Arithmetic Lower Bounds for Periodic Mizohata–Takeuchi Constants

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Abstract

We prove arithmetic lower bounds for the optimal constants in the periodic Mizohata–Takeuchi inequality for Schrödinger solutions. In dimension one we obtain

$$\mathfrak{M}_{\mathbb{T}}(N) \geq \exp\left((\log 2 - o(1))\frac{\log N}{\log\log N}\right),$$

while in every fixed dimension $d \geq 2$ we obtain

$$\mathfrak{M}_{\mathbb{T}^d}(N) \geq \exp\left((2\log 2 - o(1))\frac{\log N}{\log\log N}\right).$$

These estimates are asymptotically stronger than the lower bound $\mathfrak{M}_{\mathbb{T}^d}(N) \gtrsim_d (\log N)^d$ proved in [1, Theorem 1.5]. The argument is based on a finite convolution–fiber principle. The one-dimensional example uses the roots of $k^2 \equiv 1 \pmod{M}$, whereas the higher-dimensional example uses primitive lattice points on a circle.

1 Introduction and statement of the result

Write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, equip every torus with normalized Haar measure, and use the notation

$$\mathrm{e}(s) := \exp(2\pi i s).$$

All logarithms are natural. The symbols $\lesssim$ and $\gtrsim$ suppress positive absolute constants unless a dependence is indicated; the dimension d is always fixed. All instances of $o(1)$ are taken as $N \to \infty$. For a trigonometric polynomial u_0 on $\mathbb{T}^d$, let

$$u(x,t) := \sum_{k\in\mathbb{Z}^d} \widehat{u_0}(k)\mathrm{e}(k\cdot x + |k|^2 t), \qquad (x,t) \in \mathbb{T}^d \times \mathbb{T}, \tag{1}$$

be the corresponding periodic Schrödinger solution. For a nonnegative integrable function w on $\mathbb{T}^d \times \mathbb{T}$, set

$$\rho^*_{\mathbb{Z}^d} w(x,k) := \int_0^1 w(x - 2tk, t)\,\mathrm{d}t, \qquad (x,k) \in \mathbb{T}^d \times \mathbb{Z}^d. \tag{2}$$

Following [1], for $N \geq 1$ let $\mathfrak{M}_{\mathbb{T}^d}(N)$ be the least constant C such that

$$\int_{\mathbb{T}}\int_{\mathbb{T}^d} |u(x,t)|^2 w(x,t)\,\mathrm{d}x\,\mathrm{d}t \leq C \sup_{\substack{x\in\mathbb{T}^d \\ k\in\operatorname{supp}\widehat{u_0}}} \rho^*_{\mathbb{Z}^d} w(x,k)\|u_0\|^2_{L^2(\mathbb{T}^d)} \tag{3}$$

whenever

$$\operatorname{supp}\widehat{u_0} \subseteq [-N,N]^d \cap \mathbb{Z}^d.$$

Theorem 1.5 of [1] establishes

$$\mathfrak{M}_{\mathbb{T}^d}(N) \gtrsim_d (\log N)^d \tag{4}$$

for every fixed $d \geq 1$. Our purpose is to establish the following stronger estimates.

Theorem 1.1. *As $N \to \infty$, one has*

$$\mathfrak{M}_{\mathbb{T}}(N) \geq \exp\left((\log 2 - o(1))\frac{\log N}{\log\log N}\right). \tag{5}$$

For every fixed integer $d \geq 2$, as $N \to \infty$ one has

$$\mathfrak{M}_{\mathbb{T}^d}(N) \geq \exp\left((2\log 2 - o(1))\frac{\log N}{\log\log N}\right). \tag{6}$$

In particular, both bounds are larger than every fixed power of $\log N$.

The two constructions proving Theorem 1.1 are instances of the same elementary principle, which we isolate in the next section. An important feature of the definition of $\mathfrak{M}_{\mathbb{T}^d}(N)$ is that the cutoff N is imposed only on the Fourier support of u_0; no Fourier cutoff is imposed on w.

2 A finite convolution–fiber principle

Let

$$P(k) := (k, |k|^2) \in \mathbb{Z}^{d+1}, \qquad k \in \mathbb{Z}^d,$$

denote the integer paraboloid. If $A \subseteq \mathbb{Z}^d$ and $B \subseteq \mathbb{Z}^{d+1}$ are finite, define

$$D_k(B) := \max_{s\in\mathbb{Z}} \#\{(b,\tau) \in B : \tau - 2k\cdot b = s\}, \qquad k \in A. \tag{7}$$

Proposition 2.1. *Let A be a nonempty subset of $[-N,N]^d \cap \mathbb{Z}^d$, and let $B \subseteq \mathbb{Z}^{d+1}$ be finite and nonempty. Then*

$$\mathfrak{M}_{\mathbb{T}^d}(N) \geq \frac{|A||B|}{|P(A)+B| \max_{k\in A} D_k(B)}. \tag{8}$$

Proof. Take

$$u_0(x) := \sum_{k\in A} \mathrm{e}(k\cdot x), \qquad u(x,t) := \sum_{k\in A} \mathrm{e}\big(k\cdot x + |k|^2 t\big),$$

and set

$$f_B(x,t) := \sum_{(b,\tau)\in B} \mathrm{e}(b\cdot x + \tau t), \qquad w := |f_B|^2.$$

Parseval's identity on $\mathbb{T}^{d+1}$ gives

$$\int_{\mathbb{T}}\int_{\mathbb{T}^d} |u|^2 w = \|u f_B\|^2_{L^2(\mathbb{T}^{d+1})} = \|\mathbf{1}_{P(A)} * \mathbf{1}_B\|^2_{\ell^2(\mathbb{Z}^{d+1})}.$$

The convolution is supported on $P(A)+B$ and has ℓ^1 norm $|A||B|$. Hence Cauchy–Schwarz yields

$$\int_{\mathbb{T}}\int_{\mathbb{T}^d}|u|^2 w \ge \frac{|A|^2|B|^2}{|P(A)+B|}. \tag{9}$$

On the other hand, for every $k\in A$ we have

$$\begin{aligned}
\rho^*_{\mathbb{Z}^d} w(x,k) &= \int_0^1 \left|\sum_{(b,\tau)\in B} \mathrm{e}(b\cdot x + (\tau - 2k\cdot b)t)\right|^2 \mathrm{d}t \\
&= \sum_{s\in\mathbb{Z}} \left|\sum_{\substack{(b,\tau)\in B\\ \tau-2k\cdot b=s}} \mathrm{e}(b\cdot x)\right|^2 \\
&\le \sum_{s\in\mathbb{Z}} \#\{(b,\tau)\in B : \tau-2k\cdot b = s\}^2 \\
&\le D_k(B)|B|.
\end{aligned}$$

Finally, $\|u_0\|^2_{L^2(\mathbb{T}^d)} = |A|$. Substitution into (3), followed by comparison with (9), proves (8). $\square$

Thus it is enough to find a large set A and an auxiliary frequency set B for which $P(A)+B$ is not much larger than B, while every fiber of the linear functional $(b,\tau)\mapsto \tau-2k\cdot b$ has bounded cardinality.

3 The one-dimensional construction

We first use roots of a quadratic congruence.

Proposition 3.1. *Let $M\ge 3$ be an odd squarefree integer. If $N\ge (M-1)/2$, then*

$$\mathfrak{M}_{\mathbb{T}}(N) \gtrsim 2^{\omega(M)}, \tag{10}$$

where $\omega(M)$ denotes the number of distinct prime divisors of M. The implicit constant is absolute.

Proof. Let

$$I_M := \left[-\frac{M-1}{2}, \frac{M-1}{2}\right]\cap\mathbb{Z}$$

and define the set of centered representatives of the square roots of 1 modulo M:

$$A_M := \{k\in I_M : k^2\equiv 1 \pmod{M}\}. \tag{11}$$

Every prime divisor of M is odd. The Chinese remainder theorem therefore gives (see, for example, [3])

$$|A_M| = 2^{\omega(M)}. \tag{12}$$

Moreover, $\gcd(k,M)=1$ for every $k\in A_M$. Write

$$k^2 = 1 + Mr_k. \tag{13}$$

Since $k \in I_M$, we have

$$0 \le r_k < \frac{M}{4}. \tag{14}$$

Consider the frequency set

$$B_M := \{(b, Mc) \in \mathbb{Z}^2 : |b| \le M,\ |c| \le M\}. \tag{15}$$

It has cardinality $(2M+1)^2$. By (13), every point of $P(A_M) + B_M$ is of the form

$$(k+b, 1+M(r_k+c)).$$

The first coordinate $k+b$ and the integer r_k+c each take only $O(M)$ values, by (11), (14), and (15). Therefore

$$|P(A_M) + B_M| \lesssim M^2 \lesssim |B_M|. \tag{16}$$

It remains to control the fibers. Fix $k \in A_M$. If two elements $(b, Mc), (b', Mc') \in B_M$ lie in the same fiber, then

$$M(c-c') = 2k(b-b').$$

Because M is odd and $\gcd(k, M) = 1$, we have $\gcd(M, 2k) = 1$. Consequently there exists $h \in \mathbb{Z}$ such that

$$b - b' = Mh, \qquad c - c' = 2kh. \tag{17}$$

Since $|b|, |b'| \le M$, relation (17) forces $|h| \le 2$. After fixing one point of a nonempty fiber, every other point corresponds to a unique $h \in \{-2, -1, 0, 1, 2\}$. Each fiber consequently contains at most five points, and therefore

$$\max_{k \in A_M} D_k(B_M) \le 5. \tag{18}$$

Proposition 2.1, together with (12), (16), and (18), proves (10). □

Proof of Theorem 1.1, one-dimensional case. Let $p_1 < p_2 < \cdots$ be the odd primes, and put

$$M_s := \prod_{j=1}^{s} p_j.$$

For each sufficiently large N, choose $s = s(N)$ maximal subject to $M_s \le 2N+1$. If $\vartheta(x) := \sum_{p \le x} \log p$, then the prime number theorem gives (see [2])

$$p_s = (1+o(1))s \log s, \qquad \log M_s = \vartheta(p_s) - \log 2 = (1+o(1))s \log s. \tag{19}$$

Maximality also gives the useful sandwich

$$\frac{2N+1}{p_{s+1}} < M_s \le 2N+1. \tag{20}$$

Since $M_s \ge 3^s$, we have $s = O(\log N)$; hence $\log p_{s+1} = O(\log s) = O(\log\log N)$ by the first relation in (19). Taking logarithms in (20) now yields

$$\log M_s = (1+o(1)) \log N.$$

Combining this with (19) and solving $s \log s = (1+o(1)) \log N$ gives

$$s = (1+o(1)) \frac{\log N}{\log\log N}. \tag{21}$$

Since $(M_s - 1)/2 \le N$, Proposition 3.1 and (21) give

$$\mathfrak{M}_{\mathbb{T}}(N) \ge \exp\left((\log 2 - o(1))\frac{\log N}{\log\log N}\right).$$

Here the harmless absolute multiplicative constant in Proposition 3.1 has been absorbed into the $o(1)$ term in the exponent. □

4 The construction in dimensions two and higher

We now exploit primitive lattice points lying on a common circle.

Proposition 4.1. *Let q be a squarefree positive integer all of whose prime divisors are congruent to 1 modulo 4. If $N \ge \sqrt{q}$, then*

$$\mathfrak{M}_{\mathbb{T}^2}(N) \gtrsim 2^{\omega(q)}, \tag{22}$$

with an absolute implicit constant.

Proof. Set

$$A_q := \{k = (k_1, k_2) \in \mathbb{Z}^2 : k_1^2 + k_2^2 = q\}. \tag{23}$$

Let

$$r_2(q) := \#\{(a, b) \in \mathbb{Z}^2 : a^2 + b^2 = q\},$$

so that order and signs are counted. If χ_4 denotes the nonprincipal Dirichlet character modulo 4, then the classical identity (see [3])

$$r_2(q) = 4\sum_{m|q}\chi_4(m)$$

and the hypotheses on q give

$$|A_q| = r_2(q) = 4\tau(q) = 4 \cdot 2^{\omega(q)}. \tag{24}$$

Here $\tau(q)$ denotes the number of positive divisors of q. Every $k \in A_q$ is primitive. Indeed, if an integer $g > 1$ divided both k_1 and k_2, then g^2 would divide q, contradicting the squarefreeness of q.

Let $L := \lceil \sqrt{q} \rceil$ and define

$$B_q := \{(b, 0) \in \mathbb{Z}^2 \times \mathbb{Z} : b \in [-L, L]^2 \cap \mathbb{Z}^2\}. \tag{25}$$

Thus $|B_q| = (2L+1)^2$. Since $P(k) = (k, q)$ for every $k \in A_q$, all points of $P(A_q)+B_q$ have temporal coordinate q, and their spatial coordinates belong to $[-2L, 2L]^2$. Consequently

$$|P(A_q) + B_q| \le (4L + 1)^2 \lesssim |B_q|. \tag{26}$$

Fix $k \in A_q$. Apart from the inessential factor -2, a fiber from (7) is the intersection of $[-L, L]^2 \cap \mathbb{Z}^2$ with a line

$$k \cdot b = m, \tag{27}$$

where $m \in \mathbb{Z}$. Because k is primitive, the difference of any two integer solutions of (27) is an integer multiple of

$$k^{\perp} := (-k_2, k_1),$$

whose Euclidean length is $\sqrt{q}$. The square $[-L, L]^2$ has diameter $2\sqrt{2}L$. Hence every fiber has cardinality at most

$$1 + \frac{2\sqrt{2}L}{\sqrt{q}} = O(1),$$

uniformly in q and k. It follows that

$$\max_{k \in A_q} D_k(B_q) \lesssim 1. \tag{28}$$

Applying Proposition 2.1 and using (24), (26), and (28) proves (22). $\square$

Proof of Theorem 1.1, dimensions $d \geq 2$. Let $p_1^{(1)} < p_2^{(1)} < \cdots$ be the primes congruent to 1 modulo 4, and put

$$q_s := \prod_{j=1}^{s} p_j^{(1)}.$$

Choose $s = s(N)$ maximal subject to $q_s \leq N^2$. Writing

$$\vartheta(x; 4, 1) := \sum_{\substack{p \leq x \\ p \equiv 1 \pmod 4}} \log p,$$

the prime number theorem in arithmetic progressions gives (see [2])

$$p_s^{(1)} = (2 + o(1)) s \log s, \qquad \log q_s = \vartheta(p_s^{(1)}; 4, 1) = \left(\frac{1}{2} + o(1)\right) p_s^{(1)} = (1 + o(1)) s \log s. \tag{29}$$

Maximality gives

$$\frac{N^2}{p_{s+1}^{(1)}} < q_s \leq N^2. \tag{30}$$

Since $q_s \geq 5^s$, we have $s = O(\log N)$, and the first relation in (29) therefore implies $\log p_{s+1}^{(1)} = O(\log \log N)$. Taking logarithms in (30) gives $\log q_s = (2 + o(1)) \log N$. Combining this with (29) yields

$$s = (2 + o(1)) \frac{\log N}{\log \log N}. \tag{31}$$

Proposition 4.1 and (31) now yield

$$\mathfrak{M}_{\mathbb{T}^2}(N) \geq \exp\left((2 \log 2 - o(1)) \frac{\log N}{\log \log N}\right).$$

For $d > 2$, replace A_{q_s} by $A_{q_s} \times \{0\}^{d-2}$ and embed each $((b_1, b_2), \tau) \in B_{q_s}$ as $((b_1, b_2, 0, \ldots, 0), \tau) \in \mathbb{Z}^{d+1}$. Equivalently, the resulting trigonometric polynomials are independent of the last $d - 2$ spatial variables. The norms, the weighted integral, and all relevant X-ray transforms are unchanged. Thus

$$\mathfrak{M}_{\mathbb{T}^d}(N) \geq \mathfrak{M}_{\mathbb{T}^2}(N),$$

which completes the proof. $\square$

5 Concluding remarks

Remark 5.1. The same examples give identical lower bounds for the periodic Stein-type constants from [1]. Indeed, all Fourier coefficients of u_0 on $A = \operatorname{supp} \widehat{u_0}$ are equal to 1, and the fiber estimate is uniform over $k \in A$. Thus replacing the Mizohata–Takeuchi right-hand side by

$$\sum_{k \in A} |\widehat{u_0}(k)|^2 \sup_{x \in \mathbb{T}^d} \rho^*_{\mathbb{Z}^d} w(x, k)$$

does not change the argument.

Remark 5.2. The bounds in Theorem 1.1 are still subpolynomial in N. The constructions above do not produce a lower bound N^α for any fixed $\alpha > 0$. Whether such a power lower bound holds for the fixed periodic paraboloid is not addressed here.

Remark 5.3. The bounds in Theorem 1.1 dominate (4), because

$$\frac{\log N}{\log \log N} \gg \log \log N.$$

Equivalently, for every fixed $A > 0$,

$$\exp\left(c \frac{\log N}{\log \log N} \right) \gg (\log N)^A$$

whenever $c > 0$ is fixed.

References

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[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, sixth ed., revised by D. R. Heath-Brown and J. H. Silverman, Oxford University Press, Oxford, 2008.