A sharp nonlinear Hausdorff-Young inequality for small potentials

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(joint work with Diogo Oliveira e Silva and Jelena Rupčić)

Let us begin by describing the setting that is sometimes informally called the *nonlinear Fourier transform*. Its more precise synonyms are the $\mathrm{SU}(1,1)$-*scattering transform* and the *Dirac scattering transform*. More details can be found in the unpublished note by Tao [12], while a similar discrete-time model is studied in the lecture notes by Tao and Thiele [13].

Take a measurable, bounded, and compactly supported function $f\colon \mathbb{R}\to\mathbb{C}$ and an arbitrary number $\xi\in\mathbb{R}$. The matrix-valued initial value problem

$$\frac{d}{dx}\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} = \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix} \begin{bmatrix} 0 & f(x)e^{-2\pi i x\xi} \\ \overline{f(x)}e^{2\pi i x\xi} & 0 \end{bmatrix},$$

$$\begin{bmatrix} a(-\infty,\xi) & b(-\infty,\xi) \\ \overline{b(-\infty,\xi)} & \overline{a(-\infty,\xi)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has unique absolutely continuous solutions $a(\cdot,\xi)$ and $b(\cdot,\xi)$. This way we arrive at the functions $a,b\colon \mathbb{R}\to\mathbb{C}$ given by $a(\xi):=a(+\infty,\xi)$, $b(\xi):=b(+\infty,\xi)$, and we can study properties of the "forward transform" $f\mapsto a,b$. Observe that the matrices containing $a(x,\xi)$ and $b(x,\xi)$ remain in the matrix group $\mathrm{SU}(1,1)$, since the matrices containing $f(x)e^{-2\pi i x\xi}$ belong to its Lie algebra $\mathfrak{su}(1,1)$.

It is useful to rewrite the problem as a system of two scalar differential equations, and then in turn as a system of two integral equations,

$$a(x,\xi) = 1 + \int_{-\infty}^{x} \overline{f(y)}e^{2\pi i y\xi} b(y,\xi)dy, \quad b(x,\xi) = \int_{-\infty}^{x} f(y)e^{-2\pi i y\xi} a(y,\xi)dy.$$

Applying Picard's iteration one arrives at multilinear expansions for a and b. By the work of Christ and Kiselev [5], [6] these expansions are known to converge and extend the definition of the transform to the functions $f\in \mathrm{L}^p(\mathbb{R})$ for $1\le p<2$. However, Muscalu, Tao, and Thiele [11] showed that these expansions cannot be used for a general $f\in \mathrm{L}^2(\mathbb{R})$.

It is useful to emphasize that we are not talking about the linear Fourier transform on the group $\mathrm{SU}(1,1)$. Indeed, $f\mapsto a,b$ are "very" nonlinear transformations and, for instance, they do not allow us to use any general form of interpolation. On the other hand, they still share many symmetries with the linear Fourier transform (with respect to L^1-dilations, translations, modulations, etc.); see [12].

One source of motivation for this setting comes from the eigenproblem for the *Dirac operator*,

$$L := \begin{bmatrix} \frac{d}{dx} & -\bar{f} \\ f & -\frac{d}{dx} \end{bmatrix}, \quad L\begin{bmatrix} \varphi(\cdot,\xi) \\ \psi(\cdot,\xi) \end{bmatrix} = -\pi i\xi \begin{bmatrix} \varphi(\cdot,\xi) \\ \psi(\cdot,\xi) \end{bmatrix};$$

see [12]. This eigenvector equation for the imaginary eigenvalue $-\pi i\xi$ turns into the above system for a and b, after we substitute $a(x,\xi):=\varphi(x,\xi)e^{\pi i x\xi}$ and $b(x,\xi):=\psi(x,\xi)e^{-\pi i x\xi}$. Another source of motivation are the general AKNS-ZS systems [1], [14]. In a very special case, one can consider two bodies in a plane with mutual interactions. If $u_1(t), u_2(t) \in \mathbb{C}$ determine their positions at time t, $\omega_1 \neq \omega_2$ are given angular velocities, and $\lambda \in \mathbb{R}$ is a certain spectral parameter, then the motion of the system is governed by the differential equation

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = i\lambda \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \overline{f(t)} \\ f(t) & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

This time we substitute $a(t,\lambda) := u_1(t)e^{-i\omega_1\lambda t}$, $b(t,\lambda) := u_2(t)e^{-i\omega_2\lambda t}$ and once again we arrive at the same system for a and b as before. The central problem in this model is to determine if the system remains bounded for a.e. $\lambda \in \mathbb{R}$; it is related to the first question stated below.

One open question about our nonlinear Fourier transform is a nonlinear analogue of the Carleson theorem: if $f \in \mathrm{L}^2(\mathbb{R})$ and $\mathrm{supp}(f) \subseteq [0, +\infty)$, does the limit

$$\lim_{x\to+\infty} \begin{bmatrix} a(x,\xi) & b(x,\xi) \\ \overline{b(x,\xi)} & \overline{a(x,\xi)} \end{bmatrix}$$

exist for a.e. $\xi \in \mathbb{R}$? Even finiteness of $\sup_x |a(x,\xi)|$ for a.e. $\xi \in \mathbb{R}$ is open. Christ and Kiselev [6] showed the analogous claim for $f \in \mathrm{L}^p(\mathbb{R})$ when $1 \leq p < 2$, while Muscalu, Tao, and Thiele [10] established it in the Cantor group "toy-model", where the exponentials are replaced with characters of a different group.

Another open question is related to the nonlinear analogues of the Hausdorff-Young inequalities, also due to Christ and Kiselev [5], [6]:

$$\|(\log |a|^2)^{1/2}\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq C_p \|f\|_{\mathrm{L}^p(\mathbb{R})}$$

for $1 \leq p \leq 2$ and its conjugated exponent p'. The case $p = 1$ is trivial (with $C_1 = 1$) by Grönwall's lemma, while $p = 2$ leads to the well-known scattering identity (with $C_2 = 1$). This identity appears for instance in the work of Faddeev and Buslaev [7] and it is shown by the contour integration; see [10]. It is not known if the optimal constants C_p are bounded uniformly in $1 \leq p \leq 2$; this problem arises in the absence of any available interpolation. This uniformity was confirmed by Kovač [8], but only in the same Cantor group model mentioned before.

Let us formulate the main result by Kovač, Oliveira e Silva, and Rupčić from [9]. Fix an exponent $1 < p < 2$, a number $H > 0$ interpreted as the "height", and a number $W > 0$ interpreted as the "width". Let us also recall *Babenko-Beckner's constant* $\mathbf{B}_p := p^{1/2p} p'^{-1/2p'}$, which is known (by [2] and [3]) to be the norm of the linear Fourier transform from $\mathrm{L}^p(\mathbb{R})$ to $\mathrm{L}^{p'}(\mathbb{R})$. We only consider functions f with controlled height and width, i.e. such that $|f| \leq H$ and that f is supported in an interval of length at most W.

Theorem. There exist $\delta > 0$ and $\varepsilon > 0$ (depending on p, H, W) such that for each f satisfying $\|f\|_{\mathrm{L}^1(\mathbb{R})} \leq \delta$ one has

$$\big\|(\log |a|^2)^{\frac{1}{2}}\big\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \big(\mathbf{B}_p - \varepsilon \|f\|^2_{\mathrm{L}^1(\mathbb{R})}\big) \|f\|_{\mathrm{L}^p(\mathbb{R})}.$$

The theorem claims no uniform boundedness of the constants C_p in any sense, because δ depends on p. Indeed, a uniform estimate for functions satisfying (say)

$\|f\|_{\mathrm{L}^1(\mathbb{R})} \leq 1$ follows simply from Grönwall's lemma. Therefore, the emphasis of the theorem is on the fact that the nonlinear Hausdorff-Young ratio beats the linear one for sufficiently small values of $\|f\|_{\mathrm{L}^1(\mathbb{R})}$.

The strategy of the proof is to denote by $\mathfrak{G}$ the set of modulated Gaussians, $G(x) = Ce^{-Ax^2+Bx}$ for some $A > 0$, $B, C \in \mathbb{C}$, and to distinguish between the following two cases.

Case 1. If the relative L^p-distance of f from $\mathfrak{G}$ is greater than $\|f\|_{\mathrm{L}^1(\mathbb{R})}^{1/2}$, then we use Christ's sharpened linear Hausdorff-Young inequality [4],

$$\|\widehat{f}\|_{\mathrm{L}^{p'}(\mathbb{R})} \leq \Big(\mathbf{B}_p - c_p \frac{\mathrm{dist}_p^2(f,\mathfrak{G})}{\|f\|_{\mathrm{L}^p(\mathbb{R})}^2}\Big)\|f\|_{\mathrm{L}^p(\mathbb{R})}.$$

It compensates for the loss coming from an application of Grönwall's lemma.

Case 2. If the relative L^p-distance of f from $\mathfrak{G}$ is smaller than $\|f\|_{\mathrm{L}^1(\mathbb{R})}^{1/2}$, then we calculate a few terms of the multilinear expansion for $(\log|a|^2)^{1/2}$ and approximate f by a Gaussian. In the process of controlling the error terms we apply the standard Menshov-Paley-Zygmund estimate several times.

References

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