# Past proposals for high school and college student math competitions

*Vjekoslav Kovač*

University of Zagreb, Croatia
University of California, Los Angeles

## Introduction

This is a collection of problems I proposed to various math competitions over the past few years. They vary in the level of originality. Most of the ideas behind the problems are certainly not new (as otherwise these would be research papers), but there is a bit of creative work behind each one. They also vary in difficulty. Some of the problems proved to be very hard (consider yourself warned), having been successfully solved by only a few (if any) students at the corresponding competitions. Both problems and solutions can be distributed freely, but a reference to this document or my webpage will be appreciated. You are welcome to send me any comments.

*Vjekoslav Kovač*
`http://web.math.hr/~vjekovac/`
`http://www.math.ucla.edu/~vjekovac/`

## Contents

# 1 Problems

## 1.1 High school level competitions

1. Let $\triangle ABC$ be a triangle with sidelengths $a, b, c$ and angles $\alpha, \beta, \gamma$ inside which there exist points $P$ and $Q$ such that

$$\sphericalangle BPC = \sphericalangle CPA = \sphericalangle APB = 120^\circ,$$

$$\sphericalangle BQC = 60^\circ + \alpha, \quad \sphericalangle CQA = 60^\circ + \beta, \quad \sphericalangle AQB = 60^\circ + \gamma.$$

Prove the equality

$$\frac{AP + BP + CP}{3} \cdot \sqrt[3]{AQ \cdot BQ \cdot CQ} = \frac{1}{3}\left(\sqrt[3]{abc}\right)^2.$$

(*Croatian National Math Competition*, 2004, 4th grade, Problem 2)

2. (a) Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function satisfying the following inequalities:
   (i) $f(1) \geq 1$,
   (ii) $f(m^2 + n^3) \geq f(m)^2 + f(n)^3$ for all $m, n \in \mathbb{Z}$.
   Prove that $f(2^{3^k}) = 2^{3^k}$ for every $k \in \mathbb{N}_0$.

   (b) Show that infinitely many functions $f\colon \mathbb{Z} \to \mathbb{R}$ satisfy the conditions (i) and (ii).

(Croatian *IMO* team training, 2004)

3. Prove that there do not exist positive numbers $a_1, a_2, \ldots, a_{2004}$ satisfying

$$a_1 + a_2 + \ldots + a_{2004} \geq a_1 a_2 \ldots a_{2004} \geq a_1^2 + a_2^2 + \ldots + a_{2004}^2.$$

(*Croatian Math Olympiad*, 2004)

4. Let $A_1, A_2, \ldots, A_n$ $(n \geq 3)$ be finite sets of positive integers. Prove that

$$\frac{\sum\limits_{i=1}^{n} |A_i|}{n} + \frac{\sum\limits_{1\leq i<j<k\leq n} |A_i \cap A_j \cap A_k|}{\binom{n}{3}} \geq 2 \cdot \frac{\sum\limits_{1\leq i<j\leq n} |A_i \cap A_j|}{\binom{n}{2}}.$$

Here $|A|$ denotes the cardinality of $A$.

(*Mediterranean Math Competition*, 2005, Problem 3)

5. Let $m, n$ be positive integers and let $x_{i,j} \in [0,1]$ for all $i = 1, \ldots, m$; $j = 1, \ldots, n$. Prove that

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} x_{i,j}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1 - x_{i,j})\right) \geq 1.$$

(Proposal for *Mediterranean Math Competition*, 2005)

6. Let $(a(n))_{n\geq 1}$ be a sequence of nonnegative reals such that

$$a(m+n) \leq 2a(m) + 2a(n) \quad \text{for all } m, n \geq 1$$

and

$$a(2^k) \leq \frac{1}{k^4} \quad \text{for all } k \geq 1.$$

Prove that $(a(n))_{n\geq 1}$ must be bounded.

(*International Math Olympiad (IMO)*, 2007, Shortlist, Problem A5)

7. Suppose that functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$\frac{f(x) - f(y)}{x - y} = g\Big(\frac{x+y}{2}\Big)$$

for all different real numbers $x$ and $y$. Prove that $f$ is a polynomial of degree at most 2.

(Proposal for *Middle European Math Olympiad (MEMO)*, 2007)

8. Each positive integer is colored in one of finitely many given colors. Prove that there exist four different positive integers $a$, $b$, $c$, $d$, all in the same color, and such that:

   (1) $ad = bc$,

   (2) $\frac{b}{a}$ is a perfect power of 2,

   (3) $\frac{c}{a}$ is a perfect power of 3.

(Proposal for *Middle European Math Olympiad (MEMO)*, 2008)

9. Let $P$ be a polynomial with complex coefficients of degree at most $n-1$ and suppose that precisely $k$ of its coefficients are nonzero, $1 \leq k \leq n$. Let us also denote $Q(z) = z^n - 1$. Prove that polynomials $P$ and $Q$ have at most $n - \frac{n}{k}$ common roots, i.e. there exist at most $n - \frac{n}{k}$ different complex numbers $z$ satisfying $P(z) = 0 = Q(z)$.

(Proposal for *Middle European Math Olympiad (MEMO)*, 2008)

10. We say that a set of positive integers $S$ is *nice* if it is a nonempty subset of $\{1, 2, 3, \ldots, 2008\}$ and the product of numbers in $S$ is a perfect power of 10.

    (a) What is the size of the largest nice set?

    (b) What is the size of the largest nice set without proper nice subsets?

(Proposal for some competition, 2008)

11. On every square of a $9 \times 9$ board a light bulb is placed. In one move we are allowed to choose a square and toggle on/off the states of light bulbs on the chosen square and all its horizontally and vertically adjacent squares. (Every square has 2, 3 or 4 adjacent squares.) Initially all light bulbs are on, and suppose that after some number of moves precisely one light bulb remains on. Prove that this light bulb must be positioned in the center of the board.

(Proposal for *International Math Olympiad (IMO)*, 2009)

12. In some country there are $n$ cities, where $n \geq 5$. Some pairs of cities are connected by direct two-way flights, and there are at most $3n-7$ such flights provided by the airline. A set of 5 cities with direct two-way flights between each two of them is called a *5-city tour*. Prove that it is possible to introduce a new flight (between two cities that are not already directly connected) without making any new 5-city tours.

(Proposal for *Middle European Math Olympiad (MEMO)*, 2009)

13. Initially, only number 44 is written on the board. We repeatedly perform the following operation 30 times. At each step we simultaneously replace each number on the board, call it $a$, by four numbers $a_1, a_2, a_3, a_4$ that only have to satisfy:

- $a_1, a_2, a_3, a_4$ are four different integers.
- Average of four new numbers $(a_1 + a_2 + a_3 + a_4)/4$ is equal to the erased number $a$.

After 30 steps we end up with $n = 4^{30}$ numbers on the board, call them $b_1, b_2, \ldots, b_n$. Prove that

$$\frac{b_1^2 + b_2^2 + \ldots + b_n^2}{n} \geq 2011\,.$$

(*Middle European Math Olympiad (MEMO)*, 2011)

14. At this year's MEMO, there are $3n$ participants, there are $n$ languages spoken, and each participant speaks exactly 3 different languages. Prove that MEMO coordinators can choose at least $\frac{2n}{9}$ languages for the presentation of the official solutions, such that no participant will understand the presentation in more than 2 languages.

(*Middle European Math Olympiad (MEMO)*, 2011)

## 1.2 College level competitions

1. Let $p, q > 1$ be relatively prime positive integers.

   (a) Suppose that $f\colon \{1, 2, \ldots, p+q-1\} \to \{0, 1\}$ is a periodic function with periods both $p$ and $q$. Prove that $f$ is a constant.

   (b) Show that there exist exactly 4 functions $f\colon \{1, 2, \ldots, p+q-2\} \to \{0, 1\}$ that are periodic with periods both $p$ and $q$.

   (Proposal for *International Math Competition for University Students (IMC)*, 2003)

2. Let $A = [a_{i,j}]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ be an $m \times n$ real matrix with at least one non-zero entry. For each $i \in \{1, \ldots, m\}$ let $R_i := \sum_{j=1}^{n} a_{i,j}$ denote the sum of entries in the $i$-th row of $A$ and for each $j \in \{1, \ldots, n\}$ let $C_j := \sum_{i=1}^{m} a_{i,j}$ denote the sum of entries in the $j$-th column of $A$. Prove that there exist indices $i_0 \in \{1, \ldots, m\}$, $j_0 \in \{1, \ldots, n\}$ such that

$$\begin{aligned} & a_{i_0,j_0} > 0, \quad R_{i_0} \geq 0, \quad C_{j_0} \geq 0, \\ \text{or} \quad & a_{i_0,j_0} < 0, \quad R_{i_0} \leq 0, \quad C_{j_0} \leq 0. \end{aligned}$$

   (*Vojtěch Jarník International Math Competition*, 2003, Category I, Problem 2)

3. A sequence $(a_n)_{n\geq 0}$ of real numbers is defined recursively by

$$a_0 := 0, \quad a_1 := 1, \quad a_{n+2} := a_{n+1} + \frac{a_n}{2^n}; \; n \geq 0.$$

   Prove the following:

   (a) The sequence $(a_n)_{n\geq 0}$ is convergent.

   (b) $\lim\limits_{n\to\infty} a_n = 1 + \sum\limits_{n=1}^{\infty} \dfrac{1}{2^{\frac{n(n-1)}{2}} \cdot \prod_{k=1}^{n} (2^k - 1)}$

   (c) The limit $\lim\limits_{n\to\infty} a_n$ is an irrational number.

   (*Vojtěch Jarník International Math Competition*, 2003, Category II, Problem 3)

4. Let $f, g\colon [0,1] \to \langle 0, +\infty\rangle$ be continuous functions such that $f$ and $\frac{g}{f}$ are increasing. Prove that

$$\int_0^1 \frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt}\,dx \;\leq\; 2\int_0^1 \frac{f(t)}{g(t)}\,dt.$$

   (*Vojtěch Jarník International Math Competition*, 2003, Category II, Problem 4)

5. Let $G$ be a (multiplicatively written) group with identity $e$. If elements $a, b \in G$ satisfy the relations

$$a^3 = e, \quad ab^2 = ba^2, \quad (a^2 b)^{2003} = e,$$

   show that $a = b$.

   (Selection test for the Croatian team for *Vojtěch Jarník*, 2003)

6. Prove that there do not exist a real number $a$ and a function $f\colon \mathbb{R} \to \mathbb{R}$ continuous at 0 satisfying

$$f\left(\frac{x+a}{1-ax}\right) > f(x)$$

for every $x \in \mathbb{R}$ such that $ax \neq 1$.

(Proposal for *International Math Competition for University Students (IMC)*, 2004)

7. Let $n$ be a positive integer. A linear operator $\mathcal{T}\colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C})$ is defined as follows. For any $A \in \mathrm{M}_n(\mathbb{C})$, the $(i,j)$-th entry of $\mathcal{T}(A)$ equals the sum of the $(i,j)$-th entry of $A$ and all its neighbor entries in $A$. (Each entry has 3, 5 or 8 neighbors.) Prove that

$$\sigma(\mathcal{T}) = \left\{ \left(1 - 2\cos\frac{k\pi}{n+1}\right)\left(1 - 2\cos\frac{l\pi}{n+1}\right) : k, l = 1, \ldots, n \right\}.$$

(Proposal for *International Math Competition for University Students (IMC)*, 2004)

8. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two decreasing sequences of positive real numbers such that $\prod_{j=1}^{n} x_j \geq \prod_{j=1}^{n} y_j$ for every $n \geq 1$. Prove that $\sum_{j=1}^{n} x_j \geq \sum_{j=1}^{n} y_j$ for every $n \geq 1$.

(Proposal for *International Math Competition for University Students (IMC)*, 2005)

9. Let $R$ be a finite ring with the following property:
For any $a, b \in R$, there exists $c \in R$ (depending on $a$ and $b$) such that $a^2 + b^2 = c^2$.
Prove that:
For any $a, b, c \in R$, there exists $d \in R$ such that $2abc = d^2$.
(*Remarks.* Here $2abc$ denotes $abc + abc$. $R$ is assumed to be associative but not necessarily commutative.)

(*Vojtěch Jarník International Math Competition*, 2005, Category II, Problem 4)

10. Let $(N_n)_{n\geq 1}$ be a sequence of positive integers no smaller than 3. Inside a circle of radius $r_1$ we inscribe a regular $N_1$-gon. Next, inside the latter polygon we inscribe a circle of radius $r_2$ and in the latter circle we inscribe a regular $N_2$-gon, and so on. Continuing in this way, we obtain a sequence of circles and polygons. Prove that

$$\lim_{n\to\infty} r_n = 0 \qquad \text{if and only if} \qquad \sum_{n=1}^{\infty} \frac{1}{N_n^2} = \infty.$$

(Proposal for *Vojtěch Jarník International Math Competition*, 2005)

11. A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is defined by the equations

$$2^{x_1} = 4,\ 2^{2^{x_2}} = 4^4,\ 2^{2^{2^{x_3}}} = 4^{4^4}, \ldots$$

and generally

$$2^{2^{\cdot^{\cdot^{2^{x_n}}}}} = 4^{4^{\cdot^{\cdot^{4}}}},$$

where the left hand side contains $n$ twos and the right hand side contains $n$ fours. Prove that the sequence converges and its limit satisfies

$$3 \le \lim_{n\to\infty} x_n \le \tfrac{10}{3}.$$

(Proposal for *International Math Competition for University Students (IMC)*, 2009)

12. Let $k$ and $n$ be positive integers such that $k \le n-1$. Denote $S = \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of $S$. Prove that it is possible to color some elements of $S$ using two colors, red and blue, such that the following conditions are satisfied.

    (i) Each element of $S$ is either left uncolored or is colored red or blue.

    (ii) At least one element of $S$ is colored.

    (iii) Each set $A_i$ ($i = 1, 2, \ldots, k$) is either completely uncolored or it contains at least one red and at least one blue element.

(*Vojtěch Jarník International Math Competition*, 2009, Category I, Problem 3)

13. Let $k, m, n$ be positive integers such that $1 \le m \le n$ and denote $S = \{1, 2, \ldots, n\}$. Suppose that $A_1, A_2, \ldots, A_k$ are $m$-element subsets of $S$ with the following property. For every $i = 1, 2, \ldots, k$ there exists a partition $S = S_1^{(i)} \cup S_2^{(i)} \cup \ldots \cup S_m^{(i)}$ such that:

    (i) $A_i$ has precisely one element in common with each member of the above partition.

    (ii) Every $A_j$, $j \ne i$ is disjoint from at least one member of the above partition.

    Show that $k \le \binom{n-1}{m-1}$.

(*Vojtěch Jarník International Math Competition*, 2009, Category II, Problem 4)

14. Prove that for every complex polynomial $P(z) = a_n z^n + \ldots + a_1 z + a_0$ with $|a_n| = |a_0| = 1$, there exists a complex polynomial $Q(z) = b_n z^n + \ldots + b_1 z + b_0$ with $|b_n| = |b_0| = 1$, such that $|Q(z)| \le |P(z)|$ for every $z \in \mathbb{C}$, $|z| = 1$, and such that all complex roots of $Q$ lie on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

(Proposal for *Vojtěch Jarník International Math Competition*, 2009)

## 2 Solutions

### 2.1 High school level competitions

1. *Problem.* Let $\triangle ABC$ be a triangle with sidelengths $a, b, c$ and angles $\alpha, \beta, \gamma$ inside which there exist points $P$ and $Q$ such that

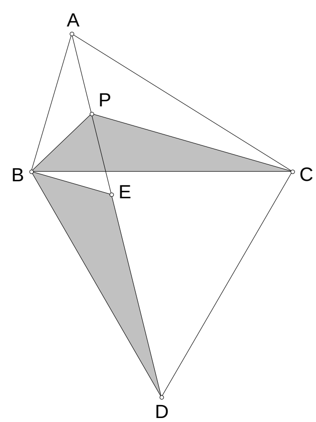
$$\sphericalangle BPC = \sphericalangle CPA = \sphericalangle APB = 120^\circ,$$

$$\sphericalangle BQC = 60^\circ + \alpha, \quad \sphericalangle CQA = 60^\circ + \beta, \quad \sphericalangle AQB = 60^\circ + \gamma.$$

Prove the equality

$$\frac{AP + BP + CP}{3} \cdot \sqrt[3]{AQ \cdot BQ \cdot CQ} = \frac{1}{3}\left(\sqrt[3]{abc}\right)^2.$$

*Solution.* Let us construct an equilateral triangle $\triangle CBD$ over the segment $\overline{BC}$, outside of $\triangle ABC$. Let $E$ be a point such that the triangles $\triangle BDE$ and $\triangle BCP$ are congruent and equally oriented.




Triangle $\triangle BEP$ is equilateral because of $BP = BE$ and $\sphericalangle PBE = 60^\circ$. From $\sphericalangle APB + \sphericalangle BPE = 120^\circ + 60^\circ = 180^\circ$ and $\sphericalangle PEB + \sphericalangle BED = 60^\circ + 120^\circ = 180^\circ$ we get that the points $A, P, E, D$ are collinear. Also note

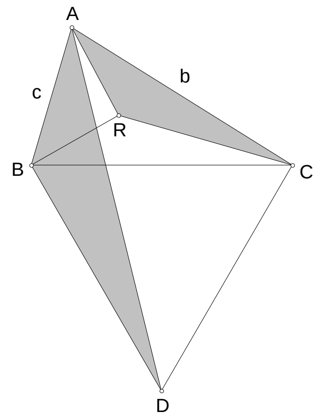
$$AP + BP + CP = AP + PE + ED = AD. \tag{1}$$

Let $R$ be a point inside $\triangle ABC$ such that the triangles $\triangle ARC$ and $\triangle ABD$ are similar, i.e. such that $\sphericalangle RAC = \sphericalangle BAD < \alpha$ and $\sphericalangle RCA = \sphericalangle BDA = \sphericalangle BCP < \gamma$.

From this similarity we conclude

$$\frac{AR}{AC} = \frac{AB}{AD}. \tag{2}$$

Furthermore, from $\sphericalangle RAB = \sphericalangle CAD$ and $\frac{AR}{AB} = \frac{AC}{AD}$ we know that $\triangle ABR$ and $\triangle ADC$ are similar too, so $\sphericalangle ARB = \sphericalangle ACD = 60^\circ + \gamma$. Now from $\sphericalangle ARB = 60^\circ + \gamma = \sphericalangle AQB$ and $\sphericalangle ARC = \sphericalangle ABD = 60^\circ + \beta = \sphericalangle AQC$ we get $R = Q$. (Namely, by the Inscribed Angle Theorem, the point $Q$ lies on the circles circumscribed around triangles $\triangle ABR$ and $\triangle ARC$.)




From (1), (2), and $Q = R$ we get

$$\frac{AQ}{b} = \frac{c}{AP + BP + CP}, \quad \text{i.e.} \quad (AP + BP + CP) \cdot AQ = bc$$

and analogously we would prove

$$(AP + BP + CP) \cdot BQ = ca, \quad (AP + BP + CP) \cdot CQ = ab.$$

Finally, multiplying we get

$$(AP + BP + CP)^3 \cdot AQ \cdot BQ \cdot CQ = a^2b^2c^2,$$

and it remains to take the third root and divide by 3. ■

*Remark.* Alternatively, we could have defined the point $R$ as the isogonal conjugate of the point $P$ and then it would be easy to verify $\sphericalangle ARB = 60^\circ + \gamma$ and $\sphericalangle ARC = 60^\circ + \beta$, from which $R = Q$ follows again.

*Another remark.* Alternative formulations of the problem could be

$$(AP + BP + CP)(AQ + BQ + CQ) = ab + bc + ca,$$

or

$$AQ + BQ + CQ \geq \sqrt{ab + bc + ca}.$$

*Yet another remark.* Nobody solved this problem at the competition. Some familiarity with triangle centers is certainly useful:
`http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

2. *Problem.*

   (a) Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function satisfying the following inequalities:

   (i) $f(1) \geq 1$,

   (ii) $f(m^2 + n^3) \geq f(m)^2 + f(n)^3$ for all $m, n \in \mathbb{Z}$.

   Prove that $f(2^{3^k}) = 2^{3^k}$ for every $k \in \mathbb{N}_0$.

(b) Show that infinitely many functions $f\colon \mathbb{Z} \to \mathbb{R}$ satisfy the conditions (i) and (ii).

*Solution.*

(a) In the chain of inequalities

$$f(1) = f(0^2 + 1^3) \geq_{\text{(ii)}} f(0)^2 + f(1)^3 \geq f(1)^3 \geq_{\text{(i)}} f(1)$$

each inequality sign can be changed into the equality sign, so $f(0) = 0$ and $f(1) = 1$. Therefore we have

$$0 = f(0) = f(1^2 + (-1)^3) \geq_{\text{(ii)}} f(1)^2 + f(-1)^3 = 1 + f(-1)^3 \ \Rightarrow\ f(-1) \leq -1, \quad (3)$$

$$1 = f(1) = f((-1)^2 + 0^3) \geq_{\text{(ii)}} f(-1)^2 + f(0)^3 = f(-1)^2 \ \Rightarrow\ |f(-1)| \leq 1, \quad (4)$$

so (3) and (4) together give $f(-1) = -1$.
Now we gradually deduce equalities:

$$f(-2) = -2, \quad f(3) = 3, \quad f(2) = 2, \quad f(8) = 8, \quad f(-4) = -4. \quad (5)$$

First,

$$f(2) = f(1^2 + 1^3) \geq_{\text{(ii)}} f(1)^2 + f(1)^3 = 2, \quad (6)$$

$$f(3) = f(2^2 + (-1)^3) \geq_{\text{(ii)}} f(2)^2 + f(-1)^3 \geq_{(6)} 4 - 1 = 3 \quad (7)$$

and then

$$1 = f(1) \geq_{\text{(ii)}} f(3)^2 + f(-2)^3 \geq_{(7)} 9 + f(-2)^3 \quad \Rightarrow \quad f(-2) \leq -2. \quad (8)$$

Next,

$$f(3) = f((-2)^2 + (-1)^3) \geq_{\text{(ii)}} f(-2)^2 + f(-1)^3 = f(-2)^2 - 1 \quad (9)$$

and observe that the right hand side in (9) is positive because of (8). Thus we can compute

$$\begin{aligned} 1 = f(1) = f(3^2 + (-2)^3) &\geq_{\text{(ii)}} f(3)^2 + f(-2)^3 \\ &\geq_{(9)} \left(f(-2)^2 - 1\right)^2 + f(-2)^3 = f(-2)^4 + f(-2)^3 - 2f(-2)^2 + 1, \end{aligned}$$

so after factoring

$$f(-2)^2\,(f(-2) - 1)\,(f(-2) + 2) \leq 0. \quad (10)$$

From (8) we get $f(-2)^2 > 0$ and $f(-2) - 1 < 0$, so (10) implies $f(-2) + 2 \geq 0$, i.e.

$$f(-2) \geq -2. \quad (11)$$

Combining (8) and (11) we get $f(-2) = -2$.
Furthermore, we relatively easily prove other equalities in (5).

$$1 = f(1) \geq_{\text{(ii)}} f(3)^2 + f(-2)^3 = f(3)^2 - 8 \quad \Rightarrow \quad |f(3)| \leq 3, \quad (12)$$

so (7) and (12) together imply $f(3)=3$. Moreover,

$$3 = f(3) \geq_{\text{(ii)}} f(2)^2 + f(-1)^3 = f(2)^2 - 1 \quad \Rightarrow \quad |f(2)| \leq 2 \tag{13}$$

so combining (6) and (13) we get $f(2)=2$. Now,

$$f(8) \geq_{\text{(ii)}} f(0)^2 + f(2)^3 = 8 \quad \Rightarrow \quad f(8) \geq 8, \tag{14}$$

$$f(-4) \geq_{\text{(ii)}} f(2)^2 + f(-2)^3 = 4 - 8 = -4 \quad \Rightarrow \quad f(-4) \geq -4 \tag{15}$$

and

$$0 = f(8^2 + (-4)^3) \geq_{\text{(ii)}} f(8)^2 + f(-4)^3 \geq_{(14),(15)} 8^2 + (-4)^3 = 0,$$

where we must have equality at every place. Thus, $f(8)=8$ and $f(-4)=-4$.
Finally, we show that for every $k \in \mathbb{N}_0$ one has

$$f(2^{3^{k+1}}) = 2^{3^{k+1}} \quad \text{and} \quad f(-2^{2\cdot 3^k}) = -2^{2\cdot 3^k}. \tag{16}$$

We have already shown the statement in the case $k=0$, since then (16) becomes $f(8)=8$ and $f(-4)=-4$. Substituting $m=0$ into (ii) we get

$$f(n^3) \geq f(n)^3 \text{ for every } n \in \mathbb{Z}$$

and then by induction on $k \in \mathbb{N}$ we easily prove

$$f(n^{3^k}) \geq f(n)^{3^k} \text{ for every } n \in \mathbb{Z} \text{ and } k \in \mathbb{N}. \tag{17}$$

By substituting particularly $n=8$ or $n=-4$ into (17), we conclude for each $k \in \mathbb{N}$,

$$f(2^{3^{k+1}}) \geq_{(17)} f(8)^{3^k} = 8^{3^k} = 2^{3^{k+1}} \quad \Rightarrow \quad f(2^{3^{k+1}}) \geq 2^{3^{k+1}}, \tag{18}$$

$$f(-2^{2\cdot 3^k}) \geq_{(17)} f(-4)^{3^k} = (-4)^{3^k} = -2^{2\cdot 3^k} \quad \Rightarrow \quad f(-2^{2\cdot 3^k}) \geq -2^{2\cdot 3^k}. \tag{19}$$

In the end, since $(2^{3^{k+1}})^2 + (-2^{2\cdot 3^k})^3 = 2^{2\cdot 3^{k+1}} - 2^{2\cdot 3^{k+1}} = 0$, from the chain of inequalities

$$\begin{aligned} 0 &= f(0) = f\left((2^{3^{k+1}})^2 + (-2^{2\cdot 3^k})^3\right) \geq_{\text{(ii)}} f(2^{3^{k+1}})^2 + f(-2^{2\cdot 3^k})^3 \geq_{(18),(19)} \\ &\geq_{(18),(19)} (2^{3^{k+1}})^2 + (-2^{2\cdot 3^k})^3 = 0 \end{aligned}$$

we derive $f(2^{3^{k+1}})^2 = (2^{3^{k+1}})^2$ and $f(-2^{2\cdot 3^k})^3 = (-2^{2\cdot 3^k})^3$. Because of $f(2^{3^{k+1}}) > 0$ we have (16). ■

(b) We first show the following auxiliary statement, which claims that at least one integer (concretely number 7) is not representable in the form $m^2+n^3$; $m,n \in \mathbb{Z}$.

*Lemma.* The equation $m^2+n^3=7$ has no solution in the integers $m,n$.

*Proof of the lemma.* Suppose that $m,n \in \mathbb{Z}$ are such that $m^2+n^3=7$. The number $n$ must be odd, since otherwise $m^2 \equiv 3 \pmod 4$. The equation can be written in the form $m^2+1 = 8-n^3$, i.e. $m^2+1 = (2-n)(4+2n+n^2)$. Take $p$ to be a prime divisor of $2-n$ that is of the form $4k+3;\ k\in\mathbb{N}_0$. Then $p$ divides $m^2+1$ and by taking powers in $m^2\equiv -1 \pmod p$ we get

$$\left(m^2\right)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \pmod p. \tag{20}$$

By Little Fermat's theorem $m^{p-1}\equiv 1 \pmod p$ so, because $\frac{p-1}{2}=2k+1$ is odd, the congruence (20) becomes $1\equiv -1 \pmod p$, which is impossible. Therefore, $2-n$ has all prime divisors of the form $4k+1;\ k\in\mathbb{N}$ and consequently $2-n$ itself has the same form, i.e.

$$2-n\equiv 1 \pmod 4 \quad\Rightarrow\quad n\equiv 1 \pmod 4. \tag{21}$$

Consider the equality $m^2+n^3=7$ modulo 4 after taking (21) into account:

$$m^2+1\equiv 3 \pmod 4 \quad\Rightarrow\quad m^2\equiv 2 \pmod 4,$$

but that is not possible. This proves the lemma.
Let us return to the original problem. We claim that for any real number $\alpha\in[-7,7]$ the function $f\colon\mathbb{Z}\to\mathbb{R}$ defined by $f(z):=\begin{cases} z & \text{if } z\neq 7\\ \alpha & \text{if } z=7\end{cases}$ satisfies the condition (ii). From the lemma, $f(m^2+n^3)=m^2+n^3$ for all $m,n\in\mathbb{Z}$, so the condition (ii) takes one of the following four shapes:

$$\begin{aligned} m^2+n^3 &\geq m^2+n^3 &\qquad& \text{when } m,n\neq 7\\ 7^2+n^3 &\geq \alpha^2+n^3 && \text{when } m=7,\ n\neq 7\\ m^2+7^3 &\geq m^2+\alpha^3 && \text{when } m\neq 7,\ n=7\\ 7^2+7^3 &\geq \alpha^2+\alpha^3 && \text{when } m,n=7 \end{aligned}$$

All these inequalities are satisfied because $\alpha$ is chosen so that $\alpha^2\leq 7^2$ and $\alpha^3\leq 7^3$. Therefore, for every $\alpha\in[-7,7]$ the corresponding function $f$ satisfies the required conditions. ■

*Remark.* Alternative formulation of part (a) could be the following.
Let $P$ be a polynomial with real coefficients satisfying:

(i) $P(1)\geq 1$,

(ii) $P(m^2+n^3)\geq P(m)^2+P(n)^3$ for all $m,n\in\mathbb{Z}$.

Prove that $P(x)\equiv x$.

*Another remark.* This problem/solution was generally received simply as "awful". Further comments are unnecessary.

3. *Problem.* Prove that there do not exist positive numbers $a_1,a_2,\ldots,a_{2004}$ satisfying

$$a_1+a_2+\ldots+a_{2004}\geq a_1a_2\ldots a_{2004}\geq a_1^2+a_2^2+\ldots+a_{2004}^2.$$

*Solution.* Suppose that such $a_1, a_2, \ldots, a_{2004}$ actually exist. By the inequality between quadratic and arithmetic means we have

$$\left(\frac{a_1^2 + a_2^2 + \ldots + a_{2004}^2}{2004}\right)^{\frac{1}{2}} \geq \frac{a_1 + a_2 + \ldots + a_{2004}}{2004},$$

i.e.

$$a_1^2 + a_2^2 + \ldots + a_{2004}^2 \geq \frac{1}{2004}\left(a_1 + a_2 + \ldots + a_{2004}\right)^2.$$

Let $\alpha$ and $\beta$ be positive numbers such that $\alpha + \beta = 2$. (We will choose them later.) By the inequality between arithmetic and geometric means,

$$\frac{a_1 + a_2 + \ldots + a_{2004}}{2004} \geq \left(a_1 a_2 \ldots a_{2004}\right)^{\frac{1}{2004}},$$

and the first inequality from the problem statement, we conclude

$$\begin{aligned}\frac{1}{2004^\alpha}\left(a_1 + a_2 + \ldots + a_{2004}\right)^2 &= \left(\frac{a_1 + a_2 + \ldots + a_{2004}}{2004}\right)^\alpha \left(a_1 + a_2 + \ldots + a_{2004}\right)^\beta \\ &\geq \left(a_1 a_2 \ldots a_{2004}\right)^{\frac{\alpha}{2004}} \left(a_1 + a_2 + \ldots + a_{2004}\right)^\beta \\ &\geq \left(a_1 a_2 \ldots a_{2004}\right)^{\frac{\alpha}{2004}} \left(a_1 a_2 \ldots a_{2004}\right)^\beta = \left(a_1 a_2 \ldots a_{2004}\right)^{\frac{\alpha}{2004}+\beta}.\end{aligned}$$

Now we also impose the condition $\frac{\alpha}{2004} + \beta = 1$, which together with $\alpha + \beta = 2$ leads to the choice $\alpha = \frac{2004}{2003}$, $\beta = \frac{2002}{2003}$. Thus, we got

$$a_1^2 + a_2^2 + \ldots + a_{2004}^2 \geq 2004^{\alpha - 1}\, a_1 a_2 \ldots a_{2004} > a_1 a_2 \ldots a_{2004}$$

(since $\alpha - 1 = \frac{1}{2003} > 0$), which is in the contradiction with the second inequality from the statement of the problem. ■

*Remark.* All members of the Croatian IMO team solved this problem. Congrats!

4. *Problem.* Let $A_1, A_2, \ldots, A_n$ $(n \geq 3)$ be finite sets of positive integers. Prove that

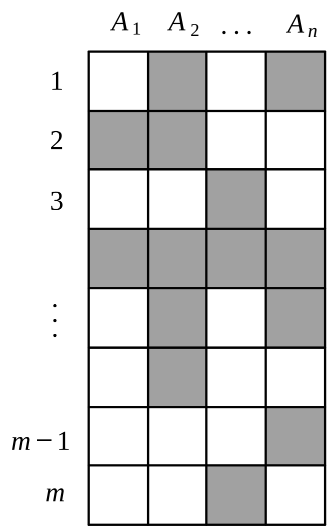
$$\frac{\sum_{i=1}^{n} |A_i|}{n} + \frac{\sum_{1\leq i<j<k\leq n} |A_i \cap A_j \cap A_k|}{\binom{n}{3}} \geq 2 \cdot \frac{\sum_{1\leq i<j\leq n} |A_i \cap A_j|}{\binom{n}{2}}.$$

Here $|A|$ denotes the cardinality of $A$.

*Solution.* Choose a positive integer $m$ large enough so that $\bigcup_{i=1}^n A_i \subseteq \{1, 2, \ldots, m\}$. Consider an $m \times n$ table, assign $1, \ldots, m$ to its rows and $A_1, \ldots, A_n$ to its columns. A square at the position $(i, j)$ is painted black if $i \in A_j$, otherwise it is painted white. Let $N_i$ denote the number of black squares in the $i$-th row of the table.

Now we count in two ways

- the total number of black squares: $\displaystyle\sum_{j=1}^{n} |A_j| = \sum_{i=1}^{m} N_i,$




- pairs of black squares in the same row: $\sum_{1\le i<j\le n} |A_i \cap A_j| = \sum_{i=1}^{m} \binom{N_i}{2}$,
- triples of black squares in the same row: $\sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| = \sum_{i=1}^{m} \binom{N_i}{3}$.

Thus, the inequality can be rewritten as

$$\frac{\sum_{i=1}^m N_i}{n} + \frac{\sum_{i=1}^m N_i(N_i-1)(N_i-2)}{n(n-1)(n-2)} \ge 2 \cdot \frac{\sum_{i=1}^m N_i(N_i-1)}{n(n-1)}.$$

It is sufficient to verify

$$\frac{N_i}{n} + \frac{N_i(N_i-1)(N_i-2)}{n(n-1)(n-2)} \ge 2 \cdot \frac{N_i(N_i-1)}{n(n-1)}$$

for $i = 1, \ldots, m$, but this is equivalent to

$$\frac{N_i(n-N_i)(n-N_i-1)}{n(n-1)(n-2)} \ge 0,$$

which obviously holds. ■

5. *Problem.* Let $m, n$ be positive integers and let $x_{i,j} \in [0,1]$ for all $i = 1, \ldots, m$; $j = 1, \ldots, n$. Prove that

$$\prod_{j=1}^{n}\Big(1 - \prod_{i=1}^{m} x_{i,j}\Big) + \prod_{i=1}^{m}\Big(1 - \prod_{j=1}^{n}(1 - x_{i,j})\Big) \ge 1.$$

*Solution.* The inequality will be proved in three steps.

*Step 1.* For $a, b, c, d \in [0,1]$ we have

$$(1-ab)(1-cd) + \Big(1-(1-a)(1-c)\Big)\Big(1-(1-b)(1-d)\Big) \ge 1. \qquad (22)$$

*Proof.*

$$
\begin{aligned}
&(1-ab)(1-cd) + \Big(1-(1-a)(1-c)\Big)\Big(1-(1-b)(1-d)\Big) - 1 \\
&= ad + bc - abc - abd - acd - bcd + 2abcd \\
&= ad(1-b)(1-c) + bc(1-a)(1-d) \geq 0
\end{aligned}
$$

*Step 2.* If $u_i, v_i \in [0,1]$; $i = 1, \ldots, m$, then

$$
\left(1-\prod_{i=1}^{m} u_i\right)\left(1-\prod_{i=1}^{m} v_i\right) + \prod_{i=1}^{m}\Big(1-(1-u_i)(1-v_i)\Big) \geq 1. \tag{23}
$$

*Proof.* We give a proof by induction on $m$. The inequality is trivial for $m = 1$ and the induction step is proved as follows.

$$
\begin{aligned}
&\prod_{i=1}^{m+1}\Big(1-(1-u_i)(1-v_i)\Big) \geq \text{[by the induction hypothesis]} \\
&\geq \left(1-\left(1-\prod_{i=1}^{m} u_i\right)\left(1-\prod_{i=1}^{m} v_i\right)\right)\Big(1-(1-u_{m+1})(1-v_{m+1})\Big) \\
&= \begin{bmatrix} a = 1-\prod_{i=1}^{m} u_i, & b = 1-\prod_{i=1}^{m} v_i \\ c = 1-u_{m+1}, & d = 1-v_{m+1} \end{bmatrix} = (1-ab)(1-cd) \geq \text{[using (22)]} \\
&\geq 1-\Big(1-(1-a)(1-c)\Big)\Big(1-(1-b)(1-d)\Big) \\
&= 1-\left(1-\prod_{i=1}^{m} u_i \cdot u_{m+1}\right)\left(1-\prod_{i=1}^{m} v_i \cdot v_{m+1}\right) \\
&= 1-\left(1-\prod_{i=1}^{m+1} u_i\right)\left(1-\prod_{i=1}^{m+1} v_i\right)
\end{aligned}
$$

*Step 3.* Statement of the problem.
*Proof.* We will prove the statement by induction on $n$. It obviously holds for $n = 1$. We turn to the induction step.

$$
\begin{aligned}
&\prod_{j=1}^{n+1}\left(1-\prod_{i=1}^{m} x_{i,j}\right) \geq \text{[by the induction hypothesis]} \\
&\geq \left(1-\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}(1-x_{i,j})\right)\right)\left(1-\prod_{i=1}^{m} x_{i,n+1}\right) \\
&= \begin{bmatrix} u_i = 1-\prod_{j=1}^{n}(1-x_{i,j}) \\ v_i = x_{i,n+1} \end{bmatrix} = \left(1-\prod_{i=1}^{m} u_i\right)\left(1-\prod_{i=1}^{m} v_i\right) \geq \text{[using (23)]} \\
&\geq 1-\prod_{i=1}^{m}\Big(1-(1-u_i)(1-v_i)\Big)
\end{aligned}
$$

$$= 1 - \prod_{i=1}^{m} \Big(1 - \prod_{j=1}^{n} (1 - x_{i,j}) \cdot (1 - x_{i,n+1})\Big)$$

$$= 1 - \prod_{i=1}^{m} \Big(1 - \prod_{j=1}^{n+1} (1 - x_{i,j})\Big)$$

The proof is complete. ■

6. *Problem.* Let $(a(n))_{n\geq 1}$ be a sequence of nonnegative reals such that

$$a(m+n) \leq 2a(m) + 2a(n) \quad \text{for all } m, n \geq 1$$

and

$$a(2^k) \leq \frac{1}{k^4} \quad \text{for all } k \geq 1.$$

Prove that $(a(n))_{n\geq 1}$ must be bounded.

*Solution.* To simplify arguments we define $a(0) = 0$. The following formulae are easy to prove by mathematical induction on $r \geq 1$:

$$a\Big(\sum_{j=1}^{r} n_j\Big) \leq \sum_{j=1}^{r} 2^j a(n_j) \tag{24}$$

$$a\Big(\sum_{j=1}^{2^r} n_j\Big) \leq 2^r \sum_{j=1}^{2^r} a(n_j) \tag{25}$$

for all nonnegative integers $n_1$, $n_2$, $n_3$, ….

Now take an arbitrary positive integer $m$ and write its binary representation:

$$m = \sum_{j\geq 0} \varepsilon_j \cdot 2^j,$$

where each $\varepsilon_j$ is 0 or 1 and only finitely many of them are nonzero. Moreover, decompose the latter sum dyadically as

$$m = \varepsilon_0 \cdot 2^0 + \sum_{k=0}^{t} \sum_{2^k \leq j < 2^{k+1}} \varepsilon_j \cdot 2^j = \varepsilon_0 + \sum_{k=0}^{t} m_k,$$

where we have put $m_k = \sum_{2^k \leq j < 2^{k+1}} \varepsilon_j \cdot 2^j$ and $t \geq 0$ is large enough.

Applying (25) we get

$$a(m_k) \leq 2^k \sum_{2^k \leq j < 2^{k+1}} a(2^j) \leq 2^k \sum_{2^k \leq j < 2^{k+1}} \frac{1}{j^4} \leq 2^k \cdot 2^k \cdot \frac{1}{(2^k)^4} = \frac{1}{2^{2k}}$$

and then applying (24) we obtain

$$a(m) \leq 2a(1) + \sum_{k=0}^{t} 2^{k+2} a(m_k) \leq 2a(1) + 4\sum_{k=0}^{t} 2^k \cdot \frac{1}{2^{2k}} =$$
$$= 2a(1) + 4\sum_{k=0}^{t} \frac{1}{2^k} = 2a(1) + 4\Big(2 - \frac{1}{2^t}\Big) < 2a(1) + 8.$$

Therefore, $a(m)$ is bounded independently of $m$. ■

*Remark.* It can be shown that the condition $a(2^k) \leq \frac{1}{k^4}$ may be relaxed to $\sum_{k=1}^{\infty} \sqrt{a(2^k)} < \infty$. For an alternative solution, see *Djukić, Janković, Matić, Petrović: The IMO Compendium.*

7. *Problem.* Suppose that functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$\frac{f(x) - f(y)}{x - y} = g\Big(\frac{x+y}{2}\Big)$$

for all different real numbers $x$ and $y$. Prove that $f$ is a polynomial of degree at most 2.

*Solution.* Observe that for any $s \in \mathbb{R}$ and $t > 0$ we have

$$f(s + 2t) - f(s - 2t) = g(s) \cdot 4t,$$

while on the other hand we have

$$f(s + 2t) - f(s) + f(s) - f(s - 2t) = g(s + t) \cdot 2t + g(s - t) \cdot 2t.$$

Therefore

$$g(s) = \frac{g(s + t) + g(s - t)}{2},$$

which gives

$$g\Big(\frac{x+y}{2}\Big) = \frac{g(x) + g(y)}{2} \qquad (26)$$

for all real numbers $x$ and $y$. (This is trivial if $x = y$.)

By taking $y = 0$ in the initial functional equation and using $g(\frac{x}{2}) = \frac{1}{2}g(x) + \frac{1}{2}g(0)$ we obtain

$$f(x) = \frac{1}{2}xg(x) + \frac{1}{2}g(0)x + f(0). \qquad (27)$$

If we prove that $g$ is an affine function (a polynomial of degree at most 1), it will immediately follow that $f$ is a polynomial of degree at most 2.

By putting (27) into the original functional equation and using (26) we obtain

$$\frac{1}{2}xg(x) + \frac{1}{2}g(0)x - \frac{1}{2}yg(y) - \frac{1}{2}g(0)y = \frac{1}{2}(g(x) + g(y))(x - y)$$

which simplifies to

$$xg(y) - yg(x) = g(0)(x - y).$$

By taking $y = 1$ we obtain

$$g(x) = (g(1) - g(0))x + g(0),$$

so $g$ is really an affine function. ■

*Remark.* Conversely, if $f(x) = ax^2 + bx + c$ is a polynomial of degree at most 2, then it satisfies the functional equation with $g(x) = 2ax + b$.

8. *Problem.* Each positive integer is colored in one of finitely many given colors. Prove that there exist four different positive integers $a$, $b$, $c$, $d$, all in the same color, and such that:

   (1) $ad = bc$,
   (2) $\frac{b}{a}$ is a perfect power of 2,
   (3) $\frac{c}{a}$ is a perfect power of 3.

   *Solution.* Let us consider only numbers of the form $2^u 3^v$ for some integers $u \geq 0$, $v \geq 0$, and to each such number assign a point from the integer lattice $(u, v) \in \mathbb{N} \times \mathbb{N}$. (Here $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.)

   The three conditions in the statement of the problem simply mean that four points assigned to numbers $a$, $b$, $c$, $d$ make vertices of a rectangle with sides parallel to coordinate axes. Indeed, if $a = 2^u 3^v$, $\frac{b}{a} = 2^s$, $\frac{c}{a} = 3^t$, then these four points are $(u, v)$, $(u + s, v)$, $(u, v + t)$, $(u + s, v + t)$.

   Suppose that the total number of colors is $k$. Consider only points in the grid $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, where $m = k+1$ and $n = k^{k+1}+1$. Every horizontal "line" $\{1, 2, \ldots, m\} \times \{j\}$ can be colored in $k^m = k^{k+1}$ ways. Since $n > k^{k+1}$, there exist two identically colored lines: $\{1, 2, \ldots, m\} \times \{j_1\}$ and $\{1, 2, \ldots, m\} \times \{j_2\}$, for some $j_1 < j_2$. Since $m > k$, there exist points on the first line $(i_1, j_1)$ and $(i_2, j_1)$, $i_1 < i_2$, that have the same color. Finally, $(i_1, j_1)$, $(i_2, j_1)$, $(i_1, j_2)$, $(i_2, j_2)$ form vertices of a rectangle and all have the same color. ■

9. *Problem.* Let $P$ be a polynomial with complex coefficients of degree at most $n - 1$ and suppose that precisely $k$ of its coefficients are nonzero, $1 \leq k \leq n$. Let us also denote $Q(z) = z^n - 1$. Prove that polynomials $P$ and $Q$ have at most $n - \frac{n}{k}$ common roots, i.e. there exist at most $n - \frac{n}{k}$ different complex numbers $z$ satisfying $P(z) = 0 = Q(z)$.

   *Solution.* If we denote $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then all roots of $Q$ are $1$, $\omega$, $\omega^2$, $\ldots$, $\omega^{n-1}$. We write $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1}z^{n-1}$ for some complex numbers $a_0, a_1, \ldots a_{n-1}$. Let us first prove the following identity:

$$\sum_{j=0}^{n-1} |P(\omega^j)|^2 = n\sum_{l=0}^{n-1} |a_l|^2. \tag{28}$$

   For the proof we first expand the left hand side as

$$\begin{aligned}\sum_{j=0}^{n-1} |P(\omega^j)|^2 &= \sum_{j=0}^{n-1} P(\omega^j)\overline{P(\omega^j)} = \sum_{j=0}^{n-1}\Big(\sum_{l=0}^{n-1} a_l\,\omega^{jl}\Big)\Big(\sum_{m=0}^{n-1}\overline{a_m}\,\omega^{-jm}\Big)\\ &= \sum_{l=0}^{n-1}\sum_{m=0}^{n-1} a_l\overline{a_m}\Big(\sum_{j=0}^{n-1}\omega^{j(l-m)}\Big)\end{aligned}$$

and then observe that for $l \neq m$

$$\sum_{j=0}^{n-1} \omega^{j(l-m)} = \frac{\omega^{n(l-m)} - 1}{\omega^{l-m} - 1} = 0,$$

while for $l = m$ the corresponding sum equals $n$. This completes the proof of (28).

If we had $P(1) = P(\omega) = P(\omega^2) = \ldots = P(\omega^{n-1}) = 0$, then $P$ would be a nonzero polynomial of degree at most $n-1$, with $n$ different roots, which is impossible. Denote by $M$ the maximum of the numbers $|P(1)|$, $|P(\omega)|$, $|P(\omega^2)|$, $\ldots$, $|P(\omega^{n-1})|$. From the previous remark we conclude $M > 0$. Also observe that simply by triangle inequality

$$|P(\omega^j)| \leq \sum_{l=0}^{n-1} |a_l \omega^{jl}| = \sum_{l=0}^{n-1} |a_l|,$$

so

$$M \leq \sum_{l=0}^{n-1} |a_l|. \tag{29}$$

Let $J$ be the set of all indices $0 \leq j \leq n-1$ such that $P(\omega^j) \neq 0$ and let $L$ be the set of all indices $0 \leq l \leq n-1$ such that $a_l \neq 0$. Observe the trivial estimate

$$\sum_{j=0}^{n-1} |P(\omega^j)|^2 = \sum_{j \in J} |P(\omega^j)|^2 \leq M^2 |J|. \tag{30}$$

Finally, using the arithmetic mean–quadratic mean inequality we get

$$\frac{1}{|L|} \sum_{l \in L} |a_l| \leq \Big( \frac{1}{|L|} \sum_{l \in L} |a_l|^2 \Big)^{\frac{1}{2}}$$

and combining this with (28), (29), (30) we obtain

$$\frac{M^2}{|L|} \leq \frac{1}{|L|} \Big( \sum_{l \in L} |a_l| \Big)^2 \leq \sum_{l \in L} |a_l|^2 = \frac{1}{n} \sum_{j \in J} |P(\omega^j)|^2 \leq \frac{M^2 |J|}{n}.$$

This gives $|J| \geq \frac{n}{|L|} = \frac{n}{k}$, i.e. at least $\frac{n}{k}$ of the numbers $P(1)$, $P(\omega)$, $P(\omega^2)$, $\ldots$, $P(\omega^{n-1})$ are nonzero, so at most $n - \frac{n}{k}$ of them are equal to 0. ■

10. *Problem.* We say that a set of positive integers $S$ is *nice* if it is a nonempty subset of $\{1, 2, 3, \ldots, 2008\}$ and the product of numbers in $S$ is a perfect power of 10.

    (a) What is the size of the largest nice set?

    (b) What is the size of the largest nice set without proper nice subsets?

*Solution.* We see that all elements of a nice set must be of the form $2^k 5^l$ for some nonnegative integers $k$ and $l$. Moreover, $2^k 5^l \leq 2008$ leaves us only the following possibilities:

$$\begin{aligned} l = 0 &\quad \text{and} \quad 0 \leq k \leq 10 \\ l = 1 &\quad \text{and} \quad 0 \leq k \leq 8 \\ l = 2 &\quad \text{and} \quad 0 \leq k \leq 6 \\ l = 3 &\quad \text{and} \quad 0 \leq k \leq 4 \\ l = 4 &\quad \text{and} \quad 0 \leq k \leq 1 \end{aligned}$$

For each number of the form $2^k 5^l$ we can consider the quantity $k - l$ and call it *value* of the number $2^k 5^l$. Observe that the value of the product of two such numbers is equal to the sum of their values. Denote by $T_j$ the set of all numbers listed above with value equal to $j$. It is easily seen that

$$|T_{-4}| = 1, \quad |T_{-3}| = |T_{-2}| = 2, \quad |T_{-1}| = 3, \quad |T_0| = |T_1| = 4$$

$$|T_2| = |T_3| = |T_4| = 3, \quad |T_5| = |T_6| = |T_7| = 2, \quad |T_8| = |T_9| = |T_{10}| = 1$$

and all other sets $T_j$ are empty. Let us also denote $T_- = \bigcup_{j<0} T_j$ and $T_+ = \bigcup_{j>0} T_j$, so that

$$|T_-| = 8, \quad |T_+| = 22.$$

Elements of $T_-$ have negative values, elements of $T_+$ have positive values, while all elements of $T_0$ have value 0.

(a) Let $S$ be a nice set. The condition that product of its elements is a power of 10 can be restated as the condition that sum of values of its elements is 0. (We simply use the fact that number of the form $2^k 5^l$ is a power of 10 if and only if $k = l$, i.e. $k - l = 0$.) The sum of values of all numbers in $T_-$ is

$$3 \cdot (-1) + 2 \cdot (-2) + 2 \cdot (-3) + 1 \cdot (-4) = -17,$$

so the total value of $S \cap T_-$ is at least $-17$, and consequently the total value of $S \cap T_+$ is at most 17. By a greedy strategy we respectively take elements from $T_1, T_2, \ldots, T_{10}$, until their total value exceeds 17. Since

$$4 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 = 19 > 17,$$

we cannot take all elements from $T_1 \cup T_2 \cup T_3$, so $S \cap T_+$ has at most $4 + 3 + 3 - 1 = 9$ elements. On the other hand if we take all elements form $T_1 \cup T_2$, one element from $T_3$ and one element from $T_4$, the total value of $S \cap T_+$ will be exactly

$$4 \cdot 1 + 3 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 = 17.$$

Therefore the largest nice set has

$$|T_-| + |T_0| + 4 + 3 + 1 + 1 = 21$$

elements.

(b) If we consider the set $S$ that consists of $T_{10} \cup T_{-1} \cup T_{-2}$ and an arbitrary element from $T_{-3}$ we get a nice set with 7 elements since

$$1 \cdot (-3) + 2 \cdot (-2) + 3 \cdot (-1) + 1 \cdot 10 = 0.$$

This set $S$ has no proper nice subsets because if we exclude the element from $T_{10}$, the total value of remaining elements would be negative, and if we include the element from $T_{10}$ but exclude some other elements, the total value would be positive.

Now suppose that $S$ is a nice set with no proper nice subsets and and that it has at least 8 elements. Then $S \cap T_0 = \emptyset$ and $S \cap T_-$ and $S \cap T_+$ are nonempty.

If $S \cap T_+$ has exactly one element, then $S \cap T_-$ has at least 7 elements. Thus $S \cap T_-$ has total value at most

$$3 \cdot (-1) + 2 \cdot (-2) + 2 \cdot (-3) = -13,$$

and since $S \cap T_+$ has total value at most 10, we get a contradiction with the fact that $S$ is nice. Therefore $S \cap T_+$ must have at least 2 elements.

If $S$ has at least one element from all 4 sets $T_{-1}$, $T_{-2}$, $T_{-3}$, $T_{-4}$, then all integers $-10, -9, \ldots, -2, -1$ appear as values of subsets of $S \cap T_-$, so none of the numbers $1, 2, \ldots, 9, 10$ can appear as a value of an element in $S \cap T_+$, which is a contradiction. Therefore $S$ does not intersect at least one of the sets $T_{-1}$, $T_{-2}$, $T_{-3}$, $T_{-4}$. In particular $S \cap T_-$ has total value at least $-14$ and consequently $S \cap T_+$ has total value at most 14.

If $S$ contains at least one element from $T_{-1}$, $T_{-2}$ and $T_{-3}$, then similarly all integers $-6, -5, -4, -3, -2, -1$ appear as values of subsets of $S \cap T_-$, so none of the numbers $1, 2, 3, 4, 5, 6$ can appear as values of elements in $S \cap T_+$. Thus, every element of $S \cap T_+$ has value at least 7 and its total value is at least 14. This is possible only when $S \cap T_+ = T_7$, but then $S \cap T_-$ could not have another element from any of sets $T_{-1}$, $T_{-2}$ and $T_{-3}$, since otherwise $-7$ would be attained as a value of one of its subsets.

In the same way we get a contradiction when $S$ intersects $T_{-1}$, $T_{-2}$ and $T_{-4}$ and in the case when $S$ contains at least 2 elements from $T_{-1}$ and at least one element from $T_{-3}$ and $T_{-4}$.

If $S$ contains exactly one elements from $T_{-1}$ and at least one element from $T_{-3}$ and $T_{-4}$, then it cannot have an element from $T_{-2}$, so the cardinality of $S \cap T_-$ is at most 4. Consequently $S \cap T_+$ has at least 4 elements and they can only be from $T_2$, $T_6$, $T_9$, $T_{10}$, since $-8, -7, -5, -4, -3, -1$ all appear as values of some subsets of $S \cap T_-$. Also note that $S$ cannot contain two elements from $T_2$ since sum of their values would be 4. Therefore the total value of $S \cap T_+$ is at least $2 + 6 + 6 + 9 = 23 > 14$, which is a contradiction.

If $S$ contains at least one element from $T_{-2}$, $T_{-3}$ and $T_{-4}$, then $S \cap T_+$ can contain only elements with values 1, 8 and 10. Since it cannot contain 2 elements from $T_1$, its total value is at least $1 + 8 + 10 = 19 > 14$, which is a contradiction.

We conclude that $S$ intersects at most 2 of the sets $T_{-1}$, $T_{-2}$, $T_{-3}$, $T_{-4}$. In particular $S \cap T_-$ has the total value at least $-10$, so $S \cap T_+$ has the total value at most 10.

Suppose that $S$ does not intersect $T_{-1}$. Then $S \cap T_-$ has at most 4 elements, so $S \cap T_+$ has at least 4 elements.

If $S$ intersects $T_{-2}$ and $T_{-3}$, then $S$ can have at most one element from $T_1$ and no elements from $T_2$ and $T_3$. Therefore $S \cap T_+$ must have total value at least $1+4+4+4 = 13 > 10$, which gives a contradiction.

If $S$ intersects $T_{-2}$ and $T_{-4}$, then $S$ can have at most one element from $T_1 \cup T_3$ and no elements from $T_2$ and $T_4$. Thus $S \cap T_+$ has the total value at least $1+5+5+5 = 16 > 10$, which is again a contradiction.

If $S$ intersects $T_{-3}$ and $T_{-4}$, then $S$ has at most one element from $T_1 \cup T_2$ and no elements from $T_3$ and $T_4$, so the value of $S \cap T_+$ is at least $1 + 5 + 5 + 5 = 16 > 10$, a contradiction.

If $S$ is a subset of $T_{-2}$, $T_{-3}$ or $T_{-4}$, then $S \cap T_-$ has at most 2 elements and its value at least $-6$. Consequently, $S \cap T_+$ has at least 6 elements and the total value at most 6, which is impossible.

We conclude that $S$ has to intersect $T_{-1}$.

If $S$ contains exactly one element from $T_{-1}$, then $S \cap T_+$ has at least 5 elements and its total value is at least $2 + 2 + 2 + 3 + 3 = 12 > 10$, which is a contradiction.

If $S$ contains exactly 2 elements from $T_{-1}$, then $S \cap T_+$ has at least 4 elements and it is disjoint from $T_1$ and $T_2$. Thus the total value of $S \cap T_+$ is least $3+3+3+4 = 13 > 10$, a contradiction.

If $S$ contains all 3 elements from $T_{-1}$, then $S \cap T_+$ has at least 3 elements and it is disjoint from $T_1$, $T_2$ and $T_3$. Consequently, the value of $S \cap T_+$ is least $4+4+4 = 12 > 10$, which is also impossible.

We have obtained a contradiction in all cases above, so $|S| \geq 8$ is impossible. Therefore, the largest set as in the problem has 7 elements.

*Remark.* At this point I got busy with problems for "grown-up" mathematicians, so I didn't ask for a feedback on this problem. I don't know whether it appeared anywhere and I am a bit skeptical about its reception.

11. *Problem.* On every square of a $9 \times 9$ board a light bulb is placed. In one move we are allowed to choose a square and toggle on/off the states of light bulbs on the chosen square and all its horizontally and vertically adjacent squares. (Every square has 2, 3 or 4 adjacent squares.) Initially all light bulbs are on, and suppose that after some number of moves precisely one light bulb remains on. Prove that this light bulb must be positioned in the center of the board.

    *Solution.* Consider all configurations on the $9 \times 9$ board consisting of a chosen square and its horizontal and vertical neighbors. (Each of those constellations contains 3, 4 or 5 squares.) Let us label squares on the board using numbers 0, 1, 2, 3, so that every such configuration: either contains odd number of 1's, odd number of 2's, and odd number of 3's; or it contains even number of 1's, even number of 2's, and even number of 3's.

    One possible construction is given in the table below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 3 |
| 1 | 1 | 0 | 2 | 2 | 2 | 0 | 3 | 3 |
| 1 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 3 |
| 0 | 3 | 3 | 1 | 2 | 3 | 1 | 1 | 0 |
| 2 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 2 |
| 2 | 3 | 1 | 3 | 2 | 1 | 3 | 1 | 2 |
| 3 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 3 | 2 | 0 | 2 | 0 | 2 | 1 | 3 |
| 1 | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 3 |

Such example is certainly not unique. To come to the above construction, one only has to guess entries for the first row: 1, 0, 0, 0, 2, 0, 0, 0, 3. After that we proceed towards the bottom row, and all other entries are uniquely determined by the above condition. However, one has to be warned that the choice of the first row is rather subtle, as we might get stuck when reaching the bottom row.

At any given moment, let $N_1$ denote the number of light bulbs on squares labeled by 1 that are currently turned on. Similarly we define quantities $N_2$ and $N_3$. Since initially all light bulbs are on, we have $N_1 = 16$, $N_2 = 18$, and $N_3 = 16$, and notice that all three of these numbers are even. At every move we either change parities of all three numbers $N_1$, $N_2$, $N_3$, or leave their parities unchanged. Since in the end we have $N_1 + N_2 + N_3 \leq 1$ and all three numbers must have the same parity, we conclude that actually $N_1 = N_2 = N_3 = 0$, i.e. the remaining light bulb must be on one of 31 squares labeled by 0.

Finally observe that the above table is "quite asymmetrical", and the same argument can be carried by labeling squares on the board using the table obtained by flipping the above one vertically or diagonally, or by rotating it through 90°, 180°, or 270°. (Actually diagonal reflections are enough.) This eliminates all 0's except for the central one, and completes the proof. ■

*Remark.* It is not very hard to see that "only the center light bulb is on" and "all light bulbs are off" are achievable positions. One can show that precisely $2^{73}$ out of $2^{81}$ positions are achievable. For more general $n \times n$ boards the situation depends on $n$ in a very complicated way.

The problem was inspired by the classic puzzle game *Lights Out*, usually played on a $5 \times 5$ board. For more references see the web page at *MathWorld*
`http://mathworld.wolfram.com/LightsOutPuzzle.html`
and *Wikipedia*
`http://en.wikipedia.org/wiki/Lights_Out_(video_game)`

12. *Problem.* In some country there are $n$ cities, where $n \geq 5$. Some pairs of cities are connected by direct two-way flights, and there are at most $3n - 7$ such flights provided by the airline. A set of 5 cities with direct two-way flights between each two of them is called a *5-city tour*. Prove that it is possible to introduce a new flight (between two cities that are not already directly connected) without making any new 5-city tours.

*Solution.* Denote the cities by $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ and let $\mathcal{F}$ be the set of all (unordered) pairs of cities that are connected by a direct two-way flight. Suppose that the statement in the theorem is false, so that for every pair $\{A_1, A_2\} \notin \mathcal{F}$ there exist cities $B_1, B_2, B_3$ such that in the 5-element set $\mathcal{S} = \{A_1, A_2, B_1, B_2, B_3\}$ all pairs except for $\{A_1, A_2\}$ belong to $\mathcal{F}$.

Imagine that a person chooses a route visiting each city precisely once. In other words, the person chooses a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ and visits cities in the order: $C_{\sigma(1)}$, $C_{\sigma(2)}$, $\ldots$, $C_{\sigma(n)}$. For a fixed pair $\{A_1, A_2\} \notin \mathcal{F}$ let the set $\mathcal{S}$ be as above. We say that a route (i.e. permutation) *respects* the pair $\{A_1, A_2\}$ if the person visits both $A_1$ and $A_2$ before visiting any of the cities from $\mathcal{C} \setminus \mathcal{S}$. In other words, $A_1$ and $A_2$ can only be preceded by $A_1, A_2, B_1, B_2, B_3$ in the above ordering of the cities for a given $\sigma$.

Let us count how many routes (i.e. permutations) respect the fixed pair $\{A_1, A_2\} \notin \mathcal{F}$. It is easier to think in terms of positions in the ordering $C_{\sigma(1)}, C_{\sigma(2)}, \ldots, C_{\sigma(n)}$. We first choose 3 positions for $B_1, B_2, B_3$ in $n(n-1)(n-2)$ ways. Among the remaining $n-3$ positions the first 2 are reserved for $A_1, A_2$, and the last $n-5$ are reserved for elements of $\mathcal{C} \setminus \mathcal{S}$. Since they can come in an arbitrary order, we see that the number of such routes is

$$n(n-1)(n-2) \cdot 2! \cdot (n-5)!\,.$$

By the condition in the problem we have $|\mathcal{F}| < 3n-6$, so the number of pairs not connected by direct flights is $|\mathcal{F}^c| > \binom{n}{2} - (3n-6) = \frac{(n-3)(n-4)}{2}$. The total number of relations "a route respects a pair" is thus

$$n(n-1)(n-2) \cdot 2! \cdot (n-5)! \cdot |\mathcal{F}^c| \;>\; n(n-1)(n-2) \cdot 2! \cdot (n-5)! \cdot \tfrac{(n-3)(n-4)}{2} = n!\,,$$

i.e. strictly greater than the total number of routes/permutations. Therefore by the pigeonhole principle there must be a permutation $\sigma$ that respects at least two different pairs from $\mathcal{F}^c$. Name those pairs $\{A_1, A_2\}$, $\{A_1', A_2'\}$, and let $\mathcal{S}$, $\mathcal{S}'$ be the corresponding sets as above. (Some of their elements could coincide.)

By definition, $A_1, A_2$ precede all cities from $\mathcal{C} \setminus \mathcal{S}$, while $A_1', A_2'$ precede all cities from $\mathcal{C} \setminus \mathcal{S}'$. We claim that either $\{A_1, A_2\} \subseteq \mathcal{S}'$, or $\{A_1', A_2'\} \subseteq \mathcal{S}$. Otherwise we could find indices $i, j \in \{1, 2\}$ such that $A_i \in \mathcal{C} \setminus \mathcal{S}'$ and $A_j' \in \mathcal{C} \setminus \mathcal{S}$, which would imply that $A_j'$ precedes $A_i$ and that $A_i$ precedes $A_j'$, a contradiction.

WLOG suppose that we are in the first case, i.e.

$$\{A_1, A_2\} \subseteq \mathcal{S}' = \{A_1', A_2', B_1', B_2', B_3'\}\,.$$

Then the set $\mathcal{S}'$ contains two different pairs not belonging to $\mathcal{F}$, namely $\{A_1, A_2\}$ and $\{A_1', A_2'\}$. This is in contradiction with its construction. ■

13. *Problem.* Initially, only number 44 is written on the board. We repeatedly perform the following operation 30 times. At each step we simultaneously replace each number on the board, call it $a$, by four numbers $a_1, a_2, a_3, a_4$ that only have to satisfy:

- $a_1, a_2, a_3, a_4$ are four different integers.
- Average of four new numbers $(a_1 + a_2 + a_3 + a_4)/4$ is equal to the erased number $a$.

After 30 steps we end up with $n = 4^{30}$ numbers on the board, call them $b_1, b_2, \dots, b_n$. Prove that

$$\frac{b_1^2 + b_2^2 + \ldots + b_n^2}{n} \geq 2011\,.$$

*Solution.* Let us first prove an auxiliary statement.

*Lemma.* If $a_1, a_2, a_3, a_4$ are four different integers such that their average $a = (a_1 + a_2 + a_3 + a_4)/4$ is also an integer, then

$$\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{4} - a^2 \geq \frac{5}{2}\,.$$

*Proof.* Note that the expression on the left hand side can be transformed as

$$\begin{aligned}
&\frac{a_1^2 + a_2^2 + a_3^2 + a_4^2}{4} - a^2 \\
&= \frac{a_1^2 + a_2^2 + a_3^2 + a_4^2 - 8a^2 + 4a^2}{4} \\
&= \frac{a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a(a_1 + a_2 + a_3 + a_4) + 4a^2}{4} \\
&= \frac{(a_1 - a)^2 + (a_2 - a)^2 + (a_3 - a)^2 + (a_4 - a)^2}{4}\,.
\end{aligned}$$

Now, $a_1 - a$, $a_2 - a$, $a_3 - a$, $a_4 - a$ are four different integers that add up to 0. We claim that sum of their squares is at least 10. If none of these integers is 0, then that sum is at least $1^2 + (-1)^2 + 2^2 + (-2)^2 = 10$. On the other hand, if one of the integers is 0, than the remaining three cannot be only from the set $\{1, -1, 2, -2\}$, because no three different elements of that set add up to 0. Therefore, the sum of their squares is at least $3^2 + 1^2 + (-1)^2 = 11$. This completes the proof of the lemma.

Returning to the given problem, we denote by $S_k$ the average of squares of the numbers on the board after $k$ steps. More precisely,

$$S_k = \frac{b_{k,1}^2 + b_{k,2}^2 + \ldots + b_{k,4^k}^2}{4^k}\,,$$

where $b_{k,1}, b_{k,2}, \dots, b_{k,4^k}$ are the numbers appearing on the board after the operation is performed $k$ times. Applying the above lemma to each of the numbers, adding up these inequalities, and dividing by $4^k$, we obtain

$$S_{k+1} - S_k \geq \frac{5}{2}\,,$$

so in particular

$$S_{30} \;\geq\; S_0 + 30 \cdot \frac{5}{2} \;=\; 44^2 + 30 \cdot \frac{5}{2} \;=\; 2011\,.$$

■

14. *Problem.* At this year's MEMO, there are $3n$ participants, there are $n$ languages spoken, and each participant speaks exactly 3 different languages. Prove that MEMO coordinators can choose at least $\frac{2n}{9}$ languages for the presentation of the official solutions, such that no participant will understand the presentation in more than 2 languages.

*Solution.* The coordinators decide to approach this problem by splitting the set of $n$ available languages into **easy**, **medium**, and **hard**, according to how they perceive the difficulty of understanding math in each of the given languages. However, they cannot agree about this classification, so they are only aware that there are $3^n$ possibilities in total, as each language can be tagged as easy, medium, or hard. For each classification, let $A$ be the number of easy languages and let $B$ be the number of students who speak 3 easy languages.

If we add up quantities $A$ over all possible classifications, the resulting sum will be $\sum A = n3^{n-1}$. In order to verify that, we realize that the result should be the same for medium and hard languages too, but all three of these sums add up to

$$3\sum A \;=\; \text{number of classifications} \;\times\; \text{number of languages} \;=\; 3^n \cdot n\,.$$

On the other hand, we use "double counting trick" to compute the sum of quantities $B$ over all possible categorizations. For each student there are $3^{n-3}$ possibilities that allow him to speak 3 easy languages, as we only have the choice to classify each of the $n-3$ languages that the student does not speak. Therefore, the desired sum is

$$\sum B \;=\; 3n \cdot 3^{n-3} \;=\; n3^{n-2}\,.$$

We claim that there exists a classification such that $A - B \geq \frac{2n}{9}$. If each possibility had $A - B < \frac{2n}{9}$, then summing over all $3^n$ of them would give

$$n3^{n-1} - n3^{n-2} \;=\; \sum A - \sum B \;<\; 3^n \cdot \frac{2n}{9}\,,$$

i.e. $2n3^{n-2} < 2n3^{n-2}$, which is contradiction.

Let us consider any categorization of languages satisfying $A - B \geq \frac{2n}{9}$. The organizers first choose all $A$ easy languages. Then they find all $B$ students who can speak 3 of these languages, and for each of them they remove one of the languages the student speaks. This leaves the organizers with a choice of at least $\frac{2n}{9}$ languages. ■

*Remark.* Classification of languages simply as **easy** or **hard** would not give the desired bound. It would lead to a choice of at least $\frac{n}{8}$ languages only. Taking more than three language classes would not be a better strategy either.

## 2.2 College level competitions

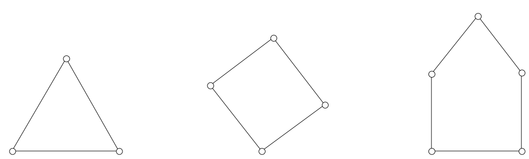
1. *Problem.* Let $p, q > 1$ be relatively prime positive integers.

   (a) Suppose that $f\colon \{1, 2, \dots, p+q-1\} \to \{0, 1\}$ is a periodic function with periods both $p$ and $q$. Prove that $f$ is a constant.

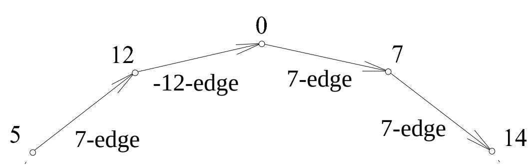
   (b) Show that there exist exactly 4 functions $f\colon \{1, 2, \dots, p+q-2\} \to \{0, 1\}$ that are periodic with periods both $p$ and $q$.

   *Solution.* Let us construct a simple undirected graph $G$ with a set of vertices $\{0, 1, 2, \dots, p+q-1\}$ and an edge between $i$ and $j$ if and only if $|i-j| = p$ or $|i-j| = q$. We claim that $G$ is indeed a cycle of length $p+q$.

   Note that every vertex $i$ has degree 2: Exactly 2 of the numbers $i-p$, $i+q$, $i-q$, $i+p$ belong to $\{0, 1, \dots, p+q-1\}$, because of the equalities $|(i+q)-(i-p)| = p+q$, $|(i+p)-(i-q)| = p+q$. Therefore, $G$ is a union of (vertex-)disjoint cycles.



   Suppose that $G$ contains a cycle $C$ of length less than $p+q$, and let $\overrightarrow{C}$ be an arbitrary orientation of $C$. A directed edge from $i$ to $j$ will be called a $k$-edge if $j - i = k$.

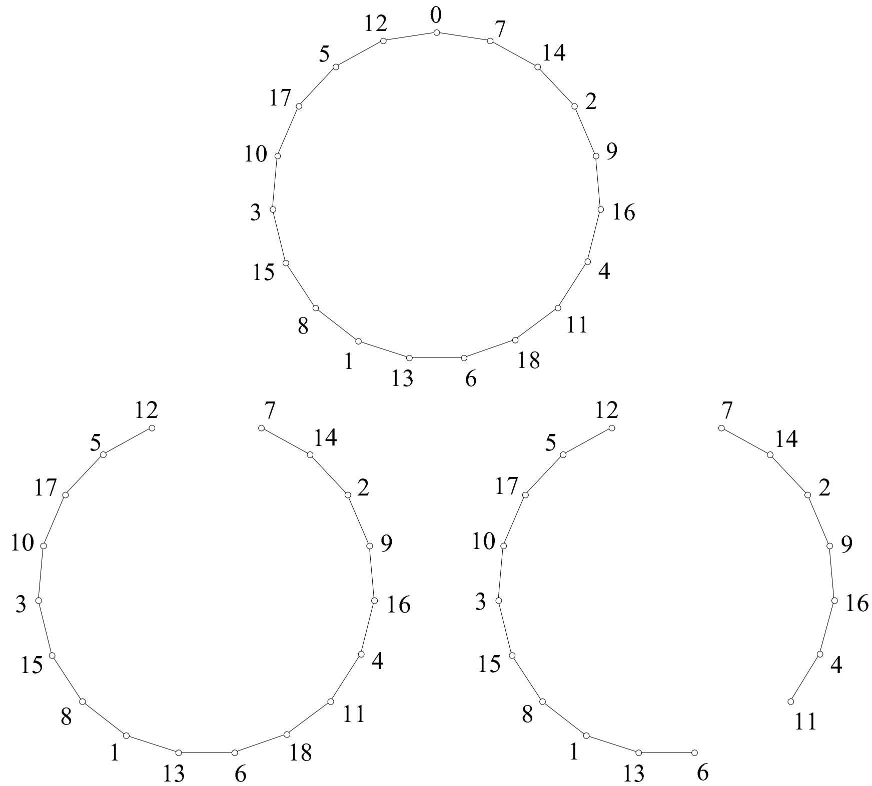



   WLOG there is a $p$-edge in $\overrightarrow{C}$. Obviously, a $p$-edge or a $-q$-edge in $\overrightarrow{C}$ is again followed by $p$-edge or $-q$-edge. Thus, $\overrightarrow{C}$ contains only $p$-edges (denote their number by $m$) and $-q$-edges (denote their number by $n$). The sum of labels along all of the edges of $\overrightarrow{C}$ equals 0, so $mp - nq = 0$. It follows that $m$ is divisible by $q$, $n$ is divisible by $p$ and $p+q \le m+n < p+q$ leads to a contradiction. Hence, we have proved that $G$ is one "big" cycle of length $p+q$. (Figures illustrate the case $p = 7$, $q = 12$.)

   Finally, we come to the proof of (a) and (b). For some $S \subseteq \{0, 1, \dots, p+q-1\}$, a function $f\colon S \to \{0, 1\}$ is both $p$-periodic and $q$-periodic if and only if it assigns the same value (0 or 1) to all vertices of the same component of $S$ (considered as the subgraph of $G$). The number of such functions is $2^{\omega(S)}$, where $\omega(S)$ denotes the number of components of $S$.

   (a) After deleting the vertex 0 from $G$, the graph becomes a path, which is connected.

   (b) Note that vertices 0 and $p+q-1$ are not adjacent in $G$. After deleting them, the graph becomes a union of 2 (vertex-)disjoint paths. The latter graph has exactly 2 components.




■

2. *Problem.* Let $A = [a_{i,j}]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ be an $m \times n$ real matrix with at least one non-zero entry. For each $i \in \{1, \dots, m\}$ let $R_i := \sum_{j=1}^{n} a_{i,j}$ denote the sum of entries in the $i$-th row of $A$ and for each $j \in \{1, \dots, n\}$ let $C_j := \sum_{i=1}^{m} a_{i,j}$ denote the sum of entries in the $j$-th column of $A$. Prove that there exist indices $i_0 \in \{1, \dots, m\}$, $j_0 \in \{1, \dots, n\}$ such that

$$\begin{aligned} & a_{i_0,j_0} > 0, \quad R_{i_0} \geq 0, \quad C_{j_0} \geq 0, \\ \text{or} \quad & a_{i_0,j_0} < 0, \quad R_{i_0} \leq 0, \quad C_{j_0} \leq 0. \end{aligned}$$

*Solution.* Consider the following sets of indices (which may be empty):

$$\begin{aligned} I^+ &:= \{i \in \{1, \dots, m\} \mid R_i \geq 0\}, \\ I^- &:= \{i \in \{1, \dots, m\} \mid R_i < 0\}, \\ J^+ &:= \{j \in \{1, \dots, n\} \mid C_j > 0\}, \\ J^- &:= \{j \in \{1, \dots, n\} \mid C_j \leq 0\}. \end{aligned}$$

Suppose that the statement of the problem does not hold. Then for every $(i, j) \in I^+ \times J^+$ we have $a_{i,j} \leq 0$ and for every $(i, j) \in I^- \times J^-$ we have $a_{i,j} \geq 0$. Let us write the sum $\displaystyle\sum_{(i,j)\in I^- \times J^+} a_{i,j}$ in two different ways.

$$\sum_{(i,j)\in I^- \times J^+} a_{i,j} = \sum_{i \in I^-} \left( \sum_{j=1}^{n} a_{i,j} - \sum_{j \in J^-} a_{i,j} \right) = \sum_{i \in I^-} R_i - \sum_{(i,j)\in I^- \times J^-} a_{i,j} \leq 0$$

$$\sum_{(i,j)\in I^-\times J^+} a_{i,j} \;=\; \sum_{j\in J^+}\left(\sum_{i=1}^{m} a_{i,j} - \sum_{i\in I^+} a_{i,j}\right) = \sum_{j\in J^+} C_j - \sum_{(i,j)\in I^+\times J^+} a_{i,j} \;\geq 0$$

Therefore, $\displaystyle\sum_{(i,j)\in I^-\times J^+} a_{i,j} = 0$ and we have only equalities in the two formulae above. This is only possible if $\displaystyle\sum_{i\in I^-} R_i = 0$, $\displaystyle\sum_{j\in J^+} C_j = 0$, so $I^- = \emptyset$, $J^+ = \emptyset$, which means $R_i \geq 0$; $i = 1,\ldots,m$ and $C_j \leq 0$; $j = 1,\ldots,n$. Moreover, from

$$0 \leq \sum_{i=1}^{m} R_i = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i,j} = \sum_{j=1}^{n} C_j \leq 0$$

we conclude $R_i = 0$; $i = 1,\ldots,m$ and $C_j = 0$; $j = 1,\ldots,n$. Because $A$ is non-zero matrix, there are indices $i_0, j_0$ such that $a_{i_0,j_0} \neq 0$, $R_{i_0} = 0$, $C_{j_0} = 0$, and this leads to a contradiction to the assumption that the statement of the problem is false. ■

*Remark.* The problem was solved by most of the students and seems to have been the easiest in that category.

3. *Problem.* A sequence $(a_n)_{n\geq 0}$ of real numbers is defined recursively by

$$a_0 := 0, \quad a_1 := 1, \quad a_{n+2} := a_{n+1} + \frac{a_n}{2^n};\; n \geq 0.$$

Prove the following:

(a) The sequence $(a_n)_{n\geq 0}$ is convergent.

(b) $\displaystyle\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \cdot \prod_{k=1}^{n}\left(2^k - 1\right)}$

(c) The limit $\displaystyle\lim_{n\to\infty} a_n$ is an irrational number.

*Solution.*

(a) Obviously, $a_n \geq 0$ for every $n \geq 0$. The sequence $(a_n)_{n\geq 0}$ is increasing since $a_{n+2} - a_{n+1} = \frac{a_n}{2^n} \geq 0$ for every $n \geq 0$. It suffices to show that $(a_n)_{n\geq 0}$ is bounded from above. For each $k \geq 0$ we have $a_{k+2} \leq a_{k+1}\left(1 + \frac{1}{2^k}\right)$. Using the inequality between geometric and arithmetic mean, for every $n \geq 1$ we obtain

$$\begin{aligned} a_{n+2} &\leq \prod_{k=0}^{n}\left(1 + \frac{1}{2^k}\right) = 2\prod_{k=1}^{n}\left(1 + \frac{1}{2^k}\right) \\ &\leq 2\left(\frac{1}{n}\left(n + \sum_{k=1}^{n}\frac{1}{2^k}\right)\right)^n \leq 2\left(\frac{n+1}{n}\right)^n \leq 2e. \end{aligned}$$

(b) Consider the power series $\sum_{n=0}^{\infty} a_n z^n$. Since $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \lim_{n\to\infty} \sqrt[n]{2e} = 1$, its radius of convergence is $R \ge 1$. Therefore, on the open unit disc with center 0 it converges to a holomorphic function $f(z) := \sum_{n=0}^{\infty} a_n z^n$. Inductively we obtain $a_{n+2} = \sum_{k=0}^{n} \frac{a_k}{2^k} + 1$; $n \ge 0$, so $\lim_{n\to\infty} a_n = \sum_{k=0}^{\infty} \frac{a_k}{2^k} + 1 = f\left(\frac{1}{2}\right) + 1$ and we have to find $f\left(\frac{1}{2}\right)$.

Now we use the recurrent relation for $(a_n)_{n\ge 0}$ to obtain a functional equation for $f$. We multiply $a_{n+2} := a_{n+1} + \frac{a_n}{2^n}$ by $z^{n+2}$ and sum over all $n \ge 0$ to get

$$\sum_{n=0}^{\infty} a_{n+2} z^{n+2} = z \sum_{n=0}^{\infty} a_{n+1} z^{n+1} + z^2 \sum_{n=0}^{\infty} a_n \left(\frac{z}{2}\right)^n,$$

that is

$$f(z) - z = z\, f(z) + z^2\, f\left(\frac{z}{2}\right),$$

or

$$(1-z) f(z) = z^2\, f\left(\frac{z}{2}\right) + z; \quad |z| < 1. \tag{31}$$

We substitute $z = \frac{1}{2^n}$; $n = 1, \ldots, N$ ($N \ge 1$ fixed) into (31), then multiply the $n$-th equality by some constant $s_n > 0$ and finally sum up those $N$ equalities.

$$\begin{array}{rcll}
\left(1-\frac{1}{2}\right) f\left(\frac{1}{2}\right) & = & \left(\frac{1}{2}\right)^2 f\left(\frac{1}{4}\right) + \frac{1}{2} & / \cdot s_1 \\
\left(1-\frac{1}{4}\right) f\left(\frac{1}{4}\right) & = & \left(\frac{1}{4}\right)^2 f\left(\frac{1}{8}\right) + \frac{1}{4} & / \cdot s_2 \\
\cdots\cdots & \cdots & \cdots\cdots & \cdots \\
\left(1-\frac{1}{2^n}\right) f\left(\frac{1}{2^n}\right) & = & \left(\frac{1}{2^n}\right)^2 f\left(\frac{1}{2^{n+1}}\right) + \frac{1}{2^n} & / \cdot s_n \\
\left(1-\frac{1}{2^{n+1}}\right) f\left(\frac{1}{2^{n+1}}\right) & = & \left(\frac{1}{2^{n+1}}\right)^2 f\left(\frac{1}{2^{n+2}}\right) + \frac{1}{2^{n+1}} & / \cdot s_{n+1} \\
\cdots\cdots & \cdots & \cdots\cdots & \cdots \\
\left(1-\frac{1}{2^N}\right) f\left(\frac{1}{2^N}\right) & = & \left(\frac{1}{2^N}\right)^2 f\left(\frac{1}{2^{N+1}}\right) + \frac{1}{2^N} & / \cdot s_N \\
\hline
\frac{s_1}{2} f\left(\frac{1}{2}\right) & = & \frac{s_N}{2^{2N}}\, f\left(\frac{1}{2^{N+1}}\right) + \sum_{n=1}^{N} \frac{s_n}{2^n} &
\end{array}$$

To obtain cancellation of terms that contain $f\left(\frac{1}{2^n}\right)$; $n = 2, \ldots, N$ we choose $(s_n)_{n\ge 0}$ such that

$$s_0 := 1, \quad \left(\frac{1}{2^n}\right)^2 s_n = \left(1 - \frac{1}{2^{n+1}}\right) s_{n+1};\ n \ge 0. \tag{32}$$

Equalities (32) lead to

$$s_n = \prod_{k=0}^{n-1} \frac{s_{k+1}}{s_k} = \prod_{k=0}^{n-1} \frac{\left(\frac{1}{2^k}\right)^2}{1 - \frac{1}{2^{k+1}}} = \prod_{k=0}^{n-1} \frac{1}{2^{k-1}\,(2^{k+1}-1)} = \frac{1}{2^{\frac{n(n-1)}{2}-n} \prod_{k=1}^{n} (2^k - 1)}$$

for every $n \ge 1$. Finally, we have

$$f\left(\frac{1}{2}\right) = \frac{s_N}{2^{2N}}\, f\left(\frac{1}{2^{N+1}}\right) + \sum_{n=1}^{N} \frac{s_n}{2^n}$$

$$= \frac{f\left(\frac{1}{2^{N+1}}\right)}{2^{\frac{N(N-1)}{2}+N}\prod_{k=1}^{N}(2^k-1)} + \sum_{n=1}^{N}\frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)}.$$

The first summand tends to 0 when $N\to\infty$, so

$$f\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)}. \tag{33}$$

(c) The proof of $\lim\limits_{n\to\infty} a_n \in \mathbb{R}\setminus\mathbb{Q}$ is based on the fact that the series in (33) converges "very rapidly". Suppose that its sum equals $\frac{p}{q}$ for some positive integers $p$ and $q$. For each integer $N\geq 1$ denote

$$q_N := 2^{\frac{N(N-1)}{2}}\prod_{k=1}^{N}\left(2^k-1\right),\quad p_N := q_N\sum_{n=1}^{N}\frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)}.$$

Obviously, $p_N$ and $q_N$ are positive integers. We manage to estimate $p\,q_N - q\,p_N$.

$$q_N = 2^{\frac{N(N-1)}{2}}\prod_{k=1}^{N}\left(2^k-1\right) < 2^{\frac{N(N-1)}{2}}\prod_{k=1}^{N}2^k = 2^{N^2}$$

$$\begin{aligned}\frac{p}{q}-\frac{p_N}{q_N} &= \sum_{n=N+1}^{\infty}\frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}(2^k-1)} \leq \sum_{n=N+1}^{\infty}\frac{1}{2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}2^{k-1}}\\ &= \sum_{n=N+1}^{\infty}\frac{1}{2^{n(n-1)}} \leq \sum_{m=N(N+1)}^{\infty}\frac{1}{2^m} = \frac{1}{2^{N^2+N-1}} < \frac{1}{2^{N-1}q_N}\end{aligned}$$

Thus, $0 < p\,q_N - q\,p_N < \frac{q}{2^{N-1}}$, so $(p\,q_N - q\,p_N)_{N\geq 1}$ is a sequence of positive integers that converges to 0. This is a contradiction and we are done.

■

*Remark.* Only part (b) was posed at the competition. Still, only one student solved it completely.

4. *Problem.* Let $f,g\colon[0,1]\to\langle 0,+\infty\rangle$ be continuous functions such that $f$ and $\frac{g}{f}$ are increasing. Prove that

$$\int_0^1 \frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt}\,dx \;\leq\; 2\int_0^1\frac{f(t)}{g(t)}\,dt.$$

*Solution.* At first, we estimate the expression under the integral sign on the left side of the given inequality. By the Chebycheff's inequality for integrals applied to increasing functions $f$ and $\frac{g}{f}$ on the segment $[0,x]$ (where $x\in\langle 0,1]$ is fixed), we get

$$\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\left(\frac{1}{x}\int_0^x \frac{g(t)}{f(t)}\,dt\right) \leq \frac{1}{x}\int_0^x g(t)\,dt,$$

that is,

$$\frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt} \le \frac{x}{\int_0^x \frac{g(t)}{f(t)}\,dt} \tag{34}$$

for every $x \in \langle 0, 1]$. From the integral form of the Cauchy-Schwarz inequality on the segment $[0, x]$, we have

$$\left(\int_0^x \frac{g(t)}{f(t)}\,dt\right)\left(\int_0^x \frac{t^2 f(t)}{g(t)}\,dt\right) \ge \left(\int_0^x t\,dt\right)^2 = \frac{x^4}{4},$$

or

$$\frac{1}{\int_0^x \frac{g(t)}{f(t)}\,dt} \le \frac{4}{x^4}\int_0^x \frac{t^2 f(t)}{g(t)}\,dt. \tag{35}$$

From (34) and (35) we obtain

$$\frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt} \le \frac{4}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,dt. \tag{36}$$

Finally, it remains to integrate (36) over $x \in \langle 0, 1]$ and to reverse the order of integration.

$$\begin{aligned}
\int_0^1 \frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt}\,dx &\le \int_0^1 \left(\int_0^x \frac{4t^2 f(t)}{x^3 g(t)}\,dt\right) dx = \int_0^1 \left(\int_t^1 \frac{4t^2 f(t)}{x^3 g(t)}\,dx\right) dt \\
&= \int_0^1 \frac{4t^2 f(t)}{g(t)}\left(\int_t^1 \frac{dx}{x^3}\right) dt = \int_0^1 \frac{4t^2 f(t)}{g(t)}\left(\frac{1}{2t^2} - \frac{1}{2}\right) dt \\
&= 2\int_0^1 \frac{f(t)}{g(t)}\left(1 - t^2\right) dt \le 2\int_0^1 \frac{f(t)}{g(t)}\,dt
\end{aligned}$$

■

*Remark.* The constant 2 on the right side of the given inequality is optimal, i.e. the least possible. Consider $f(t) := 1$, $g(t) := t + \varepsilon$, for some fixed $\varepsilon > 0$.

$$\int_0^1 \frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt}\,dx = \int_0^1 \frac{x}{\frac{1}{2}x^2 + \varepsilon x}\,dx = 2\int_0^1 \frac{dx}{x + 2\varepsilon} = 2\ln(1 + 2\varepsilon) - 2\ln 2 - 2\ln\varepsilon$$

$$\int_0^1 \frac{f(t)}{g(t)}\,dt = \int_0^1 \frac{dt}{t + \varepsilon} = \ln(1 + \varepsilon) - \ln\varepsilon$$

The quotient of those expressions can be made arbitrarily close to 2 since

$$\lim_{\varepsilon \searrow 0} \frac{2\ln(1 + 2\varepsilon) - 2\ln 2 - 2\ln\varepsilon}{\ln(1 + \varepsilon) - \ln\varepsilon} = 2\lim_{\varepsilon \searrow 0} \frac{-\frac{\ln(1+2\varepsilon)}{\ln\varepsilon} + \frac{\ln 2}{\ln\varepsilon} + 1}{-\frac{\ln(1+\varepsilon)}{\ln\varepsilon} + 1} = 2.$$

*Another remark.* Unfortunately, nobody solved this problem. Indeed, no student got more than 2 points, out of 10. Seems that the graders are too harsh sometimes.

5. *Problem.* Let $G$ be a (multiplicatively written) group with identity $e$. If elements $a, b \in G$ satisfy the relations

$$a^3 = e, \quad ab^2 = ba^2, \quad (a^2b)^{2003} = e,$$

show that $a = b$.

*Solution.* First observe that

$$(a^2b)^2 = a^2(ba^2)b = a^2(ab^2)b = a^3b^3 = b^3,$$

so

$$e = a^2b(a^2b)^{2002} = a^2b(b^3)^{1001} = a^2b^{3004} = a^{-1}b^{3004}.$$

Therefore, $a = b^{3004}$ and in particular $a$ and $b$ commute. Thus, $ab^2 = ba^2$ implies $b = a$. ∎

6. *Problem.* Prove that there do not exist a real number $a$ and a function $f\colon \mathbb{R} \to \mathbb{R}$ continuous at 0 satisfying

$$f\left(\frac{x+a}{1-ax}\right) > f(x)$$

for every $x \in \mathbb{R}$ such that $ax \neq 1$.

*Solution.* Denote $S := \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$ and define

$$g\colon S \to \mathbb{R}, \quad g(t) := f(\tan t);\ t \in S.$$

Note that $g$ is $\pi$-periodic and continuous at 0. Let $\alpha \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2}\right\rangle$ be such that $a = \tan\alpha$. If $x = \tan\beta,\ \beta \in S,\ \beta + \alpha \in S$, then we have

$$\frac{x+a}{1-ax} = \frac{\tan\beta + \tan\alpha}{1 - \tan\alpha\tan\beta} = \tan(\beta+\alpha),$$

so the inequality from the statement of the problem can be rewritten as

$$g(\beta+\alpha) > g(\beta); \quad \text{whenever } \beta \in S \text{ and } \beta+\alpha \in S.$$

We distinguish two cases.

(a) $\frac{\alpha}{\pi}$ *is rational.*
Suppose $\alpha = \frac{m}{n}\pi$, $m, n \in \mathbb{Z}$, $n > 0$. Take any $\beta \in \mathbb{R}$ such that $\frac{\beta}{\pi} \notin \mathbb{Q}$. We have

$$g(\beta) = g(\beta + m\pi) = g(\beta + n\alpha) > g(\beta),$$

a contradiction.

(b) $\frac{\alpha}{\pi}$ *is irrational.*
Since $\{n\alpha : n \in \mathbb{Z}^+\}$ is dense in $\left\langle -\frac{\pi}{2}, \frac{\pi}{2}\right\rangle$ modulo $\pi$, we can find a sequence of integers $(k_j)_j$ and a strictly increasing sequence of positive integers $(n_j)_j$ such that

$$\lim_{j\to\infty}(n_j\alpha - k_j\pi) = 0.$$

From the continuity of $g$ at 0 it follows that

$$\lim_{j\to\infty} g(n_j\alpha) = \lim_{j\to\infty} g(n_j\alpha - k_j\pi) = g(0).$$

But the sequence $(g(n_j\alpha))_j$ is increasing and $g(n_1\alpha) > g(0)$, a contradiction.

∎

7. *Problem.* Let $n$ be a positive integer. A linear operator $\mathcal{T}\colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C})$ is defined as follows. For any $A \in \mathrm{M}_n(\mathbb{C})$, the $(i,j)$-th entry of $\mathcal{T}(A)$ equals the sum of the $(i,j)$-th entry of $A$ and all its neighbor entries in $A$. (Each entry has 3, 5 or 8 neighbors.) Prove that

$$\sigma(\mathcal{T}) = \left\{ \left(1 - 2\cos\frac{k\pi}{n+1}\right)\left(1 - 2\cos\frac{l\pi}{n+1}\right) : k,l = 1,\dots,n \right\}.$$

*Solution.* Let us denote

$$J = \begin{bmatrix} 1 & 1 & & & & 0 \\ 1 & 1 & 1 & & & \\ & 1 & 1 & & & \\ & & & & 1 & \\ & & & 1 & 1 & 1 \\ 0 & & & & 1 & 1 \end{bmatrix}_{n\times n}$$

With this notation, $\mathcal{T}(A) := JAJ$ for every $A \in \mathrm{M}_n(\mathbb{C})$.

We claim that

$$\sigma(\mathcal{T}) = \{\alpha\beta : \alpha, \beta \in \sigma(J)\}.$$

Since $J$ is symmetric (hermitian), it is similar to a diagonal matrix, i.e. there exist $S, D \in \mathrm{M}_n(\mathbb{C})$, $S$ regular, $D$ diagonal, such that $J = S^{-1}DS$. If $\nu \in \sigma(\mathcal{T})$, then the equality $JAJ = \nu A$ (for some non-zero $A \in \mathrm{M}_n(\mathbb{C})$) can be rewritten as $DBD = \nu B$, where $B := SAS^{-1} \neq \mathbf{0}$. Suppose that $D = \mathrm{diag}(d_1, \dots, d_n)$ and $B = [b_{i,j}]$. Then $d_i b_{i,j} d_j = \nu b_{i,j}$ for all indices $i$,$j$. Since $B$ is non-zero, we have $\nu = d_i d_j$ for some $i,j$. Obviously, $\{d_1, \dots, d_n\} = \sigma(D) = \sigma(J)$.

Conversely, take arbitrary $\alpha, \beta \in \sigma(J)$. Let $x$, $y$ be the corresponding eigenvectors (vectors-columns). Then $A := xy^\tau$ is non-zero and

$$JAJ = JAJ^\tau = Jxy^\tau J^\tau = (Jx)(Jy)^\tau = (\alpha x)(\beta y)^\tau = (\alpha\beta)A,$$

so $\alpha\beta \in \sigma(\mathcal{T})$.

Finally, we have to calculate $\sigma(J)$. Let $p_n(\lambda)$ be the characteristic polynomial of $J$ (depending on $n$).

$$p_n(\lambda) = \det(J - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & & & & 0 \\ 1 & 1-\lambda & 1 & & & \\ & 1 & 1-\lambda & & & \\ & & & & 1 & \\ & & & 1 & 1-\lambda & 1 \\ 0 & & & & 1 & 1-\lambda \end{vmatrix}_{n\times n}$$

Using the Laplace formula we obtain a recurrent relation

$$\begin{aligned} p_n(\lambda) &= (1-\lambda)p_{n-1}(\lambda) - p_{n-2}(\lambda); \;\; n \geq 3, \\ p_1(\lambda) &= 1 - \lambda, \quad p_2(\lambda) = \lambda^2 - 2\lambda. \end{aligned}$$

By induction on $n$ we show that

$$p_n(1 - 2\cos t) = \frac{\sin(n+1)t}{\sin t}; \quad \text{for every positive integer } n \text{ and } 0 < t < \pi.$$

For $n = 1$ and $n = 2$ this is straightforward. The induction step is

$$p_n(1-2\cos t) = 2\cos t\; p_{n-1}(1-2\cos t) - p_{n-2}(1-2\cos t) = \\ = 2\cos t\,\frac{\sin nt}{\sin t} - \frac{\sin(n-1)t}{\sin t} = \frac{\sin(n+1)t}{\sin t}.$$

(Alternatively, we could solve the recurrence relation to obtain

$$p_n(1-\lambda) = \frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda+\sqrt{\lambda^2-4}}{2}\right)^{n+1} - \frac{1}{\sqrt{\lambda^2-4}}\left(\frac{\lambda-\sqrt{\lambda^2-4}}{2}\right)^{n+1}$$

and then substitute $\lambda = 2\cos t$, or we could use the recursion to prove that $p_n(1-2\lambda)$ are Chebycheff's polynomials of the second kind.)

Note that $t = \frac{k\pi}{n+1}$; $k = 1, \ldots, n$ satisfy the equality $p_n(1-2\cos t) = \frac{\sin(n+1)t}{\sin t} = 0$. Since $p_n(\lambda)$ has at most $n$ distinct roots, it follows that

$$\left\{1 - 2\cos\frac{k\pi}{n+1}\ : k = 1, \ldots, n\right\}$$

is precisely the set of zeros of $p_n(\lambda)$, i.e. the spectrum of $J$, and we are done. ■

*Remark.* The statement about $\sigma(\mathcal{T})$ can be generalized as follows. For fixed $B, C \in \mathrm{M}_n(\mathbb{C})$ the operator $\mathcal{T}\colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C})$ defined by $\mathcal{T}(A) := BAC$ has

$$\sigma(\mathcal{T}) = \{\beta\gamma : \beta \in \sigma(B), \gamma \in \sigma(C)\}\,.$$

The proof in the text is easier because $B = C = J$ are symmetric.

8. *Problem.* Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two decreasing sequences of positive real numbers such that $\prod_{j=1}^n x_j \geq \prod_{j=1}^n y_j$ for every $n \geq 1$. Prove that $\sum_{j=1}^n x_j \geq \sum_{j=1}^n y_j$ for every $n \geq 1$.

*Solution.* We first note that for $n \geq 1$ and integers $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$ we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \geq y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n}.$$

This is immediate from the problem hypothesis after rearranging factors:

$$(x_1 \ldots x_n)^{\alpha_n}(x_1 \ldots x_{n-1})^{\alpha_{n-1}-\alpha_n} \ldots (x_1x_2)^{\alpha_2-\alpha_3} x_1^{\alpha_1-\alpha_2} \\ \geq (y_1 \ldots y_n)^{\alpha_n}(y_1 \ldots y_{n-1})^{\alpha_{n-1}-\alpha_n} \ldots (y_1y_2)^{\alpha_2-\alpha_3} y_1^{\alpha_1-\alpha_2}.$$

Furthermore, note that

$$y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} \geq y_1^{\alpha_{\sigma(1)}} y_2^{\alpha_{\sigma(2)}} \ldots y_n^{\alpha_{\sigma(n)}}$$

for any permutation $\sigma$ of $\{1, 2, \ldots, n\}$.

For every integer $N \geq 1$ by using the multinomial theorem we obtain

$$
\begin{aligned}
(x_1 + x_2 + \ldots + x_n)^N &= \sum_{\substack{\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0 \text{ integers} \\ \alpha_1 + \alpha_2 + \ldots + \alpha_n = N}} \frac{N!}{\alpha_1! \alpha_2! \ldots \alpha_n!} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \\
&\geq \sum_{\substack{\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0 \text{ integers} \\ \alpha_1 + \alpha_2 + \ldots + \alpha_n = N}} \frac{N!}{\alpha_1! \alpha_2! \ldots \alpha_n!} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \\
&\geq \sum_{\substack{\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0 \text{ integers} \\ \alpha_1 + \alpha_2 + \ldots + \alpha_n = N}} \frac{N!}{\alpha_1! \alpha_2! \ldots \alpha_n!} y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} \\
&\geq \frac{1}{n!} (y_1 + y_2 + \ldots + y_n)^N
\end{aligned}
$$

since each $n$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ has at most $n!$ distinct rearrangements.

Thus $\frac{x_1 + \ldots + x_n}{y_1 + \ldots + y_n} \geq \sqrt[N]{\frac{1}{n!}}$ and by taking $N \to \infty$ we obtain $x_1 + \ldots + x_n \geq y_1 + \ldots + y_n$. ■

9. *Problem.* Let $R$ be a finite ring with the following property:
   For any $a, b \in R$, there exists $c \in R$ (depending on $a$ and $b$) such that $a^2 + b^2 = c^2$.
   Prove that:
   For any $a, b, c \in R$, there exists $d \in R$ such that $2abc = d^2$.
   (*Remarks.* Here $2abc$ denotes $abc + abc$. $R$ is assumed to be associative but not necessarily commutative.)

   *Solution.* Let us denote $S = \{x^2 : x \in R\}$. The property of $R$ can be rewritten as $S + S \subseteq S$. For each $y \in S$ the function $S \to S$, $x \mapsto x + y$ is injective, but since $S$ is finite it is indeed bijective. Therefore, $S$ is also closed under subtraction, so $S$ is an additive subgroup of $R$.

   Now for any $x, y \in R$ we have $xy + yx = (x+y)^2 - x^2 - y^2$, so

   $$xy + yx \in S.$$

   We take arbitrary $a, b, c \in R$ and substitute:

   $$
   \begin{aligned}
   x = a,\ y = bc \quad &\Rightarrow \quad abc + bca \in S \qquad (37) \\
   x = c,\ y = ab \quad &\Rightarrow \quad cab + abc \in S \qquad (38) \\
   x = ca,\ y = b \quad &\Rightarrow \quad cab + bca \in S \qquad (39)
   \end{aligned}
   $$

   If we add (37), (38) and subtract (39), we shall obtain $2abc \in S$. ■

   *Remark.* The problem was well-received and many students solved it completely.

10. *Problem.* Let $(N_n)_{n \geq 1}$ be a sequence of positive integers no smaller than 3. Inside a circle of radius $r_1$ we inscribe a regular $N_1$-gon. Next, inside the latter polygon we inscribe a circle of radius $r_2$ and in the latter circle we inscribe a regular $N_2$-gon, and so on. Continuing in this way, we obtain a sequence of circles and polygons. Prove that

    $$\lim_{n \to \infty} r_n = 0 \qquad \text{if and only if} \qquad \sum_{n=1}^{\infty} \frac{1}{N_n^2} = \infty.$$

*Solution.* Elementary trigonometry gives $\dfrac{r_{n+1}}{r_n} = \cos\dfrac{\pi}{N_n}$, so

$$r_{n+1} = r_1 \cdot \prod_{k=1}^{n} \cos\frac{\pi}{N_k}$$

and taking logarithm we get

$$\ln r_{n+1} = \ln r_1 + \sum_{k=1}^{n} \ln\cos\frac{\pi}{N_k}.$$

Note that $\ln\cos\frac{\pi}{N_k} < 0$, so

$$\lim_{n\to\infty} r_{n+1} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \ln r_{n+1} = -\infty \;\Leftrightarrow\; \sum_{k=1}^{\infty} \left|\ln\cos\frac{\pi}{N_k}\right| = +\infty.$$

We distinguish two cases.

(a) Suppose $\lim_{n\to\infty} N_n = \infty$.
We use the Limit Comparison Test to compare $\sum_{n=1}^{\infty} \frac{1}{N_n^2}$ with $\sum_{n=1}^{\infty} |\ln\cos\frac{\pi}{N_n}|$.

$$\lim_{n\to\infty} \frac{|\ln\cos\frac{\pi}{N_n}|}{\frac{1}{N_n^2}} = \lim_{x\to 0} \frac{-\ln\cos(\pi x)}{x^2} = \lim_{x\to 0} \frac{\ln\cos(\pi x)}{\cos(\pi x) - 1} \cdot \lim_{x\to 0} \frac{1-\cos(\pi x)}{(\pi x)^2} \cdot \pi^2 = \frac{\pi^2}{2}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{N_n^2}$ and $\sum_{n=1}^{\infty} |\ln\cos\frac{\pi}{N_n}|$ are either both convergent or both divergent.

(b) If $(N_n)_{n\geq 1}$ does not converge to $\infty$, then it contains a constant subsequence.
Clearly, $\sum_{n=1}^{\infty} \frac{1}{N_n^2}$ and $\sum_{n=1}^{\infty} |\ln\cos\frac{\pi}{N_n}|$ both diverge.

■

11. *Problem.* A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is defined by the equations

$$2^{x_1} = 4,\; 2^{2^{x_2}} = 4^4,\; 2^{2^{2^{x_3}}} = 4^{4^4}, \ldots$$

and generally

$$2^{2^{\cdot^{\cdot^{2^{x_n}}}}} = 4^{4^{\cdot^{\cdot^{4}}}},$$

where the left hand side contains $n$ twos and the right hand side contains $n$ fours. Prove that the sequence converges and its limit satisfies

$$3 \leq \lim_{n\to\infty} x_n \leq \tfrac{10}{3}.$$

*Solution.* Taking logarithms we immediately get $x_1 = 2$ and $x_2 = 3$.

Also from the general defining relation we obtain:

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{x_n}}}}}}_{n-1\ 2's} = \log_2\Big(\underbrace{4^{4^{\cdot^{\cdot^{4}}}}}_{n\ 4's}\Big) = 2 \cdot \underbrace{4^{4^{\cdot^{\cdot^{4}}}}}_{n-1\ 4's} = 2 \cdot \underbrace{2^{2^{\cdot^{\cdot^{2^{x_{n-1}}}}}}}_{n-1\ 2's} > \underbrace{2^{2^{\cdot^{\cdot^{2^{x_{n-1}}}}}}}_{n-1\ 2's}$$

which implies that $(x_n)_{n=1}^{\infty}$ is increasing.

Let us now prove that the sequence is bounded from above by $\frac{10}{3}$. For this we need the following auxiliary statement.

*Lemma.* Suppose that $u > v \geq 3$ and $\varepsilon > 0$ are such that $2^u - 2^v \leq \varepsilon$. Then $u - v \leq \frac{\varepsilon}{4}$.

*Proof of the lemma.* Observe that $2^x \geq 1 + \frac{1}{2}x$ for $x > 0$, which follows immediately from $2^x = 4^{\frac{x}{2}} > e^{\frac{x}{2}} = 1 + \frac{x}{2} + \sum_{k=2}^{\infty} \frac{x^k}{2^k k!} > 1 + \frac{x}{2}$. Now suppose that $u - v > \frac{\varepsilon}{4}$ and estimate: $2^u - 2^v > 2^{v+\frac{\varepsilon}{4}} - 2^v = 2^v(2^{\frac{\varepsilon}{4}} - 1) \geq 2^3 \cdot \frac{\varepsilon}{8} = \varepsilon$, but this is a contradiction.

Taking logarithms once more in the computation above we get:

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{x_n}}}}}}_{n-2\ 2'\mathrm{s}} - \underbrace{2^{2^{\cdot^{\cdot^{2^{x_{n-1}}}}}}}_{n-2\ 2'\mathrm{s}} = 1.$$

By applying the lemma $n-2$ times we obtain

$$x_n - x_{n-1} \leq \tfrac{1}{4^{n-2}} \quad \text{for } n \geq 3,$$

which inductively implies

$$x_n \leq x_2 + \textstyle\sum_{k=1}^{n-2} \frac{1}{4^k} = 3 + \frac{1}{3}(1 - \frac{1}{4^{n-2}}) < \frac{10}{3}.$$

Since $(x_n)_{n=1}^{\infty}$ is increasing and bounded from above by $\frac{10}{3}$, it has a limit and

$$3 = x_2 \leq \lim_{n\to\infty} x_n \leq \tfrac{10}{3}.$$

■

*Remark.* The sequence in question converges "very rapidly", and for instance, already the fourth term approximates the limit to more than 100 decimal places. The exact value of the limit is:

$$\lim_{n\to\infty} x_n = 3.1703761763375607\ldots$$

12. *Problem.* Let $k$ and $n$ be positive integers such that $k \leq n - 1$. Denote $S = \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of $S$. Prove that it is possible to color some elements of $S$ using two colors, red and blue, such that the following conditions are satisfied.

    (i) Each element of $S$ is either left uncolored or is colored red or blue.

    (ii) At least one element of $S$ is colored.

    (iii) Each set $A_i$ $(i = 1, 2, \ldots, k)$ is either completely uncolored or it contains at least one red and at least one blue element.

*Solution.* Consider the following system of $k$ linear equations in $n$ real variables $x_1, x_2, \ldots, x_n$:

$$\sum_{j\in A_i} x_j = 0; \qquad \text{for } i = 1, 2, \ldots, k.$$

Since $k < n$, this system has a nontrivial solution $(x_1, x_2, \ldots, x_n)$, i.e. a solution with at least one nonzero $x_j$. Now color red all elements of the set $\{j \in S : x_j > 0\}$, color blue all elements of the set $\{j \in S : x_j < 0\}$, and leave uncolored all elements of $\{j \in S : x_j = 0\}$.

Since the solution is nontrivial, at least one element is colored. If $A_i$ contains some red element $j \in S$ then $x_j > 0$, and from $\sum_{j\in A_i} x_j = 0$ we see that there exists some $j' \in A_i$ such that $x_{j'} < 0$, i.e. $j'$ is colored blue. Thus $A_i$ must have elements of both colors. Analogously we argue when $A_i$ contains a blue element. Therefore we see that the above coloring satisfies all requirements. ■

*Remark.* It is easy to see that the condition $k \le n-1$ is tight, i.e. there exist $k = n$ sets $A_1, A_2, \ldots, A_n$ that do not allow such a coloring of $S$. One can simply take $A_i = \{i\}$ for $i = 1, 2, \ldots, n$, and observe that all numbers $i$ would have to be left uncolored, which contradicts (ii).

Perhaps more interesting is that the condition $k \le n-1$ is tight even when we require that sets $A_i$ have at least 2 elements. When $n \ge 3$ we can take $A_i = \{i, n\}$ for $i = 1, 2, \ldots, n-1$ and $A_n = \{1, 2\}$. Not all of the sets $A_1, A_2, \ldots, A_{n-1}$ can be left uncolored, so we conclude in particular that number $n$ has to be colored. Without loss of generality number $n$ is colored red, but then numbers 1 and 2 have to be colored blue, which gives a completely blue set $A_n$ and contradicts (iii).

*Another remark.* The problem was solved completely by 7 students, out of 87.

13. *Problem.* Let $k, m, n$ be positive integers such that $1 \le m \le n$ and denote $S = \{1, 2, \ldots, n\}$. Suppose that $A_1, A_2, \ldots, A_k$ are $m$-element subsets of $S$ with the following property. For every $i = 1, 2, \ldots, k$ there exists a partition $S = S_1^{(i)} \cup S_2^{(i)} \cup \ldots \cup S_m^{(i)}$ such that:

    (i) $A_i$ has precisely one element in common with each member of the above partition.

    (ii) Every $A_j$, $j \ne i$ is disjoint from at least one member of the above partition.

    Show that $k \le \binom{n-1}{m-1}$.

    *Solution.* Without loss of generality assume that $1 \in S_1^{(i)}$ for all $i = 1, 2, \ldots, k$, because otherwise we simply rename members of each partition.

    For every $i = 1, 2, \ldots, k$ define the polynomial

$$P_i(x_2, x_3, \ldots, x_n) = \prod_{l=2}^{m} \Big( \sum_{s \in S_l^{(i)}} x_s \Big)$$

    and regard it as a polynomial over $\mathbb{R}$ in variables $x_2, x_3, \ldots, x_n$.

    Observe that $P_i$ is a homogenous polynomial of degree $m-1$ in $n-1$ variables. Also observe that all monomials in $P_i$ are products of different $x$'s, i.e. there are no monomials with squares or higher powers. The last statement follows simply from the fact that $S_2^{(i)}, \ldots, S_m^{(i)}$ are mutually disjoint. Such polynomials form a linear space over $\mathbb{R}$ of dimension $\binom{n-1}{m-1}$ and polynomials $P_i$ belong to that space. If we prove that polynomials $P_i$, $i = 1, 2, \ldots, k$ are linearly independent, the inequality $k \le \binom{n-1}{m-1}$ will follow from the dimension argument.

For any $i = 1, 2, \ldots, k$ let $\chi_i$ be the characteristic vector of $A \cap \{2, 3, \ldots, n\}$. In other words, $\chi_i \in \{0, 1\}^{n-1}$ where the $j$-th coordinate of $\chi_i$ equals 1 if $j + 1 \in A$, and 0 otherwise.

For every $i$ we know that each $A_i \cap S_l^{(i)}$ has exactly one element and therefore

$$P_i(\chi_i) = \prod_{l=2}^{m} |A_i \cap S_l^{(i)}| = \prod_{l=2}^{m} 1 = 1.$$

On the other hand, if $j \neq i$ then either some $A_j \cap S_l^{(i)}$, $l \geq 2$ is empty, or all $A_j \cap S_l^{(i)}$, $l \geq 2$ are nonempty but $A_j \cap S_1^{(i)} = \emptyset$. In the latter case we must have $|A_j \cap S_l^{(i)}| = 2$ for some $l \geq 2$. In any case we have at least one even factor in the following product, and so

$$P_i(\chi_j) = \prod_{l=2}^{m} |A_j \cap S_l^{(i)}| \quad \equiv 0 \pmod 2.$$

Therefore all diagonal entries in the matrix $[P_i(\chi_j)]_{i,j=1,2,\ldots,k}$ are odd, while all non-diagonal entries are even. Consequently, its determinant is an odd integer, in particular it is not 0, and thus the matrix is regular. If polynomials $P_i$ were linearly dependent, we would conclude that rows of $[P_i(\chi_j)]_{i,j=1,2,\ldots,k}$ are also linearly dependent, but this is not the case. Therefore $P_i$, $i = 1, 2, \ldots, k$ must be linearly independent and this completes the proof. ■

*Remark.* The solution is an example of the *polynomial method* in combinatorics, popularized by Noga Alon.

*Another remark.* The problem proved to be very difficult and no student solved it completely.

14. *Problem.* Prove that for every complex polynomial $P(z) = a_n z^n + \ldots + a_1 z + a_0$ with $|a_n| = |a_0| = 1$, there exists a complex polynomial $Q(z) = b_n z^n + \ldots + b_1 z + b_0$ with $|b_n| = |b_0| = 1$, such that $|Q(z)| \leq |P(z)|$ for every $z \in \mathbb{C}$, $|z| = 1$, and such that all complex roots of $Q$ lie on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

*Solution.* If $P$ is a constant polynomial, then we can simply take $Q = P$. Therefore assume $n \geq 1$. By the fundamental theorem of algebra we can write

$$P(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n),$$

where $z_j \in \mathbb{C} \setminus \{0\}$ are complex roots of $P$. If we write each $z_j$ in the polar form: $z_j = r_j \omega_j$, $r_j > 0$, $|\omega_j| = 1$, then the factorization of $P$ becomes

$$P(z) = a_n(z - r_1\omega_1)(z - r_2\omega_2)\ldots(z - r_n\omega_n).$$

Observe that by Viète's formulae

$$r_1 r_2 \ldots r_n = |z_1 z_2 \ldots z_n| = \left|\frac{a_0}{a_n}\right| = 1.$$

We define polynomial $Q$ to be

$$Q(z) = a_n(z - \omega_1)(z - \omega_2)\ldots(z - \omega_n).$$

Its roots are $\omega_j$ and they lie on the unit circle. Moreover, the first and the last coefficient in $Q$ are the same as in $P$. Finally, take some $z \in \mathbb{C}$, $|z| = 1$. For each $j = 1, 2, \ldots, n$ we estimate

$$\begin{aligned} |z - r_j\omega_j|^2 &= |z|^2 + |r_j\omega_j|^2 - 2\mathrm{Re}(\overline{z}r_j\omega_j) = 1 + r_j^2 - 2r_j\mathrm{Re}(\overline{z}\omega_j) \\ &= 1 + r_j^2 - 2r_j + r_j\Big(|z|^2 + |\omega_j|^2 - 2\mathrm{Re}(\overline{z}\omega_j)\Big) \\ &= (1 - r_j)^2 + r_j\,|z - \omega_j|^2 \;\geq\; r_j|z - \omega_j|^2. \end{aligned}$$

Multiplying for $j = 1, 2, \ldots, n$ we get

$$|P(z)|^2 = \prod_{j=1}^{n} |z - r_j\omega_j|^2 \geq \Big(\prod_{j=1}^{n} r_j\Big)\Big(\prod_{j=1}^{n} |z - \omega_j|^2\Big) = |Q(z)|^2.$$

■