Recursive Chinese remainder amplification for parabolic Mizohata–Takeuchi constants

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Abstract

For every $\lambda > 2$, we prove a lower bound

$$\mathfrak{M}_{\mathbb{T}^d}(N) \geq \exp\left(\frac{c_\lambda \log N}{\lambda^{\log^* N}}\right)$$

for all sufficiently large N and every $d \geq 1$, where $\log^* N$ is the number of iterated natural logarithms needed to reach a value at most one. The same estimate holds for the temporally localized Euclidean constants of Bennett, Kovač, Nakamura and Oliveira. The proof introduces an auxiliary energy problem over finite rings. Its gains multiply under the Chinese remainder theorem. An integer lift loses at most a factor of eight and can be reduced modulo any prime larger than sixteen times the square of the original modulus. Iterating these two operations gives the bound. The construction controls precisely the rays associated with the Fourier support. It does not establish a fixed positive power of N.

1 Definitions and statement

We use the normalization $e(s) = e^{2\pi i s}$ and normalized Haar measure on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For a trigonometric polynomial u_0, write

$$u(x,t) = \sum_{k\in\mathbb{Z}^d} \widehat{u}_0(k) e(k\cdot x + |k|^2 t), \qquad X_{\mathbb{T}} w(x,k) = \int_0^1 w(x-2tk,t)\,\mathrm{d}t.$$

For $N \geq 1$, let $\mathfrak{M}_{\mathbb{T}^d}(N)$ be the least constant C such that

$$\int_{\mathbb{T}}\int_{\mathbb{T}^d} |u(x,t)|^2 w(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq C\,\|u_0\|^2_{L^2(\mathbb{T}^d)} \sup_{\substack{x\in\mathbb{T}^d \\ k\in\operatorname{supp}\widehat{u}_0}} X_{\mathbb{T}} w(x,k) \tag{1.1}$$

for all nonnegative weights w and all nonzero u_0 with $\operatorname{supp}\widehat{u}_0 \subset [-N,N]^d \cap \mathbb{Z}^d$.

For the Euclidean version, put

$$U(x,t) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e(x\cdot\xi + |\xi|^2 t)\,\mathrm{d}\xi, \qquad X_{\mathbb{R}} w(x,v) = \int_{\mathbb{R}} w(x-2tv,t)\,\mathrm{d}t.$$

The constant $\mathfrak{M}_{\mathbb{R}^d}(N)$ is the least C for which

$$\int_0^1\int_{\mathbb{R}^d} |U(x,t)|^2 w(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq C\,\|f\|^2_{L^2(\mathbb{R}^d)} \sup_{\substack{x\in\mathbb{R}^d \\ v\in\operatorname{supp}\widehat{f}}} X_{\mathbb{R}} w(x,v) \tag{1.2}$$

for all nonnegative weights w and nonzero $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{f} \subset [-N, N]^d$. These are the constants in [1, Section 1]. In particular, the parameter N here bounds the frequencies; the left side of (1.2) is localized to the time interval $[0, 1]$.

Cairo's work [2] disproved the Mizohata–Takeuchi conjecture for non-flat C^2 hypersurfaces. In the parabolic periodic setting, [1, Theorem 1.5] proves the lower bound $\mathfrak{M}_{\mathbb{T}^d}(N) \gtrsim_d (\log N)^d$. The argument below gives a stronger rate by recursively combining finite-ring examples.

For $t > 0$, define

$$H(t) = \log^* t = \begin{cases} 0, & 0 < t \le 1, \\ 1 + H(\log t), & t > 1. \end{cases} \tag{1.3}$$

Thus H is nondecreasing, integer-valued, and finite on $(0, \infty)$. All logarithms in this note are natural.

Theorem 1.1. *For every $\lambda > 2$, there are constants $c_\lambda > 0$ and $N_\lambda \ge 1$ such that, for every integer $d \ge 1$ and every real $N \ge N_\lambda$,*

$$\min\{\mathfrak{M}_{\mathbb{T}^d}(N), \mathfrak{M}_{\mathbb{R}^d}(N)\} \ge \exp\left(\frac{c_\lambda \log N}{\lambda^{H(N)}}\right). \tag{1.4}$$

The constants can be chosen independently of d.

Equivalently, one can write $\lambda = 2 + \eta$ for any $\eta > 0$. Since $\lambda^{H(N)} = o(\log \log N)$, (1.4) eventually exceeds $\exp(C \log N / \log \log N)$ for every fixed $C > 0$. Its exponent when written as a power of N is $c_\lambda / \lambda^{H(N)}$, which tends to zero.

The contribution of the argument is the lift in proposition 3.2, together with its repeated use in proposition 5.1. The proof uses elementary finite-set counting and the prime number theorem. Only the passage to (1.2) invokes the transference theorem from [1].

2 A finite-ring energy problem

For an integer $q \ge 2$, write $R_q = \mathbb{Z}/q\mathbb{Z}$ and $P_q(a) = (a, a^2) \in R_q^2$. For nonempty sets $A \subset R_q$ and $B \subset R_q^2$, define

$$\mathcal{E}_q(A, B) = \#\{(a, a', \beta, \beta') \in A^2 \times B^2 : P_q(a) + \beta = P_q(a') + \beta'\}. \tag{2.1}$$

With counting measure and unnormalized convolution on R_q^2,

$$\mathcal{E}_q(A, B) = \left\| \mathbf{1}_{P_q(A)} * \mathbf{1}_B \right\|_{\ell^2(R_q^2)}^2 .$$

The first coordinate of P_q makes it injective, also when q is composite.

Definition 2.1. The pair (A, B) is *admissible modulo q* if, for every $a \in A$, the map

$$B \longrightarrow R_q, \qquad (b, \tau) \longmapsto \tau - 2ab \tag{2.2}$$

is injective. Define

$$G(q) = \max_{\substack{\emptyset \ne A \subset R_q,\ \emptyset \ne B \subset R_q^2 \\ (A,B) \text{ admissible}}} \frac{\mathcal{E}_q(A, B)}{|A||B|}. \tag{2.3}$$

The maximum exists because there are finitely many pairs of sets. Singletons give $G(q) \geq 1$. Also $\mathcal{E}_q(A,B) \leq |A|^2|B|$, since a, a', β determine β', and hence $G(q) \leq q$.

Lemma 2.2 (Chinese remainder multiplication)**.** *If* $\gcd(q_1, q_2) = 1$*, then*

$$G(q_1 q_2) \geq G(q_1)G(q_2). \tag{2.4}$$

Proof. Use the ring isomorphism $R_{q_1 q_2} \simeq R_{q_1} \times R_{q_2}$. For maximizing pairs (A_i, B_i), let A and B correspond to $A_1 \times A_2$ and $B_1 \times B_2$, respectively. The polynomial $a \mapsto a^2$ is respected by the ring isomorphism. Therefore the equation in (2.1) holds precisely when it holds in both factors, and

$$\mathcal{E}_{q_1 q_2}(A,B) = \mathcal{E}_{q_1}(A_1,B_1)\mathcal{E}_{q_2}(A_2,B_2).$$

Equality of two images under (2.2) implies equality in both factors, so (A,B) is admissible. Cardinalities also multiply, proving (2.4). □

3 Lifting with a fixed loss

For finite nonempty integer sets $A \subset \mathbb{Z}$ and $B \subset \mathbb{Z}^2$, define $\mathcal{E}_{\mathbb{Z}}(A,B)$ by (2.1), with integer equalities and $P(a) = (a, a^2)$.

Lemma 3.1 (Energy and ray integrals)**.** *Suppose that* $A \subset [-N, N]$ *and that* $(b,\tau) \mapsto \tau - 2ab$ *is injective on* B *for every* $a \in A$*. Then*

$$\mathfrak{M}_{\mathbb{T}}(N) \geq \frac{\mathcal{E}_{\mathbb{Z}}(A,B)}{|A||B|}. \tag{3.1}$$

Proof. Set

$$u(x,t) = \sum_{a \in A} e(ax + a^2 t), \qquad F(x,t) = \sum_{(b,\tau)\in B} e(bx + \tau t), \qquad w = |F|^2.$$

Parseval gives

$$\int_{\mathbb{T}^2} |u|^2 w = \|uF\|^2_{L^2(\mathbb{T}^2)} = \mathcal{E}_{\mathbb{Z}}(A,B), \qquad \|u(\cdot,0)\|^2_{L^2(\mathbb{T})} = |A|.$$

For $a \in A$, expansion of the ray integral gives

$$\int_0^1 w(x - 2at, t)\,\mathrm{d}t = \sum_{\substack{(b,\tau),(b',\tau')\in B \\ \tau - 2ab = \tau' - 2ab'}} e((b-b')x) = |B|.$$

The last equality follows from injectivity. Substitution in (1.1) proves the claim. □

Proposition 3.2 (Quantitative integer lift)**.** *For every integer* $q \geq 2$*, there are nonempty integer sets* $A_0 \subset [0, q)$ *and* $B_0 \subset [0, 2q) \times [0, 3q^2)$ *such that all the maps* $(b,\tau) \mapsto \tau - 2ab$*,* $a \in A_0$*, are injective on* B_0*, and*

$$\frac{\mathcal{E}_{\mathbb{Z}}(A_0,B_0)}{|A_0||B_0|} \geq \frac{G(q)}{8}. \tag{3.2}$$

Consequently,

$$\mathfrak{M}_{\mathbb{T}}(q) \geq \frac{G(q)}{8}, \qquad G(p) \geq \frac{G(q)}{8} \quad \text{if } p > 16q^2 \text{ is prime.} \tag{3.3}$$

Proof. Choose an admissible maximizing pair (A, B) modulo q. Represent A in $\{0, \dots, q-1\}$, and partition it using the two intervals

$$I_0 = \{0, \dots, \lfloor (q-1)/2 \rfloor\}, \qquad I_1 = \{\lfloor (q-1)/2 \rfloor + 1, \dots, q-1\}.$$

Each interval has diameter at most $q/2$. Write $D_i = A \cap I_i$, with finite-ring energies understood after reduction modulo q. The identity $\mathbf{1}_{P_q(A)} = \mathbf{1}_{P_q(D_0)} + \mathbf{1}_{P_q(D_1)}$ and the inequality $\|v_0 + v_1\|_2^2 \le 2(\|v_0\|_2^2 + \|v_1\|_2^2)$ imply

$$\mathcal{E}_q(A, B) \le 2\big(\mathcal{E}_q(D_0, B) + \mathcal{E}_q(D_1, B)\big).$$

It follows that some nonempty part, denoted from now on by A', satisfies

$$\operatorname{diam} A' \le q/2, \qquad \frac{\mathcal{E}_q(A', B)}{|A'|} \ge \frac{\mathcal{E}_q(A, B)}{2|A|}. \tag{3.4}$$

For $s \in \{0, \dots, q-1\}$ and $\beta = (\bar{b}, \bar{\tau}) \in B$, let $b_s(\beta)$ be the unique representative of $\bar{b}$ in $\{s, \dots, s+q-1\}$, and let $\tau(\beta)$ be the representative of $\bar{\tau}$ in $\{0, \dots, q-1\}$. Put $L = 2q+1$ and

$$B_s = \{(b_s(\beta), \tau(\beta) + qj) : \beta \in B,\ 0 \le j < L\}. \tag{3.5}$$

Reduction modulo q recovers β, and then the temporal coordinate recovers j. Hence $|B_s| = L|B|$. The set B_s is contained in $[0, 2q) \times [0, 3q^2)$.

For $a \in A'$, equality of the integer ray projections of two points of B_s implies equality of their projections modulo q. Admissibility forces the underlying points β, β' of B to be equal. Their chosen representatives are therefore equal, and the integer equality then forces $j = j'$. Thus all the required integer ray maps are injective on B_s.

We next count the energy retained by this lift. Fix a quadruple (a, a', β, β') counted by $\mathcal{E}_q(A', B)$, and put $\delta = a' - a$. Its spatial congruence is

$$\bar{b} - \bar{b}' = \delta \pmod q, \qquad |\delta| \le q/2.$$

Exactly $q - |\delta|$ choices of the cut s give

$$b_s(\beta) - b_s(\beta') = \delta \tag{3.6}$$

as an integer equality. For example, when $\delta \ge 0$, the representative of $\bar{b}'$ must occupy one of the first $q - \delta$ positions in the length-q interval; the negative case is the same with the roles reversed. Consequently, (3.6) holds with probability at least $1/2$ for a uniform cut.

The temporal congruence shows that

$$n = \frac{a'^2 + \tau(\beta') - a^2 - \tau(\beta)}{q}$$

is an integer. Since $0 \le a, a', \tau(\beta), \tau(\beta') < q$,

$$|a'^2 + \tau(\beta') - a^2 - \tau(\beta)| \le (q-1)^2 + (q-1) = q(q-1),$$

so $|n| < q$. For a cut satisfying (3.6), every pair $j, j' \in \{0, \dots, L-1\}$ with $j - j' = n$ gives a quadruple counted by $\mathcal{E}_{\mathbb{Z}}(A', B_s)$. There are $L - |n| \ge L/2$ such pairs. Different underlying quadruples or different pairs j, j' give different lifted quadruples. Therefore

$$\frac{1}{q} \sum_{s=0}^{q-1} \mathcal{E}_{\mathbb{Z}}(A', B_s) \ge \frac{L}{4} \mathcal{E}_q(A', B).$$

For at least one cut s, (3.4) now yields

$$\frac{\mathcal{E}_{\mathbb{Z}}(A', B_s)}{|A'||B_s|} \geq \frac{\mathcal{E}_q(A', B)}{4|A'||B|} \geq \frac{G(q)}{8}.$$

Take $A_0 = A'$ and $B_0 = B_s$. This proves (3.2), and lemma 3.1 proves the first inequality in (3.3).

For the second inequality, reduce A_0, B_0 modulo a prime $p > 16q^2$. The reductions preserve their cardinalities. For fixed $a \in A_0$, all integer ray projections lie in $(-4q^2, 3q^2)$, an interval of length $7q^2 < p$. Distinct projections therefore remain distinct modulo p. Every integer additive coincidence survives reduction, with no identification of its individual entries. Thus the reduced pair is admissible and has normalized energy at least that in (3.2). This proves the proposition. □

Remark 3.3. The two losses in the proof are the selection of A' and the averaged retention of additive coincidences. Their product is $1/8$, independently of q and of the number of prime factors of q. Merely choosing arbitrary representatives modulo q would not justify the energy comparison.

4 Initial examples over primes

The following elementary construction supplies arbitrarily large initial gains. It uses the same subset-sum principle as the lacunary trigonometric weights in [1, Section 5]; the choice of powers of four makes every projected coefficient nonzero.

Lemma 4.1. *For every prime $p \geq 17$,*

$$G(p) \geq \frac{1 + \lfloor \log p / \log 16 \rfloor}{4} \geq \frac{\log p}{4 \log 16}. \tag{4.1}$$

Proof. Let $m = \lfloor \log p / \log 16 \rfloor \geq 1$. Since p is prime, $16^m < p$. Use the integer sets

$$A = \{4^j : 0 \leq j < m\}, \qquad B = \left\{ \left(\sum_{j=0}^{m-1} \varepsilon_j 4^j, \sum_{j=0}^{m-1} \varepsilon_j 16^j \right) : \varepsilon_j \in \{0, 1\} \right\},$$

and then reduce them modulo p. Their coordinates are between 0 and $p-1$, so $|A| = m$ and $|B| = 2^m$.

Fix $a = 4^\ell \in A$. Its ray projection of a subset sum is

$$\sum_{j=0}^{m-1} \varepsilon_j (16^j - 2 \cdot 4^{j+\ell}).$$

The nonzero coefficients have 2-adic valuations

$$v_2(16^j - 2 \cdot 4^{j+\ell}) = \begin{cases} 4j, & j \leq \ell, \\ 2j + 2\ell + 1, & j > \ell. \end{cases} \tag{4.2}$$

These valuations are pairwise distinct. A nontrivial linear combination of the coefficients with coefficients in $\{-1, 0, 1\}$ cannot vanish: after division by the smallest power of two occurring, exactly one summand is odd. Thus the integer ray projection is injective on B.

Every such projection belongs to the interval

$$\left(-\frac{16^m}{6}, \frac{16^m}{15}\right),$$

because $b < 4^m/3$, $\tau < 16^m/15$, and $a \le 4^{m-1}$. The length of this interval is less than $16^m < p$, so injectivity persists modulo p.

For each subset $S \subset \{0, \dots, m-1\}$, the point

$$z_S = \sum_{j \in S} (4^j, 16^j)$$

has at least $|S|$ representations in $P_p(A) + B$: choose the summand supplied by A from S and supply the remaining summands from B. The points z_S are distinct modulo p, since their spatial coordinates are distinct base-four integers less than p. Hence

$$\mathcal{E}_p(A, B) \ge \sum_{S \subset \{0,\dots,m-1\}} |S|^2 = 2^{m-2} m(m+1).$$

Dividing by $m2^m$ proves the first inequality in (4.1). The second follows from $1 + \lfloor t \rfloor \ge t$. □

5 Recursive amplification

Write

$$\vartheta(x) = \sum_{\substack{r \le x \\ r \text{ prime}}} \log r, \qquad Q(x) = \prod_{\substack{r \le x \\ r \text{ prime}}} r = e^{\vartheta(x)}.$$

We use the prime number theorem in the form $\vartheta(x) \sim x$ as $x \to \infty$. Combining lemma 2.2 and proposition 3.2 gives, whenever $Q(x) \ge 2$ and $p > 16Q(x)^2$ is prime,

$$G(p) \ge \frac{1}{8} \prod_{\substack{r \le x \\ r \text{ prime}}} G(r). \tag{5.1}$$

Proposition 5.1. *Fix $\lambda > 2$, and put $K = \log 8$. There exist $c > 0$ and $P_0 \ge 17$ such that, for every prime $p \ge P_0$,*

$$\log G(p) \ge K + c \frac{\log p}{\lambda^{H(p)}}. \tag{5.2}$$

Proof. Set $h(p) = \log G(p)$, so $h(p) \ge 0$ for every prime. Choose

$$\lambda^{-1} < \beta < \alpha < \tfrac{1}{2}. \tag{5.3}$$

By lemma 4.1, choose an integer P_0 such that

$$h(r) \ge 2K \qquad \text{for every prime } r \ge P_0. \tag{5.4}$$

For a large prime p, set $x = \alpha \log p$. The prime number theorem and $2\alpha < 1$ give

$$\log(16Q(x)^2) = \log 16 + 2\vartheta(\alpha \log p) = (2\alpha + o(1)) \log p < \log p.$$

The same theorem and $\beta < \alpha$ give

$$\sum_{\substack{P_0 \le r \le \alpha \log p \\ r \text{ prime}}} \log r \ge \beta \log p$$

for all sufficiently large p. Accordingly, choose an integer $P_1 > P_0$ such that, whenever $p > P_1$ is prime,

$$16Q(\alpha \log p)^2 < p, \tag{5.5}$$

$$\sum_{\substack{P_0 \le r \le \alpha \log p \\ r \text{ prime}}} \log r \ge \beta \log p, \tag{5.6}$$

$$\#\{r \text{ prime} : P_0 \le r \le \alpha \log p\} \ge 2, \qquad \alpha \log p < p. \tag{5.7}$$

Increase P_1 if needed so that $[P_0, P_1]$ contains a prime.

There are finitely many primes in $[P_0, P_1]$. By (5.4), one can choose $c > 0$ small enough that

$$h(p) \ge K + c\frac{\log p}{\lambda^{H(p)}} \qquad (P_0 \le p \le P_1,\ p \text{ prime}). \tag{5.8}$$

For example, any positive c not exceeding

$$\min_{\substack{P_0 \le p \le P_1 \\ p \text{ prime}}} \frac{K\lambda^{H(p)}}{\log p}$$

has this property. Thus the choice of the induction threshold does not depend on c.

We now use strong induction over the primes. Let $p > P_1$ and suppose that (5.2) holds for all primes in $[P_0, p)$. If $r \le \alpha \log p$, then $r \le \log p$, because $\alpha < 1$. Monotonicity and (1.3) imply

$$H(r) \le H(\log p) = H(p) - 1. \tag{5.9}$$

Let n_p be the number of primes in $[P_0, \alpha \log p]$. Apply (5.1), omit only nonnegative terms $h(r)$ with $r < P_0$, and use the induction hypothesis:

$$\begin{aligned}
h(p) &\ge -K + \sum_{\substack{r \le \alpha \log p \\ r \text{ prime}}} h(r) \\
&\ge (n_p - 1)K + c \sum_{\substack{P_0 \le r \le \alpha \log p \\ r \text{ prime}}} \frac{\log r}{\lambda^{H(r)}} \\
&\ge K + c\lambda^{1-H(p)} \sum_{\substack{P_0 \le r \le \alpha \log p \\ r \text{ prime}}} \log r \\
&\ge K + c\lambda\beta \frac{\log p}{\lambda^{H(p)}} \\
&\ge K + c\frac{\log p}{\lambda^{H(p)}}.
\end{aligned}$$

Here the third line uses (5.7) and (5.9); the last line uses $\lambda\beta > 1$. This closes the induction. □

Remark 5.2. The reserve K in (5.2) is part of the inductive assertion. At least two copies of this reserve enter each application of (5.1); one pays for the loss $1/8$, and one remains. This avoids taking a limit of fixed-depth arguments with uncontrolled constants. The restriction $\lambda > 2$ is what permits $\lambda^{-1} < \alpha < 1/2$: the lift has quadratic size, while the CRT construction multiplies the gains.

6 Completion and scope

Proof of theorem 1.1. Fix $\lambda > 2$, and take c, P_0 from proposition 5.1. For large real N, set $x = \frac{1}{2}\log N$ and $Q = Q(x)$. The prime number theorem implies

$$Q \le N, \qquad \sum_{\substack{P_0 \le r \le x \\ r \text{ prime}}} \log r \ge \tfrac{1}{3}\log N \tag{6.1}$$

for all sufficiently large N. The last sum contains at least one prime.

The function $\mathfrak{M}_{\mathbb{T}}(N)$ is nondecreasing in N. By lemma 2.2 and proposition 3.2,

$$\log \mathfrak{M}_{\mathbb{T}}(N) \ge -K + \sum_{\substack{r \le x \\ r \text{ prime}}} h(r).$$

For $r \le x < N$, monotonicity gives $H(r) \le H(N)$. Writing $n_N = \#\{r \text{ prime} : P_0 \le r \le x\} \ge 1$ and again discarding only nonnegative small-prime terms, we obtain

$$\begin{aligned}
\log \mathfrak{M}_{\mathbb{T}}(N) &\ge (n_N - 1)K + c \sum_{\substack{P_0 \le r \le x \\ r \text{ prime}}} \frac{\log r}{\lambda^{H(r)}} \\
&\ge \frac{c}{\lambda^{H(N)}} \sum_{\substack{P_0 \le r \le x \\ r \text{ prime}}} \log r \\
&\ge \frac{c}{3}\frac{\log N}{\lambda^{H(N)}}.
\end{aligned}$$

Regard each one-dimensional pair of functions u, w as functions on $\mathbb{T}^d \times \mathbb{T}$ independent of $x_2, \dots, x_d$. The solution then has frequencies $(a, 0, \dots, 0)$, and all integrals and tested ray integrals in (1.1) are unchanged. Consequently,

$$\mathfrak{M}_{\mathbb{T}^d}(N) \ge \mathfrak{M}_{\mathbb{T}}(N) \ge \exp\left(\frac{c}{3}\frac{\log N}{\lambda^{H(N)}}\right)$$

for every $d \ge 1$, with the same threshold and constant.

The quantitative transference theorem [1, Theorem 1.3] states

$$\mathfrak{M}_{\mathbb{R}^d}(M+1) \ge \mathfrak{M}_{\mathbb{T}^d}(M) \qquad (M \ge 1). \tag{6.2}$$

For sufficiently large real N, take $M = N-1$. Since $H(N-1) \le H(N)$ and $\log(N-1) \ge \frac{1}{2}\log N$ for $N \ge 3$, it follows that

$$\mathfrak{M}_{\mathbb{R}^d}(N) \ge \exp\left(\frac{c}{6}\frac{\log N}{\lambda^{H(N)}}\right).$$

Choosing $c_\lambda = c/6$ and increasing N_λ completes the proof. □

Remark 6.1 (Size of the improvement). For each fixed $\lambda > 1$,

$$\lambda^{H(N)} = o(\log\log N).$$

One way to see this is to put $t = \log\log N$. Then $H(N) = 2 + H(t)$ for sufficiently large N, while $H(t) = O(\log\log t)$. The latter estimate follows by first observing $H(s) = O(\log s)$ from $\log s \le s/2$ for large s, and then applying $H(t) = 1 + H(\log t)$. Thus $\lambda^{H(N)}$ is bounded by a fixed power of $\log t$, which is $o(t)$. In particular, the lower bound in (1.4) dominates every fixed expression $\exp(C\log N/\log\log N)$. Nevertheless, the displayed lower bound itself is $N^{o(1)}$; the theorem supplies no positive constant γ for which $\mathfrak{M}_{\mathbb{T}^d}(N) \gtrsim N^{\gamma}$.

Remark 6.2 (Directions and the sphere). The supremum in (1.1) is restricted to $k \in \operatorname{supp} \widehat{u}_0$, and the corresponding restriction is present in (1.2). This is essential: admissibility controls the ray maps only at the selected frequencies. The proof gives no estimate for the supremum over all affine-line directions. Accordingly, this note does not assert the same lower bound for the usual local Fourier-extension constant on the sphere with an all-directions X-ray norm.

Remark 6.3 (Finite construction). The existence arguments above can be implemented by finite searches. At the lift, test the two parts of A and the q possible spatial cuts. The base-four examples, CRT products, lifts, and subsequent reductions all use finite integer sets. The theorem is asymptotic: neither c_λ nor the starting threshold is optimized, and no claim of computational efficiency is made.

References

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