}$$

$$- \tfrac{1}{2}(d^2\mathcal{B})(u,v,\ldots)(\partial_x u, \partial_x v, \ldots)$$

Heat flow paraproduct

Imitating the “heating” technique from Petermichl & Volberg (2002) or Nazarov & Volberg (2003)

Assume $f, g, h \geq 0$ and assume for simplicity that $\mathcal{B} \in \mathrm{C}^2$
Let also U, V, W be the heat extensions of f^p, g^q, h^r and define

$$b(x,t) := \mathcal{B}\big(u(x,t), v(x,t), w(x,t), U(x,t), V(x,t), W(x,t)\big)$$

$$\implies \big(\partial_t - \tfrac{1}{2}\partial_x^2\big) b(x,t) = (\nabla\mathcal{B})(u,v,\dots)\cdot \underbrace{\big(\partial_t - \tfrac{1}{2}\partial_x^2\big)(u,v,\dots)}_{=0} - \tfrac{1}{2}(d^2\mathcal{B})(u,v,\dots)(\partial_x u, \partial_x v, \dots)$$

Using ($\mathcal{B}3'$) we get

$$\pm u(x,t)\,\partial_x v(x,t)\,\partial_x w(x,t) \leq \big(\partial_t - \tfrac{1}{2}\partial_x^2\big) b(x,t)$$

Heat flow paraproduct

Integrating by parts and using $\mathcal{B} \geq 0$ we get for $\delta, T > 0$:

$$\pm \int_{\mathbb{R}\times(\delta,T-\delta)} k(x, T-t)\, u(x,t)\, \partial_x v(x,t)\, \partial_x w(x,t)\, dxdt$$

$$\leq \int_{\mathbb{R}} k(x,\delta) b(x, T-\delta) dx$$

Heat flow paraproduct

Integrating by parts and using $\mathcal{B} \geq 0$ we get for $\delta, T > 0$:

$$\pm \int_{\mathbb{R}\times(\delta,T-\delta)} k(x,T-t)\,u(x,t)\,\partial_x v(x,t)\,\partial_x w(x,t)\,dxdt$$

$$\leq \int_{\mathbb{R}} k(x,\delta) b(x,T-\delta)dx$$

Letting $\delta \to 0$ and using ($\mathcal{B}$2):

$$\pm \int_{\mathbb{R}\times(0,T)} k(x,T-t)\,u(x,t)\,\partial_x v(x,t)\,\partial_x w(x,t)\,dxdt \leq b(0,T)$$

$$\leq \frac{C_{p,q,r}}{\sqrt{2\pi T}} \Big(\int_{\mathbb{R}} f(y)^p e^{-\frac{y^2}{2T}} dy \Big)^{1/p} \Big(\int_{\mathbb{R}} g(y)^q e^{-\frac{y^2}{2T}} dy \Big)^{1/q} \Big(\int_{\mathbb{R}} h(y)^r e^{-\frac{y^2}{2T}} dy \Big)^{1/r}$$

Heat flow paraproduct

Integrating by parts and using $\mathcal{B} \geq 0$ we get for δ, $T > 0$:

$$\pm \int_{\mathbb{R}\times(\delta,T-\delta)} k(x,T-t)\,u(x,t)\,\partial_x v(x,t)\,\partial_x w(x,t)\,dxdt$$
$$\leq \int_{\mathbb{R}} k(x,\delta) b(x,T-\delta) dx$$

Letting $\delta \to 0$ and using ($\mathcal{B}2$):

$$\pm \int_{\mathbb{R}\times(0,T)} k(x,T-t)\,u(x,t)\,\partial_x v(x,t)\,\partial_x w(x,t)\,dxdt \leq b(0,T)$$
$$\leq \frac{C_{p,q,r}}{\sqrt{2\pi T}} \Big(\int_{\mathbb{R}} f(y)^p e^{-\frac{y^2}{2T}} dy \Big)^{1/p} \Big(\int_{\mathbb{R}} g(y)^q e^{-\frac{y^2}{2T}} dy \Big)^{1/q} \Big(\int_{\mathbb{R}} h(y)^r e^{-\frac{y^2}{2T}} dy \Big)^{1/r}$$

Observe $\lim_{T\to\infty} \sqrt{2\pi T}\, k(x,T-t) = 1$ uniformly over $(x,t) \in [-R,R] \times (0,T_1]$ and let $T \to \infty$, $R \to \infty$, $T_1 \to \infty$

The explicit Bellman function

An exercise:

Find the explicit Bellman function for L^p estimates for the dyadic paraproduct!

The explicit Bellman function

An exercise:

Find the explicit Bellman function for L^p estimates for the dyadic paraproduct!

Motivation:

- Very few Bellman functions for multilinear estimates in the literature

The explicit Bellman function

An exercise:

Find the explicit Bellman function for L^p estimates for the dyadic paraproduct!

Motivation:

- Very few Bellman functions for multilinear estimates in the literature
- In some contexts the explicit formula (involving powers) could be useful $\rightsquigarrow$ Carbonaro & Dragičević (2011, 2013)

The explicit Bellman function

An exercise:

Find the explicit Bellman function for L^p estimates for the dyadic paraproduct!

Motivation:

- Very few Bellman functions for multilinear estimates in the literature
- In some contexts the explicit formula (involving powers) could be useful $\rightsquigarrow$ Carbonaro & Dragičević (2011, 2013)
- We would like to find a direct proof (i.e. with no stopping times) of the estimates for the "twisted" paraproduct $\rightsquigarrow$ K. (2010) or the "twisted" quadrilinear form $\rightsquigarrow$ Durcik (2015)

The explicit Bellman function

An exercise:

Find the explicit Bellman function for L^p estimates for the dyadic paraproduct!

Motivation:

- Very few Bellman functions for multilinear estimates in the literature
- In some contexts the explicit formula (involving powers) could be useful $\rightsquigarrow$ Carbonaro & Dragičević (2011, 2013)
- We would like to find a direct proof (i.e. with no stopping times) of the estimates for the “twisted” paraproduct $\rightsquigarrow$ K. (2010) or the “twisted” quadrilinear form $\rightsquigarrow$ Durcik (2015)
- This could also extend the range of exponents for:
  - a non-adapted stochastic integral $\rightsquigarrow$ K. & Škreb (2014)
  - norm-variation of ergodic averages w.r.t. two commuting transformations $\rightsquigarrow$ Durcik, K., Škreb, Thiele (2016)

The explicit Bellman function

WLOG assume $q > r$ and use the ansatz:

$$\mathcal{B}(u,v,w,U,V,W) = C_{p,q,r}\left(\tfrac{1}{p}U + \tfrac{1}{q}V + \tfrac{1}{r}W\right) - \mathcal{A}(u,v,w)$$

$$\mathcal{A}(u,v,w) = u^p\,\gamma\Big(\underbrace{\frac{v^q}{u^p}}_{t},\underbrace{\frac{w^r}{u^p}}_{s}\Big)$$

The explicit Bellman function

WLOG assume $q > r$ and use the ansatz:

$$\mathcal{B}(u,v,w,U,V,W) = C_{p,q,r}\left(\tfrac{1}{p}U + \tfrac{1}{q}V + \tfrac{1}{r}W\right) - \mathcal{A}(u,v,w)$$

$$\mathcal{A}(u,v,w) = u^p\,\gamma\Big(\underbrace{\frac{v^q}{u^p}}_{t},\underbrace{\frac{w^r}{u^p}}_{s}\Big)$$

$$\gamma(t,s) = \begin{cases} A_1 + B_1 t + C_1 s; & 1 \le s \le t \\ A_2 + B_2 t + C_2 s^{\frac{1}{p'}}; & s \le 1 \le t \\ A_3 + B_3 t^{\frac{1}{p'}} + C_3 s^{\frac{1}{p'}}; & s \le t \le 1 \\ A_4 + B_4 t^{\frac{2}{q}} s^{\frac{1}{r}-\frac{1}{q}} + C_4 s^{\frac{1}{p'}}; & t \le s \le 1 \\ A_5 + B_5 t^{\frac{2}{q}} + C_5 t^{\frac{2}{q}} s^{1-\frac{2}{q}} + D_5 s; & t \le 1 \le s \\ A_6 + B_6 t + C_6 t^{\frac{2}{q}} s^{1-\frac{2}{q}} + D_6 s; & 1 \le t \le s \end{cases}$$

Adjust the coefficients so that γ is C^1 on $(0,\infty)^2$

The explicit Bellman function

$$\gamma(t,s)=\begin{cases} A+Bt+Cs; & 1\le s\le t\\ \frac{A(p-1)-C}{p-1}+Bt+\frac{Cp}{p-1}s^{\frac{1}{p'}}; & s\le 1\le t\\ \frac{A(p-1)-(B+C)}{p-1}+\frac{Bp}{p-1}t^{\frac{1}{p'}}+\frac{Cp}{p-1}s^{\frac{1}{p'}}; & s\le t\le 1\\ \frac{A(p-1)-(B+C)}{p-1}+\frac{Bq}{2}t^{\frac{2}{q}}s^{\frac{1}{r}-\frac{1}{q}}+\frac{2Cpr-Bp(q-r)}{2r(p-1)}s^{\frac{1}{p'}}; & t\le s\le 1\\ \frac{2Ar(p-1)-B(q+r)}{2r(p-1)}+\frac{Bq^2}{2p(q-2)}t^{\frac{2}{q}}+\frac{Bq(q-r)}{2r(q-2)}t^{\frac{2}{q}}s^{1-\frac{2}{q}} & \\ \qquad\qquad +\frac{2Cr-B(q-r)}{2r}s; & t\le 1\le s\\ A+\frac{Bq}{p(q-2)}t+\frac{Bq(q-r)}{2r(q-2)}t^{\frac{2}{q}}s^{1-\frac{2}{q}}+\frac{2Cr-B(q-r)}{2r}s; & 1\le t\le s \end{cases}$$

The explicit Bellman function

$\mathcal{B}$ contains terms:

$$U,\; V,\; W$$
$$u^p,\; v^q,\; w^r$$
$$uv^{q-\frac{q}{p}},\; uw^{r-\frac{r}{p}},\; u^{p-\frac{2p}{q}}v^2,\; v^2w^{r-\frac{2r}{q}},\; uv^2w^{1-\frac{r}{q}}$$

The explicit Bellman function

$\mathcal{B}$ contains terms:

$$U,\ V,\ W$$
$$u^p,\ v^q,\ w^r$$
$$uv^{q-\frac{q}{p}},\ uw^{r-\frac{r}{p}},\ u^{p-\frac{2p}{q}}v^2,\ v^2w^{r-\frac{2r}{q}},\ uv^2w^{1-\frac{r}{q}}$$

There are 3 critical hypersurfaces:

$$u^p = v^q,\ u^p = w^r,\ v^q = w^r$$

The explicit Bellman function

$\mathcal{B}$ contains terms:

$$U,\ V,\ W$$
$$u^p,\ v^q,\ w^r$$
$$uv^{q-\frac{q}{p}},\ uw^{r-\frac{r}{p}},\ u^{p-\frac{2p}{q}}v^2,\ v^2w^{r-\frac{2r}{q}},\ uv^2w^{1-\frac{r}{q}}$$

There are 3 critical hypersurfaces:

$$u^p = v^q,\ u^p = w^r,\ v^q = w^r$$

Similar to the function constructed by Nazarov & Treil (1995)

Thank you!

Thank you for your attention!