

## Multilinear singular integrals with entangled structure

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Multilinear singular integral operators in higher dimensions can have more complicated structure than their one-dimensional analogues. An interesting example is a two-dimensional variant of the bilinear Hilbert transform introduced by Demeter and Thiele in [5]. It is defined as the bilinear operator

$$(1) \quad T(f, g)(x, y) = \text{p.v.} \int_{\mathbb{R}^2} f((x, y) + A(s, t)) g((x, y) + B(s, t)) K(s, t) ds dt,$$

where  $K$  is a Calderón-Zygmund kernel, while  $A$  and  $B$  are  $2 \times 2$  real matrices. The same authors observed that  $L^p$  estimates for (1) imply boundedness of the Carleson operator in a certain range of exponents. This fact guarantees that the proof of any bounds for (1) has to use some techniques from time-frequency analysis.

It is natural to pair a multilinear operator with an extra function and reduce its boundedness to proving estimates for the corresponding multilinear form. We focus on a class of multilinear singular integral forms acting on two-dimensional functions that “partially share variables”. Schematically and somewhat informally we want to study forms such as

$$(2) \quad \Lambda(f_1, f_2, \dots) = \int_{\mathbb{R}^n} f_1(x_1, x_2) f_2(x_1, x_3) \dots K(x_1, \dots, x_n) dx_1 dx_2 dx_3 \dots dx_n,$$

in which a variable  $x_1$  appears in the arguments of functions  $f_1$  and  $f_2$ . It turns out that wave packet decompositions are no longer efficient for bounding such forms, because of the appearance of the pointwise product  $x_1 \mapsto f_1(x_1, x_2) f_2(x_1, x_3)$ . The structure of any such form is determined by the kernel  $K$  and a simple undirected graph  $G$  on the set of variables  $x_1, x_2, \dots$ , where two vertices  $x_i$  and  $x_j$  are joined with an edge if and only if there exists a function depending on the pair  $(x_i, x_j)$  in the definition of  $\Lambda$ .

*General results in the dyadic setting.* A rather complete theory of entangled forms is possible in the case when the graph  $G$  is bipartite and  $K$  is a “perfect” dyadic model of a multilinear Calderón-Zygmund kernel. More precisely, we take positive integers  $m, n \geq 2$  and consider the “diagonal” in  $\mathbb{R}^{m+n}$ ,

$$D = \left\{ \underbrace{(x, \dots, x)}_m, \underbrace{y, \dots, y}_n : x, y \in \mathbb{R} \right\}.$$

We require that the kernel  $K$  satisfies the usual “size condition”,

$$|K(x_1, \dots, x_m, y_1, \dots, y_n)| \leq C \left( \sum_{i_1 < i_2} |x_{i_1} - x_{i_2}| + \sum_{j_1 < j_2} |y_{j_1} - y_{j_2}| \right)^{2-m-n}$$

and also that  $K$  is constant on all  $(m+n)$ -dimensional dyadic cubes disjoint from  $D$ . The last condition is in analogy with the setting from [1]. For technical reasons we also assume that  $K$  is bounded and compactly supported. Take  $E \subseteq \{1, \dots, m\} \times \{1, \dots, n\}$  and interpret it as the set of edges of a simple bipartite

undirected graph on  $\{x_1, \dots, x_m\}$  and  $\{y_1, \dots, y_n\}$ . The corresponding  $|E|$ -linear singular form (2) becomes

$$\Lambda((F_{i,j})_{(i,j) \in E}) = \int_{\mathbb{R}^{m+n}} K(x_1, \dots, y_n) \prod_{(i,j) \in E} F_{i,j}(x_i, y_j) dx_1 \dots dy_n.$$

In order to avoid degeneracy of  $\Lambda$  we assume that there are no isolated vertices in  $G$ . Recall that there are  $|E|$  mutually adjoint  $(|E|-1)$ -linear operators  $T_{u,v}$ ,  $(u, v) \in E$  corresponding to  $\Lambda$ .

The main result from the pair of papers by Thiele and the author [6], [11] is a  $T(1)$ -type characterization of  $L^p$  boundedness.

**Theorem.** *There exist positive integers  $d_{i,j}$  such that the following holds. If*

$$|\Lambda(\mathbf{1}_Q, \dots, \mathbf{1}_Q)| \leq C_1 |Q| \text{ for every dyadic square } Q$$

and

$$\|T_{u,v}(\mathbf{1}_{\mathbb{R}^2}, \dots, \mathbf{1}_{\mathbb{R}^2})\|_{\text{BMO}(\mathbb{R}^2)} \leq C_2 \text{ for each } (u, v) \in E,$$

then

$$|\Lambda((F_{i,j})_{(i,j) \in E})| \leq C_3 \prod_{(i,j) \in E} \|F_{i,j}\|_{L^{p_{i,j}}(\mathbb{R}^2)}$$

for exponents  $p_{i,j}$  satisfying  $\sum_{(i,j) \in E} 1/p_{i,j} = 1$  and  $d_{i,j} < p_{i,j} \leq \infty$ .

Let us remark that the numbers  $d_{i,j}$  depend on the graph  $G$  in a rather complicated way, but they always determine a non-empty range of exponents in which  $L^p$  estimates hold.

The first step in the proof of the above theorem is a decomposition into two types of paraproduct-type operators. ‘‘Cancellative entangled paraproducts’’ satisfy  $L^p$  estimates provided their coefficients are bounded, while the coefficients of ‘‘non-cancellative entangled paraproducts’’ need to satisfy a Carleson-type condition. The second step consists of a certain multilinear variant of the Bellman function technique developed in [6] and [7]. This method primarily applies to multilinear forms described above, but it can occasionally also provide results on some forms possessing explicit modulation invariance, for which more involved wave packet analysis would normally be required, see [8]. However, the technique usually needs to be combined with other ideas in order to transfer the results from the perfect dyadic or other ‘‘algebraic models’’ to the actual singular integral operators.

*Applications of multilinear forms with entangled structure.* Most of the previously mentioned techniques and results were motivated by the only case of (1) that was left out from [5] as an open problem. After a change of variables it turns into

$$(3) \quad \Lambda(f, g, h) = \int_{\mathbb{R}^4} f(u, y) g(x, v) h(x, y) K(u, v, x, y) dudvxdy.$$

Since the graph  $u-y-x-v$  is bipartite, at least the dyadic model of (3) falls into the realm of the general theory from the previous section. The first bounds for (3) were established in the paper by the author [7], while the result of Bernicot [3] significantly expanded the range of  $L^p$  estimates. Later Bernicot and the author [4]

studied some Sobolev norm inequalities for the same type of bilinear multipliers. Škreb and the author [10] generalized some of the  $L^p$  estimates to the setting of general dilations found in [12].

Yet another source of motivation are quantitative convergence results on ergodic averages. Such results in ergodic theory are often established by proving relevant estimates for integral operators on the real line. Particularly interesting are double ergodic averages along orbits of two commuting invertible measure-preserving transformations, motivated by multidimensional Szemerédi's theorem. Their convergence in  $L^p$ ,  $p < \infty$  is a classical result, but recently Avigad and Rute [2] raised a question of showing any norm-variation estimates, which would quantify this convergence. The author was able to prove a finite group model for such an estimate [9], where the  $\mathbb{Z}$ -actions are replaced by two commuting measure-preserving actions of the direct sum of countably many copies of a finite abelian group. The techniques used in [9] very closely resemble the methods in the proof of the above theorem.

#### REFERENCES

- [1] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, C. Thiele, *Carleson measures, trees, extrapolation, and  $T(b)$  theorems*, Publ. Mat. 46 (2002), no. 2, 257–325.
- [2] J. Avigad, J. Rute, *Oscillation and the mean ergodic theorem for uniformly convex Banach spaces*, Ergodic Theory Dynam. Systems. 35 (2015), no. 4, 1009–1027.
- [3] F. Bernicot, *Fiber-wise Calderón-Zygmund decomposition and application to a bi-dimensional paraproduct*, Illinois J. Math. 56 (2012), 415–422.
- [4] F. Bernicot, V. Kovač, *Sobolev norm estimates for a class of bilinear multipliers*, Commun. Pure Appl. Anal. 13 (2014), no. 3, 1305–1315.
- [5] C. Demeter, C. Thiele, *On the two-dimensional bilinear Hilbert transform*, Amer. J. Math. 132 (2010), no. 1, 201–256.
- [6] V. Kovač, *Bellman function technique for multilinear estimates and an application to generalized paraproducts*, Indiana Univ. Math. J. 60 (2011), no. 3, 813–846.
- [7] V. Kovač, *Boundedness of the twisted paraproduct*, Rev. Mat. Iberoam. 28 (2012), no. 4, 1143–1164.
- [8] V. Kovač, *On a trilinear form related to the Carleson theorem*, J. Math. Anal. Appl. 405 (2013), no. 1, 220–226.
- [9] V. Kovač, *Quantitative norm convergence of double ergodic averages associated with two commuting group actions*, Ergodic Theory Dynam. Systems. 36 (2016), no. 3, 860–874.
- [10] V. Kovač, K. A. Škreb, *One modification of the martingale transform and its applications to paraproducts and stochastic integrals*, J. Math. Anal. Appl. 426 (2015), no. 2, 1143–1163.
- [11] V. Kovač, C. Thiele, *A  $T(1)$  theorem for entangled multilinear dyadic Calderón-Zygmund operators*, Illinois J. Math. 57 (2013), no. 3, 775–799.
- [12] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84 (1978), no. 6, 1239–1295.