# OPEN PROBLEMS FROM THE LECTURE SERIES: "FINITE EUCLIDEAN CONFIGURATIONS IN SETS OF POSITIVE DENSITY"

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*Problem* 1 (Large equilateral triangles in large planar sets). Does every planar set of positive upper density contain a congruent copy of the vertex set of every sufficiently large equilateral triangle? More precisely, suppose that $A \subseteq \mathbb{R}^2$ is measurable and satisfies

$$\overline{\mathrm{d}}(A) := \limsup_{R\to\infty} \frac{|A \cap [-R/2, R/2]^2|}{R^2} > 0,$$

where $|\cdot|$ denotes the Lebesgue measure. (Considering the upper Banach density instead should not make a difference.) Does there exist $\lambda_0 = \lambda_0(A) \in (0,\infty)$ such that, for every number $\lambda \geqslant \lambda_0$, there exist points $x, y, z \in A$ satisfying

$$\|x-y\|_2 = \|x-z\|_2 = \|y-z\|_2 = \lambda.$$

Here $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^2$.

The problem was implicit in the paper by Bourgain [2], who proved the analogous claim for triangles in $\mathbb{R}^d$, $d \geqslant 3$, and merely remarked:
*(...) the exact role of the dimension (...) does not seem well understood yet* [2, p. 308].
It was again implicit in the paper by Furstenberg, Katznelson, and Weiss [14], who also proved a weaker statement, namely, that there exists $\lambda_0 = \lambda_0(A)$ such that, for every $\lambda \geqslant \lambda_0$ and every $\varepsilon > 0$, the $\varepsilon$-neighborhood of $A$ necessarily contains three points forming an equilateral triangle. They wrote:
*(...) we shall have to be satisfied with results regarding [the $\varepsilon$-neighborhood of $A$]* [14, p. 185].
The problem was explicitly stated as an open problem by Croft, Falconer, and Guy in their problem book [5, Problem G14] and in several later papers, e.g., [13, p. 341].

*Problem* 2 (Large arithmetic progressions in $\ell^p$-norms). Take an integer $n \geqslant 4$ and a real number $p \in [1,\infty) \setminus \{1,2,\ldots,n-1\}$. Let $\|\cdot\|_p$ denote the $\ell^p$-norm on $\mathbb{R}^d$. Prove (or disprove) that there exists a positive integer $d_0 = d_0(n,p)$ such that the following density theorem for $n$-term arithmetic progressions holds in all dimensions $d \geqslant d_0$: if $A \subseteq \mathbb{R}^d$ is a measurable set of positive upper Banach density,

$$\overline{\delta}(A) := \lim_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|A \cap (x + [0,R]^d)|}{R^d} > 0,$$

then there exists $\lambda_0 = \lambda_0(A,n,p)$ such that, for every number $\lambda \geqslant \lambda_0$, there are $x, y \in \mathbb{R}^d$ with $\|y\|_p = \lambda$ and

$$x,\ x+y,\ x+2y,\ \ldots,\ x+(n-1)y \in A.$$

*Date*: June 5, 2026.

The conjecture is implicit but suggested in a paper by Cook, Magyar, and Pramanik [4], who proved the variant for three-term progressions, i.e., the same claim for $n = 3$. It is easy to extend Bourgain's example [2] to see that the values $p = 1, 2, \ldots, n-1$ have to be excluded [8]. Durcik and Kovač [6, Thm. 2] proved the following "compact" variant for every $n \geqslant 4$ and all $p$ as before: if $d \geqslant 2^{n+3}(n+p)$, then every measurable set $A \subseteq [0,1]^d$ of Lebesgue measure $|A| \geqslant \delta > 0$ has an interval of "scales" $I$ of positive length depending only on $\delta$ such that for every $\lambda \in I$ there exist $x, y \in \mathbb{R}^d$ satisfying $x, x+y, \ldots, x+(n-1)y \in A$ and $\|y\|_p = \lambda$. (The obtained lower bound on the length of $I$ is in fact concrete and "reasonable.")

*Problem* 3 (Rectangular boxes of a fixed large volume)*.* Take a positive integer $m \geqslant 2$. Does every set $A \subseteq \mathbb{R}^m$ of positive upper Banach density have a threshold $V_0 = V_0(A) \in (0, \infty)$ with the property that, for every number $V \geqslant V_0$, the set $A$ contains the vertices of an $m$-dimensional rectangular box (i.e., a "brick") of $m$-dimensional volume precisely $V$?

The problem was mentioned in [16]. In the same paper it was shown that sets of positive upper Banach density in $d \geqslant m+1$ dimensions have this property [16, Thm. 5 (a)]. This was a dimensional improvement over the corresponding more rigid result by Lyall and Magyar [19, Thm. 1.1 (i)] (also see the earlier results [18, Thm. 1.2] and [7, Thm. 1]), who showed that a set $A \subseteq \mathbb{R}^{2m}$ of positive upper Banach density contains isometric copies of the vertex sets of all sufficiently large cubes. Conversely, no obstructions are known that would prevent even this more rigid result from holding in $\mathbb{R}^m$, but this is expected to be much more difficult. (Squares in the plane are more difficult to find than fixed-shape triangles, which were themselves objects of a previous open problem.)

*Problem* 4 (Parallelograms of prescribed area)*.* Does every finite coloring of the Euclidean plane have a color class containing the vertices of a parallelogram of area 1?

The question was posed by Erdős and Graham [12, p. 15], along with its variants for several other planar shapes. A discrete variant of the problem has also been asked by Graham [15, p. 96]. The analogue of the problem for rectangles is known to have a negative answer [16, Thm. 3]. Partial results [16, Thm. 6] show that any positive proof must deal with nearly degenerate parallelograms and with infinitely many side directions.

*Problem* 5 (Unit-area triangles in sets of large finite measure)*.* The following is an open problem posed by Erdős, who asked:
*Is it true that there is an absolute constant $C$ so that if $S$ has planar measure greater than $C$ then $S$ contains the vertices of a triangle [of] area 1?* [10, p. 122]
Erdős even suspected that one can take $C = 4\pi/\sqrt{27}$, which is the area of the smallest disk containing an equilateral triangle of area 1.

Previously, in the popular Hungarian magazine *Matematikai Lapok*, Erdős posed, as an exercise, the problem of showing that every measurable set $S$ of infinite measure has this property. One of the readers proved this under the weaker assumptions that $S$ is unbounded and has positive measure. (This turned out to be an easy consequence of the Steinhaus theorem.) Erdős kept asking this question in several subsequent papers [9, 11]. Freiling and Mauldin [21, 20] proved this when $S$ is a union of at most 3 convex sets. Bulj and Kovač [3] proved that, for sufficiently large $R$, a measurable subset of $[0,R]^2$ that does not contain a triple of points spanning a triangle of area 1 necessarily has measure

$$O\bigg(R^2\Big(\frac{\log\log R}{\log R}\Big)^{1/2}\bigg).$$

The bound $o(R^2)$ as $R\to\infty$, which only slightly beats the trivial bound $O(R^2)$, can be deduced from a much older comment by Graham [15, p. 96]. Erdős's problem can be restated as the conjectural bound $O(1)$, i.e., a bound independent of $R$. Active discussion of the problem can be found on the *Erdős problems* forum [1, Problem 352].

*Problem* 6 (Sharp density threshold for $n$-point configurations). Fix positive integers $d,n\geqslant 2$. Let $\rho(d,n)\in[0,1]$ be the smallest threshold for a Euclidean density theorem of the form: if a measurable set $A\subseteq\mathbb{R}^d$ has $\overline{\delta}(A)>\rho(d,n)$, then it contains all sufficiently large similar copies of all $n$-point configurations. More precisely, $\rho(d,n)$ is the supremum of upper Banach densities $\overline{\delta}(A)$ of all measurable sets $A\subseteq\mathbb{R}^d$ for which there exist a set $P\subseteq\mathbb{R}^d$ of cardinality $n$ and a sequence of positive numbers $(\lambda_j)_{j=1}^{\infty}$ converging to infinity such that, for every index $j\geqslant 1$, the set $A$ does not contain an isometric copy of $\lambda_j P$. (Considering the usual upper density should not make a big difference.) What is the asymptotic behavior of $\rho(d,n)$ as $n\to\infty$?

The question was posed by Falconer, Kovač, and Yavicoli [13, Question 1.6]. A combination of papers [13, 17] shows the best currently known bounds:

$$1-C\frac{\log n}{n}\leqslant\rho(d,n)\leqslant 1-\frac{1}{n-1}$$

for $d\geqslant 2$ and sufficiently large $n$. The analogous problem in $\mathbb{R}^d$ with the $\ell^p$-norm for $p\in(1,\infty)$, $p\neq 2$, was resolved [17] with the expected sharp asymptotics

$$1-\frac{1}{n}+O_d\Big(\frac{1}{n^2}\Big).$$

## References


[1] T. F. Bloom, *Erdős problems*, `https://www.erdosproblems.com/`, accessed: June 5, 2026.

[2] J. Bourgain, *A Szemerédi type theorem for sets of positive density in* $\mathbf{R}^k$, Israel J. Math. **54** (1986), no. 3, 307–316.

[3] A. Bulj and V. Kovač, *On hyperbolic corners and unit-area triangles in planar sets of large measure*, preprint, `https://arxiv.org/abs/2605.30033`.

[4] B. Cook, Á. Magyar, and M. Pramanik, *A Roth-type theorem for dense subsets of* $\mathbb{R}^d$, Bull. London Math. Soc. **49** (2017), no. 4, 676–689.

[5] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved problems in geometry*, Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics II. Springer-Verlag, New York, 1991.

[6] P. Durcik and V. Kovač, *A Szemerédi-type theorem for subsets of the unit cube*, Anal. PDE **15** (2022), no. 2, 507–549.

[7] P. Durcik and V. Kovač, *Boxes, extended boxes and sets of positive upper density in the Euclidean space*, Math. Proc. Cambridge Philos. Soc. **171** (2021), no. 3, 481–501.

[8] P. Durcik, V. Kovač, and L. Rimanić, *On side lengths of corners in positive density subsets of the Euclidean space*, Int. Math. Res. Not. IMRN **2018**, no. 22, 6844–6869.

[9] P. Erdős, *My Scottish Book 'Problems'*, in *The Scottish Book*, Birkhäuser, Boston, 1981, pp. 27–33.

[10] P. Erdős, *Set-theoretic, measure-theoretic, combinatorial, and number-theoretic problems concerning point sets in Euclidean space*, Real Anal. Exchange **4** (1978/79), no. 2, 113–138.

[11] P. Erdős, *Some combinatorial, geometric and set theoretic problems in measure theory*, in *Measure Theory, Oberwolfach 1983*, Lecture Notes in Math. **1089**, Springer, 1984, pp. 321–327.

[12] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique **28**, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[13] K. J. Falconer, V. Kovač, and A. Yavicoli, *The density of sets containing large similar copies of finite sets*, J. Anal. Math. **148** (2022), no. 1, 339–359.

[14] H. Furstenberg, Y. Katznelson, and B. Weiss, *Ergodic theory and configurations in sets of positive density*, in *Mathematics of Ramsey Theory*, Algorithms Combin. **5**, Springer, Berlin, 1990, pp. 184–198.

[15] R. L. Graham, *On partitions of* $\mathbb{E}^n$, J. Combin. Theory Ser. A **28** (1980), no. 1, 89–97.

[16] V. Kovač, *Coloring and density theorems for configurations of a given volume*, Proc. Lond. Math. Soc. (3) **132** (2026), no. 3, Paper No. e70143.

[17] V. Kovač and A. A. Santos Sepčić, *Near-optimal density theorems for large dilates of large point configurations*, preprint, `https://arxiv.org/abs/2604.18544`.

[18] N. Lyall and Á. Magyar, *Product of simplices and sets of positive upper density in* $\mathbb{R}^d$, Math. Proc. Cambridge Philos. Soc. **165** (2018), no. 1, 25–51.

[19] N. Lyall and Á. Magyar, *Weak hypergraph regularity and applications to geometric Ramsey theory*, Trans. Amer. Math. Soc. Ser. B **9** (2022), 160–207.

[20] R. D. Mauldin, *Some problems and ideas of Erdős in analysis and geometry*, in *Erdős Centennial*, Bolyai Soc. Math. Stud. **25**, János Bolyai Math. Soc., Budapest, 2013, pp. 365–376.

[21] R. D. Mauldin, *Some problems in set theory, analysis and geometry*, in *Paul Erdős and his mathematics, I (Budapest, 1999)*, Bolyai Soc. Math. Stud. **11**, János Bolyai Math. Soc., Budapest, 2002, pp. 493–506.

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