Abstracts

Large copies of large configurations in large sets

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There exist patterns in large but otherwise arbitrary structures.

This is the main maxim of the Ramsey theory, but it is also widespread in other areas of mathematics. Results that are addressed here can be collectively called the *Euclidean density theorems* and they belong to the intersection of the arithmetic combinatorics and the geometric measure theory. These results study *large sets*, as this is what the word "density" stands for. In the present context, a measurable subset A of the unit cube $[0,1]^d$ is considered *large* if its Lebesgue measure is positive. On the other hand, a measurable subset A of the whole space $\mathbb{R}^d$ is considered *large* if it occupies a positive portion of the space, i.e., its (appropriately defined) upper (Banach) density $\overline{\delta}(A)$ is positive. The Euclidean density theorems search inside A for congruent (i.e., isometric) copies of given *configurations* (*patterns*) from a prescribed family $\mathcal{P} = \{P_\lambda : \lambda \in (0,\infty)\}$, indexed by a certain "size" parameter λ. Typically, P_λ is the dilate by λ of a fixed point configuration P, i.e., $P_\lambda = \lambda P$.

Using the Lebesgue density theorem one can easily find all kinds of finite configurations inside a positive measure set A. Moreover, generalizing the Steinhaus theorem one can even find inside A all sufficiently small dilates of a given finite point configuration P. Therefore, we have to ask for more in order to obtain a meaningful result in this setting. There are two types of results that we are generally aiming for. The first one is the *"all large scales" formulation.*

ALS: *For every measurable set $A \subseteq \mathbb{R}^d$ satisfying $\overline{\delta}(A) > 0$ there exists a number $\lambda_0 = \lambda_0(\mathcal{P}, A) > 0$ such that for every $\lambda \geq \lambda_0$ the set A contains a congruent copy of P_λ.*

This is a rather strong but only qualitative claim, as the number λ_0 depends on more than just the density $\overline{\delta}(A)$. The second one is *"an interval of scales" formulation*, sometimes also known as a *compact formulation* (after [1]).

IOS: *Take a number $0 < \delta \ll 1$ and a measurable set $A \subseteq [0,1]^d$ with measure at least δ. Then the set of scales $\lambda \in (0,\infty)$ such that A contains a congruent copy of P_λ contains an interval of length at least $\varepsilon = F_{\mathcal{P},d}(\delta)$.*

This is a weaker but quantitative claim and it enables a competition to find better dependencies of ε on δ.

This whole topic was initiated by a question of Székely [17] on whether a positive upper density set $A \subseteq \mathbb{R}^2$ realizes all sufficiently large distances (i.e., in our terminology, whether an ALS result holds for $P = \{0,1\}$), which has been subsequently popularized by Erdős [8]. It was answered affirmatively by Furstenberg, Katznelson, and Weiss [11], and independently also by Falconer and Marstrand

[10] and Bourgain [1]. Since then, a lot of work has been done in the aforementioned natural special case, when a fixed pattern P is scaled by the usual Euclidean dilations. The most general known positive result, in both ALS and IOS formulations, is due to Lyall and Magyar [16] and it holds when $P = \Delta_1 \times \cdots \times \Delta_m$ is a Cartesian product of vertex-sets Δ_j of nondegenerate simplices. The most general negative result is still due to Graham [12], who showed that ALS (and similarly IOS) results fail for configurations that cannot be inscribed in a sphere.

The purpose of this note is to inform the reader on where to look for the most recent developments on the topic, which have become possible primarily due to recent breakthroughs in the field of the multilinear harmonic analysis. We can "change the rules" slightly in one of the following ways, in order to open new interesting research directions.

Quantitative bounds. We might want to improve bounds in the IOS formulations. Already when P is a set of vertices of an n-dimensional rectagular box and A is a measurable subset of $[0,1]^{2n}$, the approach of Lyall and Magyar [16] gives an interval of a very small length; namely ε^{-1} is a tower of exponentials of height n of the number $\delta^{-3\cdot 2^n}$. Durcik and the author [4] have increased this to a "more reasonable" bound, $\varepsilon = (\exp(\delta^{-C(n,P)}))^{-1}$. A bound of the same type was later shown, more generally, for products of simplices by Durcik and Stipčić [7].

Anisotropic dilations. One can start with a configuration P and generate the collection $\mathcal{P}$ by applying to it anisotropic power-type scalings, namely $(x_1, \ldots, x_n) \mapsto (\lambda^{a_1} b_1 x_1, \ldots, \lambda^{a_n} b_n x_n)$, where a_j, b_j are fixed positive parameters. It was shown in [13] that analogues of many classical results from [1, 15, 16] remain valid in this modified context.

Sizes in ℓ^p. Already Bourgain [1] noted that ALS results fail for a triple of collinear points P. Cook, Magyar, and Pramanik [2] came up with an idea to study three-term arithmetic progressions $x, x+t, x+2t \in \mathbb{R}^d$, but evaluate sizes of their gaps t in other ℓ^p norms. Thew showed the ALS formulation whenever $p \neq 1, 2, \infty$ and d is sufficiently large. This was generalized to *corners* (x,y), $(x+t,y)$, $(x,y+t) \in (\mathbb{R}^d)^2$ by Durcik, Rimanić, and the author [5], but longer arithmetic progressions are still an open problem at the time of writing. As opposed to that, the IOS formulation (with a still "reasonable" length ε) was shown by Durcik and the author [4]. It turns out that for n-term arithmetic progressions one needs to avoid precisely the values $1, 2, \ldots, n-1, \infty$ for p. Finally, certain mixtures of three-term progressions or corners and product-type configurations were explored by the same authors in [3].

Very dense sets. Falconer, Yavicoli, and the author [9] considered measurable sets with density $\overline{\delta}(A)$ sufficiently close to 1 that A must contain all large dilates of all n-point configurations. Nontrivial upper and lower bounds for the critical density were shown in that paper, but its sharp asymptotics as $n \to \infty$ is currently still unknown.

Nonlinear configurations. Kuca, Orponen, and Sahlsten [14] showed that every compact set $K \subseteq \mathbb{R}^2$ of Hausdorff dimension sufficiently close to 2 contains a pair of distinct points of the form (x,y), $(x,y)+(u,u^2)$. This can be thought of as a continuous variant of the Furstenberg–Sárközy theorem (on $\mathbb{R}^2$ instead of $\mathbb{Z}$). One naturally wonders what stronger property of this type holds for sets of positive Lebesgue measure. Durcik, Stipčić, and the author [6] showed, among other things, that a positive measure set $A \subseteq [0,1]^2$ contains a point $(x_0, y_0) \in A$ such that A nontrivially intersects parabolae $y - y_0 = a(x - x_0)^2$ for a whole interval $I \subseteq (0, \infty)$ of parameters $a \in I$. Larger nonlinear configurations could be an interesting topic to study.

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