Finite Euclidean configurations in sets of positive density

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Abstract

Euclidean density theorems ask which finite point configurations must occur in measurable subsets of Euclidean space that are large in the sense of positive measure or positive upper density. Motivated by the Ramsey-theoretic principle that large structures contain patterns, the central problems are to prove either that all sufficiently large scales of a configuration appear or, in compact settings, that the set of realized scales contains an interval whose length can be bounded quantitatively in terms of the density. Beginning with Székely's distance problem and Bourgain's simplex theorem, the area has developed into a meeting point of geometric measure theory, arithmetic combinatorics, and real multilinear harmonic analysis. The basic analytic strategy is to encode configurations using counting forms and decompose them into structured, error, and uniform components, with the latter two controlled through Fourier decay, oscillatory integrals, Littlewood–Paley theory, and multilinear singular or Brascamp–Lieb-type inequalities. Conversely, negative results are often proved by constructing funny colorings.

Contents

Chapter 1

What are Euclidean density theorems?

1.1 The guiding problems

A typical Euclidean density theorem has one of the following forms. Let P be a finite configuration in $\mathbb{R}^d$, and let $A \subseteq \mathbb{R}^d$ be measurable and "large." Must A contain a certain copy of P? Or copies of P at all sufficiently large scales? Or copies of P at all scales forming an interval? There are several other choices hidden in all these formulations:

- the meaning of "large": positive fraction of a cube or a ball, positive upper density, or positive upper Banach density;
- the allowed "copies": translates, similar copies, congruent copies, or isometric copies but in another norm;
- the "rigidity" of the configuration: fixed edge lengths, flexible graph embeddings, prescribed volume, etc.;
- the definition of a "scale": a certain size of a configuration, an eccentricity, etc..

Positive results are usually obtained when the configuration has enough degrees of freedom relative to the ambient dimension. Negative results exploit algebraic invariants, modular obstructions, or appropriate colorings of the ambient space.

Throughout the notes, d denotes the ambient Euclidean dimension. The letter A usually denotes the measurable set in which configurations are being sought, whether $A \subseteq \mathbb{R}^d$ is unbounded or A lies in a fixed cube. When a positive-density set is localized to a large cube, the localized set is denoted by $A_R \subseteq [0, R]^d$ and its indicator by $f = \mathbb{1}_{A_R}$.

Euclidean balls are denoted by $\mathrm{B}(a, R) = \{x \in \mathbb{R}^d : |x - a| < R\}$, and $\mathrm{B}_R = \mathrm{B}(0, R)$, while cubes are rather written explicitly as $x + [0, R]^d$. The Lebesgue measure of $A \subseteq \mathbb{R}^d$ will be written simply as $|A|$. This will cause no confusion with the absolute value or the Euclidean length or the cardinality, and the (implicit) ambient dimension d will be understood.

See Section A.1 for the explanation of the notation $\lesssim$, $\gtrsim$, O, and Ω.

1.2 Density notions

Definition 1.1 (Asymptotic densities). For a measurable set $A \subseteq \mathbb{R}^d$, define

$$\overline{\mathrm{d}}(A) := \limsup_{R\to\infty} \frac{|A \cap [-R/2, R/2]^d|}{R^d}, \qquad \underline{\mathrm{d}}(A) := \liminf_{R\to\infty} \frac{|A \cap [-R/2, R/2]^d|}{R^d},$$

called the *upper density* and the *lower density* of A, respectively. If the actual limit exists, it is denoted $\mathrm{d}(A)$ and called the *density* of A.

Definition 1.2 (Upper Banach density). The *upper Banach density* of a measurable set $A \subseteq \mathbb{R}^d$ is

$$\overline{\delta}(A) := \lim_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|A \cap (x + [0,R]^d)|}{R^d}.$$

The existence of this limit, and its invariance under replacing the cube by any fixed compact convex body with nonempty interior, are proved in Appendix A.2. Thus one may equivalently use large balls, ellipsoids, simplices, or other convex averaging windows $\mathcal{B}$, as long as we are stretching them isotropically by factor R:

$$R\mathcal{B} := \{Ry : y \in \mathcal{B}\},$$
$$\overline{\delta}(A) = \lim_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|A \cap (x + R\mathcal{B})|}{|R\mathcal{B}|}.$$

We always have

$$\underline{\mathrm{d}}(A) \leqslant \overline{\mathrm{d}}(A) \leqslant \overline{\delta}(A).$$

Definition 1.3 (Compact density). If $A \subseteq [0,R]^d$ is measurable, its *density in the cube* is $|A|/R^d$. Compact density theorems usually assume $|A| \geqslant \delta R^d$ and produce quantitative information depending on δ. Similarly we would define density in a fixed ball.

1.3 A near-full-density translation lemma

The following elementary lemma is often the quickest way to pass from a high density of a set on a fixed ball to a translated copy of a finite pattern. It is simply the union bound applied to the complement, but it is useful enough to be recorded explicitly.

Lemma 1.4 (Near-full-density translation lemma). *Let $A \subseteq \mathbb{R}^d$ be measurable, let $\mathrm{B}(a,R)$ be the Euclidean ball of radius R centered at a, and suppose that*

$$\frac{|A \cap \mathrm{B}(a,R)|}{|\mathrm{B}(a,R)|} \geqslant 1 - \eta.$$

Let $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$, and put $D = \operatorname{diam} P$. If

$$D < R \qquad \text{and} \qquad n\eta < \left(1 - \frac{D}{R}\right)^d, \tag{1.1}$$

then A contains a translate of P.

Proof. Translate P in advance so that $p_1 = 0$. Then $|p_i| \leqslant D$ for every i. Let

$$H = \mathrm{B}(a,R) \setminus A, \qquad B' = \mathrm{B}(a, R - D).$$

If $t \in B'$, then $t + p_i \in \mathrm{B}(a,R)$ for each i. Suppose, for contradiction, that no translate $t + P$ is contained in A. Then for every $t \in B'$ at least one of the points $t + p_i$ lies in H, and hence

$$B' \subseteq \bigcup_{i=1}^{n} (H - p_i).$$

Taking measures and using translation-invariance gives

$$|B'| \leqslant \sum_{i=1}^{n} |H - p_i| = n|H| \leqslant n\eta|\mathrm{B}(a,R)|.$$

On the other hand,

$$|B'| = (1 - D/R)^d|\mathrm{B}(a,R)|,$$

which contradicts (1.1). Therefore some $t \in B'$ satisfies $t + p_i \in A$ for all i. □

Corollary 1.5 (Positive measure gives small homothetic copies). *Let $A \subseteq \mathbb{R}^d$ be measurable with $|A| > 0$, and let $P \subseteq \mathbb{R}^d$ be finite. Then there is $\lambda_0 > 0$ such that, for every $0 < \lambda \leqslant \lambda_0$, the set A contains a translate of λP.*

Proof. Let $n = |P|$ and choose $\eta < 1/(n2^d)$. By the Lebesgue density theorem, there is a density point a of A and a (small) radius $R > 0$ such that

$$|A \cap \mathrm{B}(a,R)| \geqslant (1-\eta)|\mathrm{B}(a,R)|.$$

Lemma 1.4 applies to λP whenever $\lambda \operatorname{diam} P \leqslant R/2$. □

Things are not so easy for infinite configurations P, but we do not discuss those issues in these notes.

Corollary 1.6 (The threshold $1 - 1/n$ for translated copies). *Let $P \subseteq \mathbb{R}^d$ be an n-point configuration. If $A \subseteq \mathbb{R}^d$ is measurable and*

$$\overline{\delta}(A) > 1 - \frac{1}{n},$$

then, for every $\lambda > 0$, the set A contains a translate of λP.

Proof. Choose η with

$$1 - \overline{\delta}(A) < \eta < \frac{1}{n}.$$

By the ball formulation of upper Banach density proved in Appendix A.2, there are arbitrarily large balls $\mathrm{B}(a,R)$ satisfying

$$|A \cap \mathrm{B}(a,R)| \geqslant (1-\eta)|\mathrm{B}(a,R)|.$$

For the fixed configuration λP, let $D = \operatorname{diam}(\lambda P)$. Since $\eta < 1/n$, we can choose such a ball with R so large that (1.1) holds. Lemma 1.4 then gives a translate of λP inside A. □

Density theorems for cubes, simplices, and distance graphs

This chapter collects positive model results for isotropic configurations, i.e., those that will be scaled equally in "all directions." We will principally study all-large-scale theorems: every measurable set of positive upper Banach density contains the prescribed configuration at every sufficiently large scale. We will also briefly discuss their "interval of scales" variants, which are interesting mainly because they have quantitative formulations.

The examples gathered here will also provide the basic prototypes for the later chapters. The order is deliberately methodological, even if not historically faithful. After a general "abstract nonsense" discussion, we begin with distances, then move to rectangular boxes, simplices, and hypercube graphs. Historically, the first large-distance theorem was motivated by a question of Székely [46], and it is due to Furstenberg–Katznelson–Weiss [20] and also (independently) to Falconer–Marstrand [18]. Bourgain introduced the analytic large-scale strategy for larger configurations [3], and Lyall–Magyar developed product and graph versions for simplices, boxes, and distance graphs [38, 39, 40]. The proof language used below is the so-called *largeness–smoothness method*, close in spirit to Bourgain's original argument and to later multilinear formulations of Cook–Magyar–Pramanik and the heat-flow variants of Durcik and the author [8, 10, 29].

2.1 The largeness–smoothness template

Much of the following is probably incomprehensible until one sees concrete examples, so either proceed with caution, or skip to the next section (but be sure to come back).

For a geometric family $\mathcal{C}_\lambda$, let $\mathcal{N}_\lambda^0$ denote the exact *counting form*. Positivity of $\mathcal{N}_\lambda^0(A)$ means that A contains a copy of $\mathcal{C}_\lambda$. The exact measure (or measures) involved in the definition of the counting form is usually singular: it is a spherical measure, an iterated product of spherical measures, or a measure constrained by several distance equations. One therefore introduces a *smoothed count* $\mathcal{N}_\lambda^\varepsilon(A)$, $0 < \varepsilon \leqslant 1$, and writes

$$\mathcal{N}_\lambda^0 = \mathcal{N}_\lambda^1 + \big(\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1\big) + \big(\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon\big). \tag{2.1}$$

The terms in (2.1) have different meanings.

- The *structured part* $\mathcal{N}_\lambda^1$ is blurred at the same scale as the configuration. At this resolution curvature is no longer important, and density alone forces many configurations (usually via certain combinatorial "counting").
- The *error part* $\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1$ compares two positive smoothing scales. It is seldom small at every scale, but heat-flow cancellation and lacunarity make it small on average over a long list of scales.

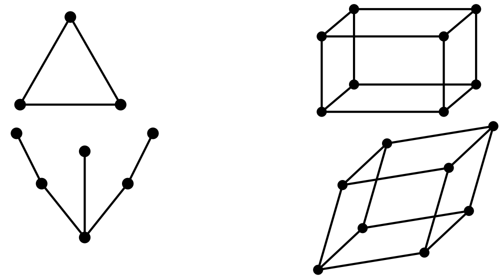

Figure 2.1: Typical configurations in the all-large-scales theory: a simplex, a rectangular box, a distance tree, and a hypercube graph.

- The *uniform part* $\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon$ removes the final smoothing. It is controlled uniformly in λ by Fourier decay, some other oscillatory phenomena, or by a multilinear smoothing inequality.

This decomposition was already implicit in Bourgain's Euclidean Szemerédi theorem and has become a standard organizing principle in later multilinear work [3, 8, 12, 10, 29].

Proposition 2.1 (Abstract all-large-scales criterion). *Let $\lambda_1 < \cdots < \lambda_J$ be lacunary, say $\lambda_{j+1} \geqslant 2\lambda_j$. Suppose that, for every $A_R \subseteq [0,R]^d$ with $|A_R| \geqslant \delta R^d$ and all R sufficiently large compared with λ_J, one has*

$$\mathcal{N}_\lambda^1(A_R) \geqslant c_0(\delta) R^d,$$

$$\sum_{j=1}^{J} \left|\mathcal{N}_{\lambda_j}^\varepsilon(A_R) - \mathcal{N}_{\lambda_j}^1(A_R)\right| \leqslant \eta_\varepsilon(J) J R^d, \qquad \eta_\varepsilon(J) \to 0 \text{ as } J \to \infty,$$

$$\left|\mathcal{N}_\lambda^0(A_R) - \mathcal{N}_\lambda^\varepsilon(A_R)\right| \leqslant u(\varepsilon) R^d, \qquad u(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Then every measurable set of positive upper Banach density contains the corresponding configuration at all sufficiently large scales.

Proof. Fix a measurable set A with $\overline{\delta}(A) > 0$ and choose δ with $0 < \delta < \overline{\delta}(A)$. We prove that the set of bad scales is bounded. Choose $\varepsilon > 0$ so small that

$$u(\varepsilon) \leqslant \frac{1}{4} c_0(\delta),$$

and then choose J so large that

$$\eta_\varepsilon(J) \leqslant \frac{1}{4} c_0(\delta).$$

Assume, toward a contradiction, that there are arbitrarily large bad scales. We may then select bad scales

$$\lambda_1 < \lambda_2 < \cdots < \lambda_J, \qquad \lambda_{j+1} \geqslant 2\lambda_j,$$

with λ_1 as large as needed for the estimates. By the definition of upper Banach density there are arbitrarily large cubes on which A has density at least δ. Choose such a cube $Q = x_0 + [0,R]^d$ with R larger than the threshold in the hypotheses and with $R \gg \lambda_J$. After translating, set

$$A_R = (A - x_0) \cap [0,R]^d, \qquad |A_R| \geqslant \delta R^d.$$

Because each λ_j is bad for A, the localized exact count is zero:

$$\mathcal{N}^0_{\lambda_j}(A_R) = 0, \qquad j = 1, \dots, J.$$

Indeed, any configuration counted inside A_R would translate back to a configuration of scale λ_j inside A.

The averaged error estimate implies that at least one index $j \in \{1, \dots, J\}$ satisfies

$$\left|\mathcal{N}^\varepsilon_{\lambda_j}(A_R) - \mathcal{N}^1_{\lambda_j}(A_R)\right| \leqslant \eta_\varepsilon(J) R^d \leqslant \frac{1}{4} c_0(\delta) R^d.$$

For this index, the decomposition (2.1) gives

$$\begin{aligned}\mathcal{N}^0_{\lambda_j}(A_R) &\geqslant \mathcal{N}^1_{\lambda_j}(A_R) - \left|\mathcal{N}^\varepsilon_{\lambda_j}(A_R) - \mathcal{N}^1_{\lambda_j}(A_R)\right| - \left|\mathcal{N}^0_{\lambda_j}(A_R) - \mathcal{N}^\varepsilon_{\lambda_j}(A_R)\right| \\ &\geqslant c_0(\delta) R^d - \frac{1}{4} c_0(\delta) R^d - \frac{1}{4} c_0(\delta) R^d > 0,\end{aligned}$$

contradicting the badness of λ_j. Thus the bad scales are bounded. Since the argument works for every $0 < \delta < \overline{\delta}(A)$, the corresponding threshold may depend on A, but it is finite. $\square$

2.2 Basic analytic tools: heat flow, curvature, and lacunarity

The Fourier normalization is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x.$$

We use the Gaussian

$$g(x) = e^{-\pi |x|^2},$$

with derivatives $h^{(l)} = \partial_l g$ and Laplacian $k = \Delta g$. Thus

$$\widehat{g}(\xi) = e^{-\pi|\xi|^2}, \qquad \widehat{h^{(l)}}(\xi) = 2\pi i \xi_l e^{-\pi|\xi|^2}, \qquad \widehat{k}(\xi) = -4\pi^2 |\xi|^2 e^{-\pi|\xi|^2}.$$

For $\alpha > 0$ write $f_\alpha(x) = \alpha^{-d} f(x/\alpha)$, so $\widehat{f_\alpha}(\xi) = \widehat{f}(\alpha\xi)$. In this parametrization the heat equation is

$$\frac{\partial}{\partial t} g_t(x) = \frac{1}{2\pi t} k_t(x). \tag{2.2}$$

Consequently a difference between two smoothed counts is an integral in which one Gaussian factor has been replaced by the cancellative factor $k_{t\lambda}$.

Curvature enters through Fourier decay. For normalized circle measure σ in $\mathbb{R}^2$,

$$|\widehat{\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1/2},$$

and therefore

$$|\widehat{\sigma}(\zeta) \widehat{k}(t\zeta)| \lesssim (1 + |\zeta|)^{-1/2} (t|\zeta|)^2 e^{-\pi t^2 |\zeta|^2} \lesssim t^{1/2}, \qquad 0 < t \leqslant 1. \tag{2.3}$$

Higher-dimensional analogues follow from stationary phase for curved hypersurfaces [45]. The factor $t^{1/2}$ in (2.3) is the typical uniform-part gain.

We shall also use the normalized ball average

$$\varphi_r(x) = |\mathrm{B}(0, r)|^{-1} \mathbb{1}_{\mathrm{B}(0,r)}(x).$$

The exact choice of ball may be replaced by a cube or any fixed bounded convex body; only positivity on a definite neighborhood matters.

Finally, lacunarity converts square-function control into a scale average. If $\lambda_{j+1} \geqslant 2\lambda_j$ and $\theta > 0$ is fixed, then for each fixed t the intervals

$$[\theta t\lambda_j, e\theta t\lambda_j]$$

have uniformly bounded overlap. In Fourier variables this is the elementary estimate

$$\sup_{\xi\in\mathbb{R}^d} \sum_{j=1}^{J} (t\lambda_j|\xi|)^4 e^{-2\pi t^2\lambda_j^2|\xi|^2} \lesssim 1, \qquad 0 < t \leqslant 1, \tag{2.4}$$

because the function $u^4 e^{-2\pi u^2}$ is concentrated where $u \simeq 1$ and the numbers $t\lambda_j|\xi|$ form a lacunary sequence. This is the basic Littlewood–Paley mechanism behind the averaged estimates below.

2.3 Two points: Székely's problem

The first benchmark is the two-point configuration.

Theorem 2.2 (Falconer–Marstrand; Furstenberg–Katznelson–Weiss [18, 20]). *Let $A \subseteq \mathbb{R}^d$, $d \geqslant 2$, be measurable with $\overline{\delta}(A) > 0$. Then there is $\lambda_0 = \lambda_0(A) > 0$ such that for every $\lambda \geqslant \lambda_0$ there exist $x, x' \in A$ with*

$$|x - x'| = \lambda.$$

For $d = 2$ this is the theorem of Falconer–Marstrand and Furstenberg–Katznelson–Weiss [18, 20]; the higher-dimensional statement follows by the same spherical-averaging argument.

Let σ be normalized surface measure on S^{d-1}. For $f = \mathbb{1}_{A_R}$ define

$$\begin{aligned}
\mathcal{N}_\lambda^0(f) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)f(x+y)\,\mathrm{d}\sigma_\lambda(y)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)(f * \sigma_\lambda)(x)\,\mathrm{d}x,\\
\mathcal{N}_\lambda^\varepsilon(f) &:= \int_{\mathbb{R}^d} f(x)\big(f * \sigma_\lambda * g_{\varepsilon\lambda}\big)(x)\,\mathrm{d}x.
\end{aligned}$$

Here σ_λ is the pushforward of σ under dilation by λ; this is simply the normalized surface measure on the sphere of radius λ about the origin. Thus, $\mathcal{N}_\lambda^0(f) > 0$ means that the set contains a pair of points separated by exactly λ.

Proposition 2.3 (Distance estimates). *Let $A_R \subseteq [0, R]^d$, $|A_R| \geqslant \delta R^d$, and $R \gg \lambda_J$. For every lacunary list $\lambda_1 < \cdots < \lambda_J$ and every $0 < \varepsilon \leqslant 1$,*

$$\mathcal{N}_\lambda^1(\mathbb{1}_{A_R}) \geqslant c_d\delta^2 R^d, \tag{2.5}$$

$$\sum_{j=1}^{J} |\mathcal{N}_{\lambda_j}^\varepsilon(\mathbb{1}_{A_R}) - \mathcal{N}_{\lambda_j}^1(\mathbb{1}_{A_R})| \leqslant C_d \log(1/\varepsilon) J^{1/2} R^d, \tag{2.6}$$

$$|\mathcal{N}_\lambda^0(\mathbb{1}_{A_R}) - \mathcal{N}_\lambda^\varepsilon(\mathbb{1}_{A_R})| \leqslant C_d \varepsilon^{c_d} R^d. \tag{2.7}$$

Proof. We prove the three estimates in the same order in which they are used.

Structured part. The kernel $\sigma * g$ is continuous and strictly positive on compact sets. Hence, after dilation, there are constants $a, c > 0$ depending only on d such that

$$\sigma_\lambda * g_\lambda(z) \geqslant c\lambda^{-d} \mathbb{1}_{\mathrm{B}(0,a\lambda)}(z).$$

Partition $[0,R]^d$ into half-open cubes Q of side length $a\lambda/(4\sqrt{d})$, discarding only a boundary layer of total measure $O(\lambda R^{d-1})$, which is harmless when $R \gg \lambda$. If $x, y \in Q$, then $|x-y| \leqslant a\lambda/4$. Therefore

$$\mathcal{N}^1_\lambda(\mathbb{1}_{A_R}) \geqslant c\lambda^{-d} \sum_Q |A_R \cap Q|^2.$$

Since $|Q| \simeq \lambda^d$ and the number of cubes is $\simeq (R/\lambda)^d$, Cauchy's inequality gives

$$\sum_Q |A_R \cap Q|^2 \geqslant \frac{(\sum_Q |A_R \cap Q|)^2}{|\{Q\}|} \gtrsim \frac{\delta^2 R^{2d}}{(R/\lambda)^d} = \delta^2 R^d \lambda^d.$$

This proves (2.5).

Uniform part. By Plancherel,

$$\mathcal{N}^0_\lambda(f) - \mathcal{N}^\varepsilon_\lambda(f) = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \widehat{\sigma}(\lambda\xi)\big(1 - \widehat{g}(\varepsilon\lambda\xi)\big)\,\mathrm{d}\xi.$$

Put $u = \lambda|\xi|$ and $\alpha = (d-1)/2$. Stationary phase gives

$$|\widehat{\sigma}(\lambda\xi)| \lesssim (1+u)^{-\alpha},$$

while $|1 - e^{-\pi\varepsilon^2 u^2}| \lesssim \min(1, \varepsilon^2 u^2)$. The product

$$(1+u)^{-\alpha} \min(1, \varepsilon^2 u^2)$$

is bounded by $C_d \varepsilon^{c_d}$ for some $c_d > 0$: split at $u = \varepsilon^{-2/(\alpha+2)}$. Since $\|f\|_2^2 \leqslant R^d$, (2.7) follows.

Error part. The heat equation (2.2) gives

$$\mathcal{N}^\varepsilon_\lambda(f) - \mathcal{N}^1_\lambda(f) = -\frac{1}{2\pi} \int_\varepsilon^1 \langle f, f * \sigma_\lambda * k_{t\lambda} \rangle \,\frac{\mathrm{d}t}{t}.$$

Fix $t \in [\varepsilon, 1]$ and set

$$T_{j,t} f = f * \sigma_{\lambda_j} * k_{t\lambda_j}.$$

Cauchy–Schwarz in the scale index gives

$$\sum_{j=1}^J |\langle f, T_{j,t} f \rangle| \leqslant J^{1/2} \|f\|_2 \Big(\sum_{j=1}^J \|T_{j,t} f\|_2^2 \Big)^{1/2}.$$

By Plancherel and $|\widehat{\sigma}| \leqslant 1$,

$$\sum_{j=1}^J \|T_{j,t} f\|_2^2 = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \sum_{j=1}^J |\widehat{\sigma}(\lambda_j \xi)|^2 |\widehat{k}(t\lambda_j \xi)|^2 \,\mathrm{d}\xi$$

$$\lesssim \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi = \|f\|_2^2 \, ,$$

where the multiplier sum is bounded by (2.4). Hence

$$\sum_{j=1}^{J} |\, \langle f, T_{j,t} f \rangle \,| \lesssim J^{1/2} \, \|f\|_2^2 \leqslant J^{1/2} R^d.$$

Integrating $t^{-1}\,\mathrm{d}t$ from ε to 1 proves (2.6). □

Proof of Theorem 2.2. Apply Proposition 2.1 with $\mathcal{N}_\lambda^\varepsilon = \mathcal{N}_\lambda^\varepsilon$. The structured constant is $c_0(\delta) = c_d \delta^2$, the uniform function is $u(\varepsilon) = C_d \varepsilon^{c_d}$, and the error estimate gives

$$\eta_\varepsilon(J) = C_d \log(1/\varepsilon) J^{-1/2}.$$

For fixed ε, this tends to 0 as $J \to \infty$. Therefore the abstract criterion gives all sufficiently large distances. Positivity of the exact count produces actual points because the integrand is nonnegative and the measure $\mathrm{d}x \; \mathrm{d}\sigma_\lambda(y)$ is supported precisely on pairs $(x, x+y)$ with $|y| = \lambda$. □

This is not the shortest possible proof of the density theorem for two points, but it fits well to our more general philosophy.

2.4 Rectangular boxes

Let $q \geqslant 1$ and

$$\square = \{0, b_1\} \times \cdots \times \{0, b_q\} \subseteq \mathbb{R}^q, \qquad b_1, \ldots, b_q > 0.$$

The ambient space is $(\mathbb{R}^2)^q$. The kth side of the box is allowed to rotate only in the kth distinguished copy of $\mathbb{R}^2$.

Theorem 2.4 (Lyall–Magyar, rectangular-box case [38, 40]). *Let $A \subseteq (\mathbb{R}^2)^q$ be measurable with $\overline{\delta}(A) > 0$. Then there is $\lambda_0 = \lambda_0(A, \square) > 0$ such that, for every $\lambda \geqslant \lambda_0$, there exist $x_1, \ldots, x_q, y_1, \ldots, y_q \in \mathbb{R}^2$ with*

$$|y_k| = \lambda b_k, \qquad k = 1, \ldots, q,$$

and

$$\{(x_1 + r_1 y_1, \ldots, x_q + r_q y_q) : (r_1, \ldots, r_q) \in \{0,1\}^q\} \subseteq A.$$

This is a particular case of Lyall–Magyar's product and hypergraph results [38, 40]; closely related heat-flow proofs for box configurations appear in work of Durcik and the author [10, 11].

Counting forms

For $\mathbf{x} = (x_1^0, x_1^1, \ldots, x_q^0, x_q^1) \in (\mathbb{R}^2)^{2q}$ write

$$\mathcal{F}(\mathbf{x}) := \prod_{r \in \{0,1\}^q} f(x_1^{r_1}, \ldots, x_q^{r_q}).$$

The exact and smoothed forms are

$$\mathcal{N}_\lambda^0(f) := \int_{(\mathbb{R}^2)^{2q}} \mathcal{F}(\mathbf{x}) \prod_{k=1}^{q} \mathrm{d}\sigma_{\lambda b_k}(x_k^1 - x_k^0) \prod_{k=1}^{q} \mathrm{d}x_k^0,$$

$$\mathcal{N}_\lambda^\varepsilon(f) := \int_{(\mathbb{R}^2)^{2q}} \mathcal{F}(\mathbf{x}) \prod_{k=1}^{q} (\sigma * g_\varepsilon)_{\lambda b_k}(x_k^0 - x_k^1)\, \mathrm{d}\mathbf{x}.$$

The variables x_k^0 and x_k^1 represent the two endpoints of the kth side. Choosing all 2^q combinations of endpoints gives all vertices of the box.

Proposition 2.5 (Box estimates). *For $A_R \subseteq ([0,R]^2)^q$ with $|A_R| \geqslant \delta R^{2q}$ and $R \gg \lambda_J$,*

$$\mathcal{N}_\lambda^1(\mathbb{1}_{A_R}) \geqslant c_\square \delta^{2^q} R^{2q}, \tag{2.8}$$

$$\sum_{j=1}^{J} |\mathcal{N}_{\lambda_j}^\varepsilon(\mathbb{1}_{A_R}) - \mathcal{N}_{\lambda_j}^1(\mathbb{1}_{A_R})| \leqslant C_{\square,q} \varepsilon^{-C_q} \log(1/\varepsilon) R^{2q}, \tag{2.9}$$

$$|\mathcal{N}_\lambda^0(\mathbb{1}_{A_R}) - \mathcal{N}_\lambda^\varepsilon(\mathbb{1}_{A_R})| \leqslant C_{\square,q} \varepsilon^{c_q} R^{2q}. \tag{2.10}$$

Proof. Again write $f = \mathbb{1}_{A_R}$.

Structured part. Partition the kth coordinate plane into squares Q_k of side length comparable to λb_k. In the smoothest form, restrict x_k^0 and x_k^1 to lie in the same square Q_k. Since $(\sigma * g)_{\lambda b_k}$ is bounded below by $c(\lambda b_k)^{-2}$ on a definite ball around the origin, this restriction gives a positive contribution:

$$\mathcal{N}_\lambda^1(f) \gtrsim_\square \sum_{Q_1,\dots,Q_q} |Q_1| \cdots |Q_q| \fint_{Q_1 \times Q_1 \times \cdots \times Q_q \times Q_q} \mathcal{F}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

For each product cell, the box Cauchy–Schwarz inequality gives

$$\fint_{Q_1 \times Q_1 \times \cdots \times Q_q \times Q_q} \prod_{r \in \{0,1\}^q} f(x_1^{r_1}, \dots, x_q^{r_q})\, \mathrm{d}\mathbf{x} \geqslant \left(\fint_{Q_1 \times \cdots \times Q_q} f \right)^{2^q}. \tag{2.11}$$

For $q = 1$ this is Jensen. For the induction step, write the left side as an average over the last pair (x_q^0, x_q^1) of a product of two $(q-1)$-dimensional box forms. Cauchy–Schwarz in the variables indexed by the first $q-1$ coordinates bounds this average below by the square of the corresponding $(q-1)$-dimensional average, and the induction hypothesis applies. This is the continuum form of the Gowers–Cauchy–Schwarz inequality proved in Proposition A.5.

Summing (2.11) over all product cells gives

$$\mathcal{N}_\lambda^1(f) \gtrsim_\square \sum_{Q_1,\dots,Q_q} |Q_1| \cdots |Q_q| \alpha_{Q_1,\dots,Q_q}^{2^q},$$

where $\alpha_{Q_1,\dots,Q_q}$ is the density of A_R in $Q_1 \times \cdots \times Q_q$. Jensen's inequality over the product partition gives

$$\sum |Q_1| \cdots |Q_q| \alpha_{Q_1,\dots,Q_q}^{2^q} \geqslant R^{2q} \left(\frac{|A_R|}{R^{2q}} \right)^{2^q},$$

which is (2.8).

Error part. By differentiating the product of the q smoothed circular factors, the contribution of the derivative falling on the mth edge is

$$(\sigma * k_t)_{\lambda b_m}(x_m^0 - x_m^1) \prod_{k \neq m} (\sigma * g_t)_{\lambda b_k}(x_k^0 - x_k^1).$$

It suffices to treat $m = q$. Insert the average $1 = \int_{\theta t\lambda}^{e\theta t\lambda} \mathrm{d}s/s$. For $s \simeq t\lambda$, the Fourier identity for the Laplacian gives

$$(\sigma_{\lambda b_q} * k_{t\lambda b_q}) = \sum_{l=1}^{2} \sigma_{\lambda b_q} * g_r * h_s^{(l)} * h_s^{(l)}, \qquad r \simeq s \simeq t\lambda, \tag{2.12}$$

up to constants depending on $\square$. The other smoothed circle kernels and the factor $\sigma_{\lambda b_q} * g_r$ are dominated, for $t \geqslant \varepsilon$, by superpositions of Gaussian kernels at scale $s\gamma$, $\gamma \geqslant 1$, with total mass $O_\square(\varepsilon^{-C_q})$.

Expand the two derivatives in (2.12). The variables with $r_q = 0$ form one face of the box and the variables with $r_q = 1$ form the opposite face. Applying Cauchy–Schwarz between these two faces converts the absolute value of the derivative term into a positive cubical heat-flow form. More explicitly, one obtains the bound

$$\sum_{j=1}^{J} |\mathcal{L}_{\lambda_j}^q(f)| \lesssim_{\square,q} \varepsilon^{-C_q} \log(1/\varepsilon) \int_{[1,\infty)^{q-1}} \Theta^{q,q}_{\gamma_1,\dots,\gamma_{q-1},\sqrt{2}}(f) \prod_{k=1}^{q-1} \frac{\mathrm{d}\gamma_k}{\gamma_k^2}, \tag{2.13}$$

where

$$\Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f) := -\int_0^\infty \int \prod_{r\in\{0,1\}^q} f(x_1^{r_1},\dots,x_q^{r_q})\, k_{s\gamma_m}(x_m^0 - x_m^1) \times \prod_{k\neq m} g_{s\gamma_k}(x_k^0 - x_k^1)\,\mathrm{d}\mathbf{x}\frac{\mathrm{d}s}{s}.$$

The minus sign is chosen because $k = \Delta g$ and the corresponding quadratic form is nonnegative. Indeed, writing the two m-faces as a function F_m of the common mth coordinate, the identity

$$k_{s\gamma_m} = 2\sum_{l=1}^{2} h^{(l)}_{s\gamma_m/\sqrt{2}} * h^{(l)}_{s\gamma_m/\sqrt{2}}$$

gives

$$\Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f) = 2\sum_{l=1}^{2} \int_0^\infty \int \left\| F_m * h^{(l)}_{s\gamma_m/\sqrt{2}} \right\|_2^2 \prod_{k\neq m} g_{s\gamma_k}(x_k^0 - x_k^1)\frac{\mathrm{d}s}{s} \geqslant 0.$$

Summing these nonnegative forms over m telescopes by the heat equation:

$$\sum_{m=1}^{q} \Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f) \lesssim \|f\|_{2^q}^{2^q}.$$

To see the telescoping, differentiate the fully Gaussian cubical average

$$\int \prod_{r\in\{0,1\}^q} f(x_1^{r_1},\dots,x_q^{r_q}) \prod_{k=1}^{q} g_{s\gamma_k}(x_k^0 - x_k^1)\,\mathrm{d}\mathbf{x}$$

with respect to s and integrate from $s = 0$ to $s = \infty$. At $s = \infty$ the normalized Gaussians vanish weakly on compactly supported data, while at $s = 0$ all paired coordinates coalesce and the limit is $\int f^{2^q} \leqslant R^{2q}$. Since each Θ is nonnegative, (2.13) is bounded by $C\varepsilon^{-C_q}\log(1/\varepsilon)R^{2q}$. Summing over the possible derivative positions proves (2.9). This is the cubical singular-integral estimate in its soft heat-flow form; related estimates appear in [9, 14].

Uniform part. Let the heat derivative fall on the mth edge and keep all other endpoint variables fixed. The product of vertex factors splits into two faces,

$$F_0(u) = \prod_{r_m=0} f(\ldots, u, \ldots), \qquad F_1(v) = \prod_{r_m=1} f(\ldots, v, \ldots).$$

The relevant bilinear form in $u, v \in \mathbb{R}^2$ is

$$\int F_0(u)F_1(v)(\sigma_{\lambda b_m} * k_{t\lambda b_m})(u-v)\,\mathrm{d}u\,\mathrm{d}v.$$

By Plancherel it is equal to

$$\int \widehat{F}_0(\xi)\overline{\widehat{F}_1(\xi)}\widehat{\sigma}(\lambda b_m \xi)\widehat{k}(t\lambda b_m \xi)\,\mathrm{d}\xi.$$

The multiplier is $O(t^{1/2})$ by (2.3). Hence the bilinear form is bounded by $Ct^{1/2}\,\|F_0\|_2\,\|F_1\|_2$. After integrating the remaining variables against probability kernels and using $0 \leqslant f \leqslant 1$ and $\operatorname{supp} f \subseteq ([0,R]^2)^q$, this gives

$$|\mathcal{L}^m_{\lambda,t}(f)| \lesssim t^{1/2}R^{2q}.$$

Integrating $0 < t < \varepsilon$ and summing over m proves (2.10). □

Proof of Theorem 2.4. Apply Proposition 2.1 to $\mathcal{N}^\varepsilon_\lambda = \mathcal{N}^\varepsilon_\lambda$. The structured constant is $c_\square \delta^{2^q}$, the uniform function is $C_{\square,q}\varepsilon^{c_q}$, and the error estimate gives

$$\eta_\varepsilon(J) = C_{\square,q}\varepsilon^{-C_q}\log(1/\varepsilon)J^{-1}.$$

The stronger J^{-1} gain comes from the positive cubical telescoping in the error estimate. The abstract contradiction argument therefore gives all sufficiently large scales. Positivity of $\mathcal{N}^0_\lambda(\mathbb{1}_A)$ produces the required box because each singular factor is supported on the circle $|x^0_k - x^1_k| = \lambda b_k$. □

With no additional effort one can prove the following compact counterpart to the last (cubical all-large-scale) theorem.

Theorem 2.6 (Quantified cube interval [10, 11]). *For every positive integer q there exists $C'(q) < \infty$ such that, whenever $A \subseteq ([0,1]^2)^q$ is measurable with $|A| \geqslant \delta$, $0 < \delta \leqslant 1/2$, there is an interval $I \subseteq (0,1]$ of length at least*

$$\exp(-\delta^{-C'(q)})$$

such that for every $\lambda \in I$ one can find $x_i, y_i \in \mathbb{R}^2$, $1 \leqslant i \leqslant q$, satisfying

$$(x_1 + r_1 y_1, \ldots, x_q + r_q y_q) \in A$$

for all $(r_1, \ldots, r_q) \in \{0,1\}^q$, with

$$\|y_i\|_2 = \lambda \quad \textit{for all } i.$$

Namely, one considers the intervals $(2^{-j}, 2^{-j+1}]$, $1 \leqslant j \leqslant J$, for a sufficiently large number of scales J. If every one of them contained a scale λ_j at which no configuration is found, then we would obtain a contradiction just as in the previous proof. Thus, in one of these intervals, every scale is realized by a cube in A. Its length is at least $2^{-J} \geqslant \exp(-\delta^{-C'(q)})$ and we are done.

We omit the details, since the whole next chapter will be concerned with a much more involved compact density theorem.

2.5 Bourgain's simplex theorem

Let $d \geqslant 2$, put $q = d-1$, and let

$$\Delta = \{0, u_1, \dots, u_q\} \subseteq \mathbb{R}^q$$

be a non-degenerate q-simplex, meaning that $u_1, \dots, u_q$ are linearly independent. We view Δ inside $\mathbb{R}^d$ by adding one zero coordinate. The increase from the affine dimension q to the ambient dimension $d = q+1$ supplies one extra degree of rotational freedom.

Theorem 2.7 (Bourgain [3]). *Let $A \subseteq \mathbb{R}^d$ be measurable with $\overline{\delta}(A) > 0$. Then there is $\lambda_0 = \lambda_0(A, \Delta) > 0$ such that, for every $\lambda \geqslant \lambda_0$, the set A contains an isometric copy of $\lambda\Delta$.*

Bourgain proved this in [3]. Later papers gave related direct or product-type proofs and extensions [38, 27, 39, 40].

Counting forms

Let μ be Haar probability measure on $\mathrm{SO}(d)$. For compactly supported $0 \leqslant f \leqslant 1$ set

$$\mathcal{N}^0_\lambda(f) := \int_{\mathbb{R}^d} \int_{\mathrm{SO}(d)} f(x) \prod_{k=1}^{q} f(x + \lambda U u_k)\, \mathrm{d}\mu(U)\, \mathrm{d}x,$$
$$\mathcal{N}^\varepsilon_\lambda(f) := \int_{\mathbb{R}^d} \int_{\mathrm{SO}(d)} f(x) \prod_{k=1}^{q} (f * g_{\varepsilon\lambda})(x + \lambda U u_k)\, \mathrm{d}\mu(U)\, \mathrm{d}x.$$

The exact count is positive precisely when f sees a translated and rotated copy of the simplex at scale λ.

The rotation integral may be written as an iterated integration over spheres. Choose $y_1 = \lambda U u_1$ first; it lies on a sphere. Once $y_1, \dots, y_{k-1}$ have been chosen, the vector $y_k = \lambda U u_k$ is constrained to a sphere inside the affine plane determined by the prescribed inner products with the previous vectors. The dimension of this sphere decreases by one at each step. The last step is a circle, and it is precisely the decay of this remaining circle that drives the uniform estimate.

Proposition 2.8 (Simplex estimates). *For $A_R \subseteq [0,R]^d$ with $|A_R| \geqslant \delta R^d$ and $R \gg \lambda_J$, one has*

$$\mathcal{N}^1_\lambda(\mathbb{1}_{A_R}) \geqslant c_\Delta \delta^d R^d, \tag{2.14}$$

$$\sum_{j=1}^{J} |\mathcal{N}^\varepsilon_{\lambda_j}(\mathbb{1}_{A_R}) - \mathcal{N}^1_{\lambda_j}(\mathbb{1}_{A_R})| \leqslant C_{\Delta,q} \varepsilon^{-C_q} \log(1/\varepsilon) J^{1/2} R^d, \tag{2.15}$$

$$|\mathcal{N}^0_\lambda(\mathbb{1}_{A_R}) - \mathcal{N}^\varepsilon_\lambda(\mathbb{1}_{A_R})| \leqslant C_{\Delta,q} \varepsilon^{c_q} R^d. \tag{2.16}$$

Proof. Write $f = \mathbb{1}_{A_R}$; the ambient dimension is $d = q+1$.

Structured part. Since g is strictly positive, for each fixed vertex vector u_k there are constants $c_k, a_k > 0$ depending only on Δ such that, uniformly in $U \in \mathrm{SO}(d)$,

$$(f * g_\lambda)(x + \lambda U u_k) \geqslant c_k (f * \varphi_{a_k \lambda})(x).$$

Indeed, if $|z - x| \leqslant a_k \lambda$ with a_k small, then

$$|x + \lambda U u_k - z| \leqslant (|u_k| + a_k)\lambda,$$

so the Gaussian kernel $g_\lambda(x+\lambda U u_k - z)$ is bounded below by a constant multiple of λ^{-d} on the ball $\mathrm{B}(x, a_k\lambda)$. Therefore

$$\mathcal{N}_\lambda^1(f) \gtrsim_\Delta \int f(x) \prod_{k=1}^q (f * \varphi_{a_k\lambda})(x)\,\mathrm{d}x.$$

Choose dyadic partitions of $[0,R]^d$ so that each cube of the kth partition has diameter at most $a_k\lambda/10$. Denote the associated conditional expectation by $E_k f$. Away from a boundary layer of size $O(\lambda R^{d-1})$, the average $f * \varphi_{a_k\lambda}$ dominates $E_k f$ up to a constant. Thus

$$\mathcal{N}_\lambda^1(f) \gtrsim_\Delta \int f \prod_{k=1}^q E_k f.$$

After relabeling, the sigma-algebras defining the E_k are nested. Let them be $\mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_q$, and write $F_k = \mathbb{E}(f \mid \mathcal{G}_k)$. Then

$$\int f \prod_{k=1}^q F_k = \int F_q^2 \prod_{k=1}^{q-1} F_k.$$

Conditioning on $\mathcal{G}_{q-1}$ and using Jensen gives

$$\mathbb{E}(F_q^2 \mid \mathcal{G}_{q-1}) \geqslant F_{q-1}^2.$$

Consequently

$$\int f \prod_{k=1}^q F_k \geqslant \int F_{q-1}^3 \prod_{k=1}^{q-2} F_k.$$

Repeating this argument moves successively to coarser sigma-algebras and increases the power at each step. At the end one obtains

$$\int f \prod_{k=1}^q F_k \geqslant (\mathbb{E}f)^{q+1} R^d.$$

Since $\mathbb{E}f = |A_R|/R^d \geqslant \delta$, this proves (2.14).

Error part. By the heat equation, for $0<\alpha<\beta\leqslant 1$,

$$\mathcal{N}_\lambda^\alpha(f) - \mathcal{N}_\lambda^\beta(f) = -\sum_{m=1}^q \int_\alpha^\beta \mathcal{L}_{\lambda,t}^m(f)\frac{\mathrm{d}t}{t},$$

where $\mathcal{L}_{\lambda,t}^m$ is obtained from $\mathcal{N}_\lambda^t$ by replacing the mth smoothed factor $(f * g_{t\lambda})(x+\lambda U u_m)$ by $(f * k_{t\lambda})(x+\lambda U u_m)$ and leaving the other smoothed factors unchanged.

We prove the required averaged bound for one value of m; summing over m only changes the constant. In the iterated-sphere parametrization, choose the order so that the mth vertex is selected last. The last integration is over a circle in a two-dimensional orthogonal plane. Insert the harmless identity

$$1 = \int_{\theta t\lambda_j}^{e\theta t\lambda_j} \frac{\mathrm{d}s}{s}$$

with a fixed small $\theta > 0$. For $s \simeq t\lambda_j$, the Gaussian identities imply a decomposition of the cancellative kernel of the form

$$k_{t\lambda_j} = \sum_{l=1}^{d} h_s^{(l)} * h_s^{(l)} * g_{r(j,t,s)}, \qquad r(j,t,s) \simeq s, \tag{2.17}$$

up to constants depending only on θ. This follows directly on the Fourier side from

$$|\xi|^2 e^{-\pi(t\lambda_j)^2|\xi|^2} = \sum_{l=1}^{d} \xi_l^2 e^{-2\pi s^2|\xi|^2} e^{-\pi r^2|\xi|^2}$$

with $r^2 = (t\lambda_j)^2 - 2s^2 > 0$.

Using (2.17), one derivative $h_s^{(l)}$ is placed on f and the other derivative is absorbed into the final circular average. The remaining spherical and Gaussian factors are positive probability kernels with rapidly decreasing tails. They are dominated by a finite superposition of ball or Gaussian averages at scale comparable to s, and hence by Hardy–Littlewood maximal functions. Applying Cauchy–Schwarz in the base variable and in the already chosen directions gives the estimate

$$|\mathcal{L}^m_{\lambda_j,t}(f)| \lesssim_{\Delta,q} \varepsilon^{-C_q} \sum_{l=1}^{d} \left(\int_{\theta t\lambda_j}^{e\theta t\lambda_j} \left\| f * h_s^{(l)} \right\|_2^2 \frac{\mathrm{d}s}{s} \right)^{1/2} R^{d/2}. \tag{2.18}$$

Here the factor ε^{-C_q} comes from comparing all smoothing parameters between $t\lambda_j$ and s while $t \geqslant \varepsilon$; the maximal-function factors are bounded in L^{2q} and use $0 \leqslant f \leqslant 1$ and $\operatorname{supp} f \subseteq [0,R]^d$.

Sum (2.18) over j and apply Cauchy–Schwarz in j. For fixed t, lacunarity gives bounded overlap of the intervals $[\theta t\lambda_j, e\theta t\lambda_j]$, and therefore

$$\sum_{j=1}^{J} |\mathcal{L}^m_{\lambda_j,t}(f)| \lesssim_{\Delta,q} \varepsilon^{-C_q} J^{1/2} R^{d/2} \left(\int_0^\infty \sum_{l=1}^{d} \left\| f * h_s^{(l)} \right\|_2^2 \frac{\mathrm{d}s}{s} \right)^{1/2}.$$

Plancherel's identity gives

$$\int_0^\infty \sum_{l=1}^{d} \left\| f * h_s^{(l)} \right\|_2^2 \frac{\mathrm{d}s}{s} \lesssim_d \|f\|_2^2 \leqslant R^d.$$

Thus $\sum_j |\mathcal{L}^m_{\lambda_j,t}(f)| \lesssim \varepsilon^{-C_q} J^{1/2} R^d$. Integrating $t^{-1}\,\mathrm{d}t$ over $[\varepsilon, 1]$ and summing over m proves (2.15).

Uniform part. It suffices to control the heat-flow tail between smoothing parameters 0 and ε. Fix the derivative on the final vertex in the iterated-sphere description. After the first $q-1$ simplex directions have been chosen, the last direction ranges over a circle in a two-dimensional plane E. The relevant piece of the heat derivative has the schematic form

$$\int f_0(x)\,(f * k_{t\lambda} * \sigma_{\lambda,E})(x)\,\mathrm{d}x,$$

where $\sigma_{\lambda,E}$ is circle measure in the plane E and $0 \leqslant f_0 \leqslant 1$ is a product of the already selected vertex factors. Cauchy–Schwarz in all outer variables and Plancherel in x reduce the square of this expression to an integral bounded by

$$R^d \int |\widehat{f}(\xi)|^2 |\widehat{k}(t\lambda\xi)|^2 \mathcal{I}(\lambda\xi)\,\mathrm{d}\xi.$$

Here $\mathcal{I}(\zeta)$ is the average, over the possible first $q-1$ directions, of the reciprocal distance from ζ to their span. This factor is exactly what remains from the stationary-phase estimate for the last circle: the Fourier transform of a circle in the plane E decays like the inverse square root of the length of the projection of ζ onto E.

The averaging over rotations removes the singularity. By rotational invariance, fix the span of the first $q-1$ directions to be the coordinate $(q-1)$-plane and average the direction of ζ over S^{d-1}. In spherical coordinates the singular factor is

$$\frac{1}{|\zeta| \sin \phi_1 \cdots \sin \phi_{q-1}},$$

whereas the surface element contributes

$$\sin^{q-1} \phi_1 \sin^{q-2} \phi_2 \cdots \sin \phi_{q-1} \, \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_q.$$

The quotient is integrable, hence

$$\mathcal{I}(\zeta) \lesssim_\Delta |\zeta|^{-1}. \tag{2.19}$$

Combining (2.19) with the formula for $\widehat{k}$ gives

$$|\widehat{k}(t\zeta)|^2 \mathcal{I}(\zeta) \lesssim (t|\zeta|)^4 e^{-2\pi t^2 |\zeta|^2} |\zeta|^{-1} \lesssim t,$$

because $u^3 e^{-2\pi u^2}$ is bounded. Thus the derivative at scale t is $O(t^{1/2} R^d)$. Integrating from 0 to ε gives

$$|\mathcal{N}_\lambda^0(f) - \mathcal{N}_\lambda^\varepsilon(f)| \lesssim R^d \int_0^\varepsilon t^{1/2} \frac{\mathrm{d}t}{t} \lesssim \varepsilon^{1/2} R^d,$$

which proves (2.16). □

Proof of Theorem 2.7. Use Proposition 2.1 with $\mathcal{N}_\lambda^\varepsilon = \mathcal{N}_\lambda^\varepsilon$ and the estimates in Proposition 2.8. The structured constant is $c_\Delta \delta^d$, the uniform function is $C_{\Delta,q}\varepsilon^{c_q}$, and the averaged error has

$$\eta_\varepsilon(J) = C_{\Delta,q} \varepsilon^{-C_q} \log(1/\varepsilon) J^{-1/2}.$$

After choosing ε and then J as in the abstract criterion, a bad lacunary list of scales would force a positive exact count at one of those scales, which is impossible. Positivity of $\mathcal{N}_\lambda^0(\mathbb{1}_A)$ gives an actual congruent copy of $\lambda\Delta$ because Haar measure is supported on rotations preserving all distances among the vertices of Δ. □

2.6 The hypercube distance graph

There will be no time to cover this section in detail in class, so the reader can read it primarily as a useful practice of the previously seen techniques. We still include the full proofs for reader's convenience.

Let Γ_q be the graph isomorphic to the 1-skeleton of the q-dimensional unit hypercube. A planar set A contains an isometric copy of $\lambda\Gamma_q$ if there are $x \in \mathbb{R}^2$ and vectors $y_1, \ldots, y_q \in \mathbb{R}^2$ with $|y_i| = \lambda$ such that

$$x + r_1 y_1 + \cdots + r_q y_q \in A$$

for all $(r_1, \ldots, r_q) \in \{0,1\}^q$, and the 2^q points are distinct.

Theorem 2.9 (All large dilates of Γ_q [31]). *For every positive integer q and every measurable $A \subseteq \mathbb{R}^2$ of positive upper Banach density, there exists $\lambda_0(A,q) > 0$ such that for every $\lambda \geqslant \lambda_0(A,q)$ the set A contains an isometric copy of $\lambda\Gamma_q$.*

Theorem 2.10 (Compact interval version [31]). *For every q there is $C(q) > 0$ such that if $A \subseteq [0,1]^2$ is measurable with $|A| \geqslant \delta$, $0 < \delta \leqslant 1/2$, then there is an interval $I \subseteq (0,1]$ of length at least*

$$\exp(-\delta^{-C(q)})$$

such that A contains an isometric copy of $\lambda\Gamma_q$ for every $\lambda \in I$.

These are dimensionally sharp: the graph is embedded in the plane even though its degeneracy is q.

Let $f = \mathbb{1}_{A_R}$ and define

$$\mathcal{F}_q(x; y_1, \dots, y_q) = \prod_{\omega \in \{0,1\}^q} f(x + \omega_1 y_1 + \dots + \omega_q y_q).$$

With σ_λ circle measure of radius λ,

$$\mathcal{N}^0_\lambda(f) = \int_{\mathbb{R}^2} \int_{(\mathbb{R}^2)^q} \mathcal{F}_q(x; y_1, \dots, y_q)\, \mathrm{d}\sigma_\lambda^{\otimes q}(y)\, \mathrm{d}x.$$

The smoothed form replaces each σ_λ by $\sigma_\lambda * g_{\varepsilon\lambda}$. Positivity of $\mathcal{N}^0_\lambda(\mathbb{1}_A)$ implies a nondegenerate copy because degenerate coincidences are described by finitely many linear relations among $y_1, \dots, y_q$ and therefore have zero $\sigma_\lambda^{\otimes q}$-measure.

Proposition 2.11 (Hypercube graph structured estimate [31]). *For $R \geqslant \lambda > 0$ and $A_R \subseteq [0,R]^2$,*

$$\mathcal{N}^1_\lambda(\mathbb{1}_{A_R}) \geqslant c_{\mathrm{str}} \left(\frac{|A_R|}{R^2} \right)^{2^q} R^2.$$

Proof. The smoothed circle kernel has a fixed positive lower bound on a square of side comparable to λ:

$$\sigma_\lambda * g_\lambda(y) \gtrsim_q \lambda^{-2} \mathbb{1}_{[-c\lambda, c\lambda]^2}(y).$$

Thus

$$\mathcal{N}^1_\lambda(\mathbb{1}_{A_R}) \gtrsim_q \lambda^{-2q} \int \mathcal{F}_q(x; y_1, \dots, y_q) \prod_{k=1}^q \mathbb{1}_{[-c\lambda, c\lambda]^2}(y_k)\, \mathrm{d}y\, \mathrm{d}x.$$

Partition $[0,R]^2$ into squares Q of side length comparable to λ. We restrict the last integral further by asking that all vertices $x + \omega \cdot y$ lie in the same square Q. Under this restriction the bounds $y_k \in [-c\lambda, c\lambda]^2$ are automatic after changing constants. Therefore

$$\mathcal{N}^1_\lambda(\mathbb{1}_{A_R}) \gtrsim_q \lambda^{-2q} \sum_Q \int_{(\mathbb{R}^2)^{q+1}} \prod_{\omega \in \{0,1\}^q} \mathbb{1}_{A_R \cap Q}(x + \omega \cdot y)\, \mathrm{d}x\, \mathrm{d}y.$$

The integral is the 2^qth power of the continuum Gowers U^q norm of $\mathbb{1}_{A_R \cap Q}$. The Gowers–Cauchy–Schwarz inequality, recorded as Proposition A.5, implies the lower bound

$$\int \prod_\omega \mathbb{1}_{A_R \cap Q}(x + \omega \cdot y)\, \mathrm{d}x\, \mathrm{d}y \geqslant |Q|^{q+1} \left(\frac{|A_R \cap Q|}{|Q|} \right)^{2^q}.$$

Since $|Q| \simeq \lambda^2$, multiplication by λ^{-2q} leaves a contribution comparable to

$$|Q| \left(\frac{|A_R \cap Q|}{|Q|} \right)^{2^q}.$$

Summing over Q and applying Jensen gives

$$\mathcal{N}^1_\lambda(\mathbb{1}_{A_R}) \gtrsim_q R^2 \left(\frac{|A_R|}{R^2} \right)^{2^q},$$

as claimed. $\square$

Proposition 2.12 (Hypercube graph error estimate [31]). *For $0 < \lambda_1 < \cdots < \lambda_J$ with $\lambda_{j+1} \geqslant 2\lambda_j$, $R \geqslant 2\lambda_J$, and $\varepsilon \in (0,1]$,*

$$\sum_{j=1}^{J} \left| \mathcal{N}^\varepsilon_{\lambda_j}(\mathbb{1}_{A_R}) - \mathcal{N}^1_{\lambda_j}(\mathbb{1}_{A_R}) \right| \leqslant C_{\mathrm{err}} \varepsilon^{-3q} \log(1/\varepsilon) R^2.$$

Proof. Let $f = \mathbb{1}_{A_R}$. For $0 < \alpha < \beta \leqslant 1$, the product rule and the heat equation give

$$\mathcal{N}^\alpha_\lambda(f) - \mathcal{N}^\beta_\lambda(f) = \sum_{m=1}^{q} \mathcal{L}^{\alpha,\beta,m}_\lambda(f),$$

where

$$\begin{aligned} \mathcal{L}^{\alpha,\beta,m}_\lambda(f) = -\frac{1}{2\pi} \int_\alpha^\beta \int \mathcal{F}_q(x; y_1, \ldots, y_q)(\sigma_\lambda * k_{t\lambda})(y_m) \\ \times \prod_{k \neq m} (\sigma_\lambda * g_{t\lambda})(y_k) \, \mathrm{d}x \, \mathrm{d}y_1 \cdots \mathrm{d}y_q \frac{\mathrm{d}t}{t}. \end{aligned}$$

It is enough to consider $m = q$.

Insert $1 = \int_{\theta t\lambda_j}^{e\theta t\lambda_j} \mathrm{d}s/s$ and use the factorization

$$k_{t\lambda_j} = \frac{(t\lambda_j)^2}{s^2} \sum_{l=1}^{2} h^{(l)}_s * h^{(l)}_s * g_r, \qquad r^2 = (t\lambda_j)^2 - 2s^2 \simeq s^2.$$

For $t \geqslant \varepsilon$ the kernels $\sigma_{\lambda_j} * g_{t\lambda_j}$ and $\sigma_{\lambda_j} * g_r$ are dominated by Gaussian averages at scale $s\gamma$, $\gamma \geqslant 1$, with total cost $O(\varepsilon^{-3})$ for each edge. After expanding the two derivatives $h^{(l)}_s$, the two opposite $(q-1)$-faces of the hypercube appear:

$$\mathcal{F}_{q-1}(x; y_1, \ldots, y_{q-1}) \quad \text{and} \quad \mathcal{F}_{q-1}(x + y_q; y_1, \ldots, y_{q-1}).$$

Cauchy–Schwarz between these faces removes the absolute value and produces a positive heat-flow form. Using the bounded overlap of the intervals $[\theta t\lambda_j, e\theta t\lambda_j]$ as j varies, we obtain

$$\sum_{j=1}^{J} |\mathcal{L}^{\varepsilon,1,q}_{\lambda_j}(f)| \lesssim_q \varepsilon^{-3q} \log(1/\varepsilon) \int_{[1,\infty)^{q-1}} \Theta^{q,q}_{\gamma_1,\ldots,\gamma_{q-1},\sqrt{2}}(f) \prod_{k=1}^{q-1} \frac{\mathrm{d}\gamma_k}{\gamma_k^2}. \tag{2.20}$$

Here

$$\Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f) := -\int_0^\infty \int_{(\mathbb{R}^2)^{q+1}} \mathcal{F}_q(x;y_1,\dots,y_q) k_{s\gamma_m}(y_m) \times \prod_{k\neq m} g_{s\gamma_k}(y_k)\,\mathrm{d}x\,\mathrm{d}y_1\cdots\mathrm{d}y_q\frac{\mathrm{d}s}{s}.$$

The form is nonnegative. Indeed, write

$$F_m(u) = \mathcal{F}_{q-1}(u;y_1,\dots,y_{m-1},y_{m+1},\dots,y_q).$$

Using $k_{s\gamma_m} = 2\sum_{l=1}^2 h^{(l)}_{s\gamma_m/\sqrt{2}} * h^{(l)}_{s\gamma_m/\sqrt{2}}$ and the oddness of $h^{(l)}$, the preceding definition becomes

$$\Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f) = 2\sum_{l=1}^2 \int_0^\infty \int \left\| F_m * h^{(l)}_{s\gamma_m/\sqrt{2}} \right\|_2^2 \prod_{k\neq m} g_{s\gamma_k}(y_k)\,\mathrm{d}y_{k\neq m}\frac{\mathrm{d}s}{s} \geqslant 0.$$

Moreover, the sum over m telescopes. Differentiate the fully smoothed Gowers-cube average

$$G(s) = \int \mathcal{F}_q(x;y_1,\dots,y_q)\prod_{k=1}^q g_{s\gamma_k}(y_k)\,\mathrm{d}x\,\mathrm{d}y.$$

By (2.2), $-sG'(s)$ is a constant multiple of $\sum_m \Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f)$ at scale s. Integrating in s from 0 to ∞ gives

$$\sum_{m=1}^q \Theta^{q,m}_{\gamma_1,\dots,\gamma_q}(f) \lesssim_q \lim_{s\to 0^+} G(s) - \lim_{s\to\infty} G(s).$$

The second limit is zero for compactly supported f, while the first coalesces all increments y_k and equals $\int f(x)^{2^q}\,\mathrm{d}x$. Since $f = \mathbb{1}_{A_R}$, this is $|A_R| \leqslant R^2$. Combining this with (2.20), integrating the finite measure $\prod \gamma_k^{-2}\,\mathrm{d}\gamma_k$, and summing over m proves the stated estimate. □

Proposition 2.13 (Hypercube graph uniform estimate [31]). *For all $\lambda > 0$ and $\varepsilon \in (0,1]$,*

$$\left|\mathcal{N}^0_\lambda(\mathbb{1}_{A_R}) - \mathcal{N}^\varepsilon_\lambda(\mathbb{1}_{A_R})\right| \leqslant C_{\mathrm{uni}}\varepsilon^{1/2}R^2.$$

Proof. It is enough to compare $\mathcal{N}^\theta_\lambda$ and $\mathcal{N}^\varepsilon_\lambda$ for $0<\theta<\varepsilon$ and then let $\theta\to 0^+$. The same heat-flow expansion as above gives derivative terms $\mathcal{L}^{\theta,\varepsilon,m}_\lambda(f)$. Fix m. For fixed values of the other edge vectors, the variables x and y_m enter as

$$\int F_m(x)F_m(x+y_m)(\sigma_\lambda * k_{t\lambda})(y_m)\,\mathrm{d}x\,\mathrm{d}y_m,$$

where

$$F_m(u) = \mathcal{F}_{q-1}(u;y_1,\dots,y_{m-1},y_{m+1},\dots,y_q).$$

After the change of variables $v = x+y_m$, this is

$$\int (F_m * \sigma_\lambda * k_{t\lambda})(v)\overline{F_m(v)}\,\mathrm{d}v.$$

Plancherel gives

$$\int |\widehat{F}_m(\xi)|^2 \widehat{\sigma}(\lambda\xi)\widehat{k}(t\lambda\xi)\,\mathrm{d}\xi.$$

By (2.3), the multiplier is bounded by $Ct^{1/2}$. Hence the absolute value of the inner integral is at most $Ct^{1/2}\,\|F_m\|_2^2$. Since $0 \leqslant f \leqslant 1$ and f is supported in $[0,R]^2$, one has $\|F_m\|_2^2 \leqslant R^2$. The remaining kernels $\sigma_\lambda * g_{t\lambda}$ are probability kernels, so their integration does not change this bound. Thus

$$|\mathcal{L}_\lambda^{\theta,\varepsilon,m}(f)| \lesssim_q R^2 \int_\theta^\varepsilon t^{1/2}\frac{\mathrm{d}t}{t} \lesssim_q R^2(\varepsilon^{1/2} - \theta^{1/2}).$$

Summing over m and passing to the limit $\theta \to 0^+$ gives the proposition. □

Proof of Theorem 2.9. Let $A \subseteq \mathbb{R}^2$ have positive upper Banach density and choose $0 < \delta < \overline{\delta}(A)$. Suppose that arbitrarily large bad scales exist. Passing to a subsequence, choose bad scales $\lambda_1 < \cdots < \lambda_J$ with $\lambda_{j+1} \geqslant 2\lambda_j$. Choose ε so that

$$C_{\mathrm{uni}}\varepsilon^{1/2} < \frac{1}{3}c_{\mathrm{str}}\delta^{2^q},$$

and then J so large that

$$J^{-1}C_{\mathrm{err}}\varepsilon^{-3q}\log(1/\varepsilon) < \frac{1}{3}c_{\mathrm{str}}\delta^{2^q}.$$

Localize A to a cube $A_R \subseteq [0,R]^2$ with $|A_R| \geqslant \delta R^2$ and $R \geqslant 2\lambda_J$. By Proposition 2.12, one of the first J scales satisfies

$$|\mathcal{N}_{\lambda_j}^{\varepsilon}(\mathbb{1}_{A_R}) - \mathcal{N}_{\lambda_j}^{1}(\mathbb{1}_{A_R})| \leqslant J^{-1}C_{\mathrm{err}}\varepsilon^{-3q}\log(1/\varepsilon)R^2.$$

For that j, the decomposition (2.1) and Propositions 2.11 and 2.13 yield

$$\mathcal{N}_{\lambda_j}^{0}(\mathbb{1}_{A_R}) \geqslant c_{\mathrm{str}}\delta^{2^q}R^2 - \frac{1}{3}c_{\mathrm{str}}\delta^{2^q}R^2 - \frac{1}{3}c_{\mathrm{str}}\delta^{2^q}R^2 > 0.$$

This gives a copy of $\lambda_j\Gamma_q$ inside A_R and hence inside A, contradicting the choice of λ_j. Therefore all sufficiently large scales are good. □

Proof of Theorem 2.10. Let $A \subseteq [0,1]^2$ with $|A| \geqslant \delta$. Consider the intervals

$$I_j = [2^{-2j}, 2^{-2j+1}), \qquad j = 1, 2, \ldots.$$

If each of the first J intervals contained a bad scale, choose one such scale $\lambda_j \in I_j$. Then $\lambda_{j+1} \leqslant \lambda_j/2$, so after reversing the finite order the selected scales are lacunary increasing. The structured, error, and uniform estimates above apply with $R = 1$ and with the compact set A itself.

Choose ε so that

$$C_{\mathrm{uni}}\varepsilon^{1/2} < \frac{1}{3}c_{\mathrm{str}}\delta^{2^q},$$

and then choose

$$J \simeq_q \delta^{-C(q)}$$

large enough that

$$J^{-1}C_{\mathrm{err}}\varepsilon^{-3q}\log(1/\varepsilon) < \frac{1}{3}c_{\mathrm{str}}\delta^{2^q}.$$

The same pigeonhole argument as in the large-scale proof gives an index $j \leqslant J$ for which the error term is small. The decomposition then forces

$$\mathcal{N}^0_{\lambda_j}(\mathbb{1}_A) > 0,$$

contradicting the choice of λ_j as a bad scale. Hence at least one of the intervals $I_1, \ldots, I_J$ contains no bad scale. For that interval I_j, every $\lambda \in I_j$ is realized by A.

Finally,

$$|I_j| = 2^{-2j} \geqslant 2^{-2J} \geqslant \exp(-C'J) \geqslant \exp(-\delta^{-C(q)})$$

after increasing $C(q)$ if necessary. This proves the compact interval theorem. □

2.7 Open problem

Problem 2.14 (Equilateral triangles in the plane). A classical problem asks whether every planar set of positive upper density contains a congruent copy of every sufficiently large equilateral triangle; see Bourgain and the discussion surrounding Euclidean obstructions [3, 17]. Higher-dimensional analogues are known, but the dimensionally sharp planar case remains open. The obstruction is not a lack of density or curvature in isolation; rather, present analytic methods do not simultaneously exploit the one-dimensional family of rotations available in the plane and control the possible annular obstructions.

Chapter 3

Density theorems for arithmetic progressions

This chapter treats the compact, quantitative analogue of the large-scale Euclidean Szemerédi problem. Instead of asking whether a set of positive upper Banach density contains all sufficiently large configurations, one fixes a dense set inside a unit cube and asks for a whole interval of realized scales. The analytic proof follows Durcik and the author [10]; the three-term predecessor is due to Cook–Magyar–Pramanik [8], and the scale-pigeonholing philosophy goes back to Bourgain [3]. The proof combines the Szemerédi–Varnavides averaging principle [47, 50], Euclidean Gowers norms [21, 15], and multiscale cancellation ideas related to the multilinear Hilbert transform [34, 35, 49, 13, 52]. The quantitative statement below is obtained by inserting the finitary Szemerédi bounds recorded in (3.2): Raghavan's three-term estimate, building on Kelley–Meka and Bloom–Sisask [28, 2, 42], the Green–Tao polylogarithmic bound for four-term progressions [25], and the Leng–Sah–Sawhney improvement for all lengths at least five [36]. The geometric obstruction and the harmonic-analysis estimates are the same; the finitary input enters only through the structured Szemerédi–Varnavides lower bound.

3.1 Bourgain's annular obstruction for three-term gaps

The first issue is that the Euclidean norm cannot satisfy a compact interval theorem with a lower bound depending only on density. Bourgain's obstruction [3] is already visible for three-term progressions. For a measurable set $A \subseteq [0,1]^d$ write

$$G_2(A) = \{\lambda \geqslant 0 : \exists x, y \in \mathbb{R}^d,\ x, x+y, x+2y \in A,\ \|y\|_2 = \lambda\}$$

for the set of Euclidean three-term gap lengths.

Fix $0 < \eta < 1/16$ and $0 < \varepsilon < 1$. Define the compact annular set

$$A_{\varepsilon,\eta} = \{x \in [0,1]^d : \operatorname{dist}(\varepsilon^{-2} \|x\|_2^2, \mathbb{Z}) \leqslant \eta\}.$$

Thus $A_{\varepsilon,\eta}$ is the union of thin spherical shells on which the phase $\varepsilon^{-2} \|x\|_2^2$ lies within distance η of an integer. The shells become thinner and less separated as the radius grows, because consecutive level surfaces of $\|x\|_2^2$ are separated in the radial variable by an amount comparable to ε^2/r.

Proposition 3.1 (Bourgain's obstruction for length 3). *For every fixed $d \geqslant 1$ and $0 < \eta < 1/16$, the sets $A_{\varepsilon,\eta}$ satisfy*

$$|A_{\varepsilon,\eta}| \longrightarrow 2\eta \qquad (\varepsilon \to 0).$$

Moreover, every nonzero gap $\lambda \in G_2(A_{\varepsilon,\eta})$ obeys

$$\operatorname{dist}(2\varepsilon^{-2}\lambda^2, \mathbb{Z}) \leqslant 4\eta. \tag{3.1}$$

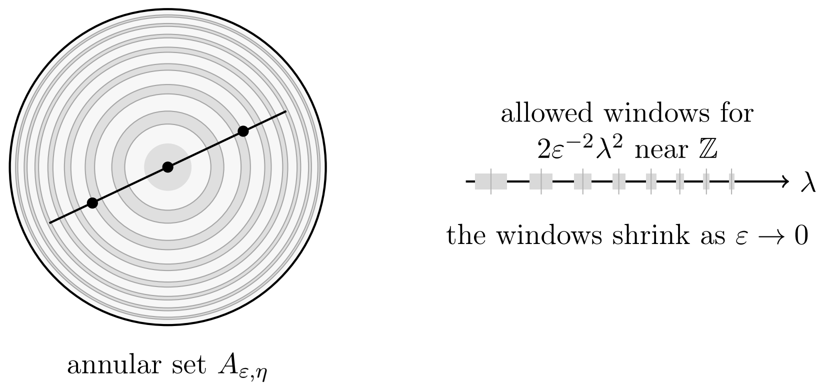

Figure 3.1: Bourgain's annular obstruction: three points in arithmetic progression inside the annular set force the Euclidean gap length into small windows separated by forbidden gaps.

Consequently the possible gaps are trapped in a union of short intervals, and the longest interval contained in $G_2(A_{\varepsilon,\eta}) \cap [0, \sqrt{d}]$ tends to 0 with ε. In particular, no positive lower bound for the length of an interval of Euclidean three-term gaps can depend only on the density of A.

Proof. The density statement is an equidistribution calculation. If ϕ is a smooth 1-periodic function with mean $\int_0^1 \phi(t)\,\mathrm{d}t$, then expanding ϕ into Fourier series gives

$$\int_{[0,1]^d} \phi(\varepsilon^{-2}\|x\|_2^2)\,\mathrm{d}x = \int_0^1 \phi(t)\,\mathrm{d}t + \sum_{m\neq 0} \widehat{\phi}(m) \int_{[0,1]^d} e^{2\pi i m \varepsilon^{-2}\|x\|_2^2}\,\mathrm{d}x.$$

The nonzero oscillatory integrals tend to 0 as $\varepsilon \to 0$: split off a ball of radius ρ about the origin, whose measure is $O(\rho^d)$, and integrate by parts on its complement, where $|\nabla \|x\|_2^2| \geqslant 2\rho$. Then let $\rho \downarrow 0$. Approximating the indicator of the interval $\{t \in \mathbb{R}/\mathbb{Z} : \mathrm{dist}(t, \mathbb{Z}) \leqslant \eta\}$ from above and below by smooth periodic functions gives $|A_{\varepsilon,\eta}| \to 2\eta$.

Now suppose that $x, x+y, x+2y \in A_{\varepsilon,\eta}$ and set $\lambda = \|y\|_2$. For $j = 0, 1, 2$ choose integers m_j and errors $\theta_j \in [-\eta, \eta]$ such that

$$\varepsilon^{-2}\|x + jy\|_2^2 = m_j + \theta_j.$$

Taking the second finite difference and using

$$\|x+2y\|_2^2 - 2\|x+y\|_2^2 + \|x\|_2^2 = 2\|y\|_2^2$$

gives

$$2\varepsilon^{-2}\lambda^2 = (m_2 - 2m_1 + m_0) + (\theta_2 - 2\theta_1 + \theta_0).$$

Since $|\theta_2 - 2\theta_1 + \theta_0| \leqslant 4\eta$, this proves (3.1).

It remains only to translate (3.1) into gaps on the line of possible lengths. Put

$$E_{\varepsilon,\eta} = \{\lambda \geqslant 0 : \mathrm{dist}(2\varepsilon^{-2}\lambda^2, \mathbb{Z}) \leqslant 4\eta\}.$$

Then $G_2(A_{\varepsilon,\eta}) \subseteq E_{\varepsilon,\eta}$. The set $E_{\varepsilon,\eta}$ is contained in the union of intervals

$$I_0 = \big[0, \varepsilon\sqrt{2\eta}\,\big], \qquad I_k = \frac{\varepsilon}{\sqrt{2}}\big[\sqrt{k - 4\eta}, \sqrt{k + 4\eta}\,\big] \quad (k \geqslant 1).$$

The intervals are disjoint because $\eta < 1/16$. Also

$$|I_k| = \frac{8\eta\varepsilon}{\sqrt{2}\left(\sqrt{k+4\eta}+\sqrt{k-4\eta}\right)} \qquad (k \geqslant 1),$$

while $|I_0| = \varepsilon\sqrt{2\eta}$. Hence the largest component of $E_{\varepsilon,\eta} \cap [0,\sqrt{d}]$ has length $O_\eta(\varepsilon)$, which tends to 0. Since $|A_{\varepsilon,\eta}| \geqslant \eta$ for all sufficiently small ε, any asserted density-only lower bound for an interval of Euclidean three-term gaps would be contradicted by choosing ε small enough. □

3.2 The correct problem

Let $A \subseteq [0,1]^d$ be measurable. For an integer $n \geqslant 3$ define

$$\ell^p\text{-gaps}_n(A) = \{\lambda \geqslant 0 : \exists x, y \in \mathbb{R}^d,\ x, x+y, \ldots, x+(n-1)y \in A,\ \|y\|_p = \lambda\}.$$

A positive-measure set always realizes very small gaps: a Steinhaus-type argument shows that the full vector gap set contains a neighborhood of the origin. The compact theorem is much stronger and asks for an interval of gap lengths whose size is bounded from below using only the density.

There is an essential restriction on the norm. Section 3.1 gives Bourgain's Euclidean obstruction in detail for three-term progressions. The same finite-difference mechanism adapts to ℓ^p whenever $p \in \{1, 2, \ldots, n-1\}$: working in a fixed orthant, the n-fold finite difference of the polynomial $t \mapsto t^p$ vanishes, so an annular congruence construction again forces the gap parameter into short separated windows. (The details can be found in [12].) Thus the positive theorem needs to exclude exactly these exponents.

Theorem 3.2 (Compact interval theorem for ℓ^p gaps). *Let $n \geqslant 3$,*

$$p \in [1,\infty) \setminus \{1, 2, \ldots, n-1\},$$

and let

$$D(n,p) = 2^{n+3}(n+p).$$

If $d \geqslant D(n,p)$, then there exists $C = C(n,p,d) < \infty$ such that every measurable $A \subseteq [0,1]^d$ with $|A| \geqslant \delta$, $0 < \delta \leqslant 1/2$, has ℓ^p-gaps$_n(A)$ containing an interval of length at least the following quantity. Put

$$L_\delta = \log(e/\delta), \qquad L_\delta^{(2)} = \log(eL_\delta).$$

Then the interval length is bounded from below by

$$\begin{cases} \exp\Big(-\exp\Big(CL_\delta^6 L_\delta^{(2)}\Big)\Big), & n = 3, \\ \exp\big(-\exp(\delta^{-C})\big), & n = 4, \\ \exp\big(-\exp\big(\exp(L_\delta^C)\big)\big), & n \geqslant 5. \end{cases}$$

The geometric and harmonic-analysis parts are those of Durcik and the author [10]. The three quantitative regimes come solely from the finitary Szemerédi input: three-term progressions have the strongest Roth-type estimate, four-term progressions have the Green–Tao polylogarithmic estimate, and the best general bounds for lengths at least five are those of Leng–Sah–Sawhney.

3.3 Szemerédi's theorem as the structured lower bound

Let $N(n, \eta)$ be the least integer such that every subset of $\{0, 1, \dots, N-1\}$ with at least ηN elements contains a nontrivial n-term arithmetic progression. It is convenient to record the finitary input in the inverse form needed by the Varnavides argument. For each fixed $n \geqslant 3$ there is $C_n < \infty$ such that

$$N(n, \eta) \leqslant \begin{cases} \exp\Big(C_n L_\eta^6 L_\eta^{(2)}\Big), & n = 3, \\ \exp(\eta^{-C_n}), & n = 4, \\ \exp\big(\exp(L_\eta^{C_n})\big), & n \geqslant 5. \end{cases} \tag{3.2}$$

Indeed, if $r_n(M)$ denotes the largest cardinality of an n-term-progression-free subset of $\{1, \dots, M\}$, then Raghavan's improvement of Roth's theorem gives

$$r_3(M) \leqslant M \exp\{-c(\log M)^{1/6}(\log\log M)^{-1/6}\}$$

for an absolute constant $c > 0$ [42]. Taking $\log M = C L_\eta^6 L_\eta^{(2)}$ with C large makes the exponential saving at least e^{-L_η}, and hence forces a nontrivial three-term progression in every subset of density at least η. The second line follows from Green–Tao's bound $r_4(M) \ll M(\log M)^{-c}$ [25]. For $n \geqslant 5$, Leng–Sah–Sawhney prove $r_n(M) \ll_n M \exp[-(\log\log M)^{c_n}]$ [36], which gives the third line. Gowers's quantitative proof [21] provided the earlier general benchmark; Green–Tao supplies the four-term polylogarithmic input used in the second line, and Leng–Sah–Sawhney replaces the former $N(n, \eta) \leqslant \exp\exp(\eta^{-C_n})$ bound for $n \geqslant 5$ by the third line.

The continuous ingredient needed below is a Varnavides averaging version of Szemerédi's theorem. Notice that the lemma itself is qualitative in form and keeps the exact dependence on $N(n, \delta/4)$; the estimates in (3.2) are substituted only afterwards.

Lemma 3.3 (Continuous Varnavides lower bound). *For $n \geqslant 3$, $d \geqslant 1$, $0 < \delta \leqslant 1/2$, $0 < \lambda \leqslant 1$, and every measurable $A \subseteq [0,1]^d$ with $|A| \geqslant \delta$,*

$$\lambda^{-d} \int_{[0,\lambda]^d} \int_{[0,1]^d} \prod_{i=0}^{n-1} \mathbb{1}_A(x + iy)\,\mathrm{d}x\,\mathrm{d}y \gtrsim_d \delta^{d+1} N(n, \delta/4)^{-d-2}. \tag{3.3}$$

Proof. Put

$$N = N(n, \delta/4), \qquad \theta = \frac{\delta}{4d},$$

and consider the parameter set

$$T = \Big(\frac{\theta\lambda}{2N}, \frac{\theta\lambda}{N}\Big]^d \times [0, 1-\theta]^d,$$

whose elements are denoted by (t, u). Since $0 < \lambda \leqslant 1$, for every $0 \leqslant k \leqslant N-1$ and $(t, u) \in T$ the point $u + kt$ lies in $[0,1]^d$; indeed $0 \leqslant kt \leqslant \theta\lambda \leqslant \theta$ coordinatewise.

Average the density of A along the N-point progression $u, u+t, \dots, u+(N-1)t$:

$$⨍_T \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{1}_A(u + kt)\,\mathrm{d}t\,\mathrm{d}u.$$

For a fixed k and t, the change of variables $v = u + kt$ sends $u \in [0, 1-\theta]^d$ to the box $[0, 1-\theta]^d + kt$. This box contains $[\theta, 1-\theta]^d$, because $0 \leqslant kt \leqslant \theta$. Hence

$$\fint_T \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{1}_A(u + kt)\, \mathrm{d}t\, \mathrm{d}u \geqslant |A \cap [\theta, 1-\theta]^d|$$

$$\geqslant |A| - \big|[0,1]^d \setminus [\theta, 1-\theta]^d\big|$$
$$\geqslant \delta - 2d\theta = \delta/2.$$

Let

$$T_{\text{large}} = \Big\{(t,u) \in T : \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{1}_A(u + kt) \geqslant \delta/4\Big\}.$$

Since the averaged discrete density is bounded by 1 on T_{large} and by $\delta/4$ on its complement, the preceding inequality implies

$$\frac{\delta}{2} \leqslant \frac{|T_{\text{large}}|}{|T|} + \frac{\delta}{4},$$

and therefore

$$|T_{\text{large}}| \geqslant \frac{\delta}{4}|T| \gtrsim_d \delta^{d+1} N^{-d} \lambda^d. \tag{3.4}$$

For every $(t,u) \in T_{\text{large}}$ the index set

$$S_{t,u} = \{0 \leqslant k \leqslant N-1 : u + kt \in A\}$$

has cardinality at least $(\delta/4)N$. By the choice of N, it contains a nontrivial n-term arithmetic progression. Thus there exist integers k and $l \geqslant 1$ with $k + (n-1)l \leqslant N-1$ such that

$$\prod_{i=0}^{n-1} \mathbb{1}_A(u + (k+il)t) = 1.$$

Consequently,

$$|T_{\text{large}}| \leqslant \sum_{k=0}^{N-1} \sum_{1 \leqslant l \leqslant (N-1-k)/(n-1)} \int_T \prod_{i=0}^{n-1} \mathbb{1}_A(u + (k+il)t)\, \mathrm{d}t\, \mathrm{d}u.$$

In a summand perform the change of variables

$$x = u + kt, \qquad y = lt.$$

The Jacobian is l^{-d}, and the new y lies in $(0, \theta\lambda]^d \subseteq [0, \lambda]^d$. Enlarging the x-domain to $[0,1]^d$, and using $l^{-d} \leqslant 1$, gives

$$|T_{\text{large}}| \leqslant N^2 \int_{[0,\lambda]^d} \int_{[0,1]^d} \prod_{i=0}^{n-1} \mathbb{1}_A(x + iy)\, \mathrm{d}x\, \mathrm{d}y.$$

Combining this with (3.4) yields (3.3). $\square$

Proposition 3.4 (Continuous Szemerédi–Varnavides bound with quantitative input). *For integers $n \geqslant 3$ and $d \geqslant 1$, there exists $C = C(n,d)$ such that every measurable $A \subseteq [0,1]^d$ with $|A| \geqslant \delta$, $0 < \delta \leqslant 1/2$, satisfies*

$$\int_{[0,1]^d} \int_{[0,1]^d} \prod_{i=0}^{n-1} \mathbb{1}_A(x+iy)\,\mathrm{d}y\,\mathrm{d}x \geqslant \Theta_{n,d}(\delta),$$

where, with $L_\delta = \log(e/\delta)$ and $L_\delta^{(2)} = \log(eL_\delta)$,

$$\Theta_{n,d}(\delta) = \begin{cases} \exp\left(-CL_\delta^6 L_\delta^{(2)}\right), & n = 3, \\ \exp(-\delta^{-C}), & n = 4, \\ \exp\left(-\exp(L_\delta^C)\right), & n \geqslant 5. \end{cases}$$

Proof. Apply Lemma 3.3 with $\lambda = 1$. The lower bound is

$$\gtrsim_d \delta^{d+1} N(n, \delta/4)^{-d-2}.$$

It remains to insert (3.2). The factor $\delta^{d+1} = \exp[-(d+1)\log(1/\delta)]$ is smaller than the displayed main losses and can be absorbed by increasing C.

For $n = 3$, Raghavan's estimate gives

$$N(3, \delta/4) \leqslant \exp\left(CL_\delta^6 L_\delta^{(2)}\right),$$

and hence the continuous count is at least

$$\exp\left(-CL_\delta^6 L_\delta^{(2)}\right).$$

For $n = 4$, Green–Tao gives $N(4, \delta/4) \leqslant \exp(\delta^{-C})$, so the lower bound is $\exp(-\delta^{-C})$. For $n \geqslant 5$, Leng–Sah–Sawhney gives $N(n, \delta/4) \leqslant \exp\exp(L_\delta^C)$, so the lower bound is $\exp[-\exp(L_\delta^C)]$. This proves the three cases. $\square$

This proposition is the structured estimate at the coarsest level: it says that after one has blurred the gap sphere at the same scale as the gap itself, positive density forces many progressions.

3.4 Exact and smoothed counting forms

Let $S_p^{d-1} = \{y \in \mathbb{R}^d : \|y\|_p = 1\}$. Following Cook–Magyar–Pramanik and the work of Durcik and the author [8, 10], let σ denote the natural hypersurface measure on S_p^{d-1}, equivalently the vague limit of smooth densities

$$\sigma^\eta(y) = \psi_\eta(\|y\|_p^p - 1)$$

with ψ a fixed smooth approximate identity on the line. Let φ be a nonnegative even C^∞ bump on $\mathbb{R}^d$, supported in $[-3,3]^d$, positive on $[-2,2]^d$, and normalized to have integral one. For $\lambda > 0$ write σ_λ and $\varphi_{\varepsilon\lambda}$ for L^1-normalized dilates.

The exact counting form is

$$\mathcal{N}_\lambda^0(A) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=0}^{n-1} \mathbb{1}_A(x+iy)\,\mathrm{d}\sigma_\lambda(y)\,\mathrm{d}x,$$

so $\mathcal{N}^0_\lambda(A) > 0$ implies that A contains an n-term progression with ℓ^p gap λ. The smoothed form is

$$\mathcal{N}^\varepsilon_\lambda(A) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\prod_{i=0}^{n-1}\mathbb{1}_A(x+iy)\,(\sigma_\lambda * \varphi_{\varepsilon\lambda})(y)\,\mathrm{d}y\,\mathrm{d}x.$$

For fixed A and λ, $\mathcal{N}^\varepsilon_\lambda(A) \to \mathcal{N}^0_\lambda(A)$ as $\varepsilon \to 0^+$. Indeed, if

$$F(x,y) = \prod_{i=0}^{n-1}\mathbb{1}_A(x+iy), \qquad f(y) = \int F(x,y)\,\mathrm{d}x,$$

then f is continuous: translating y to y' changes each factor by an L^1-small translate of $\mathbb{1}_A$, and the product difference is bounded by a telescoping sum of these L^1 differences. By Fubini,

$$\mathcal{N}^\varepsilon_\lambda(A) = \int (f * \varphi_{\varepsilon\lambda})(y)\,\mathrm{d}\sigma_\lambda(y),$$

and dominated convergence with respect to the finite measure σ_λ gives the claim.

The proof decomposes

$$\mathcal{N}^0_\lambda(A) = \mathcal{N}^1_\lambda(A) + \big(\mathcal{N}^\varepsilon_\lambda(A) - \mathcal{N}^1_\lambda(A)\big) + \big(\mathcal{N}^0_\lambda(A) - \mathcal{N}^\varepsilon_\lambda(A)\big). \tag{3.5}$$

The first term is structured, the second is the error term between very coarse and moderately fine smoothing, and the third is the uniform term measuring the cost of replacing the sharp sphere by a thin smoothed annulus.

3.5 The three estimates for arithmetic progressions

Proposition 3.5 (Structured part [10], with quantitative Szemerédi input). *There exists $E = E(n,p,d)$ such that, for every $\lambda \in (0,1]$ and every $A \subseteq [0,1]^d$ with $|A| \geqslant \delta$,*

$$\mathcal{N}^1_\lambda(A) \geqslant \Theta_{n,p,d}(\delta),$$

where, with $L_\delta = \log(e/\delta)$ and $L^{(2)}_\delta = \log(eL_\delta)$,

$$\Theta_{n,p,d}(\delta) = \begin{cases} \exp\Big(-EL_\delta^6 L_\delta^{(2)}\Big), & n=3,\\ \exp(-\delta^{-E}), & n=4,\\ \exp\big(-\exp(L_\delta^E)\big), & n \geqslant 5.\end{cases}$$

Proof. The only point is to compare the coarse annular kernel with normalized Lebesgue measure on a cube of gap vectors. Since φ is positive on $[-2,2]^d$ and every point $z \in S_p^{d-1}$ has coordinates in $[-1,1]$, for every $u \in [0,1]^d$ one has

$$(\sigma * \varphi)(u) = \int_{S_p^{d-1}}\varphi(u-z)\,\mathrm{d}\sigma(z) \geqslant \sigma(S_p^{d-1})\min_{w\in[-2,2]^d}\varphi(w) > 0.$$

After dilation this becomes

$$(\sigma_\lambda * \varphi_\lambda)(y) \gtrsim_{p,d} \lambda^{-d}\mathbb{1}_{[0,\lambda]^d}(y), \qquad 0 < \lambda \leqslant 1.$$

Therefore

$$\mathcal{N}^1_\lambda(A) \gtrsim_{p,d} \lambda^{-d} \int_{[0,\lambda]^d} \int_{[0,1]^d} \prod_{i=0}^{n-1} \mathbb{1}_A(x+iy)\,\mathrm{d}x\,\mathrm{d}y.$$

Lemma 3.3 bounds the right hand side below by

$$\gtrsim_{p,d} \delta^{d+1} N(n,\delta/4)^{-d-2}.$$

Substitution of (3.2), with polynomial factors in δ and constants depending on p, d absorbed by increasing E, gives precisely the three cases in the statement. $\square$

The other two estimates require harmonic analysis. We write

$$g(x) = e^{-\pi\|x\|_2^2}, \qquad h^{(\ell)} = \partial_\ell g, \qquad k^{(\ell)} = \partial_\ell^2 g, \qquad k = \Delta g = \sum_{\ell=1}^d k^{(\ell)}.$$

Thus $\widehat{g} = g$,

$$\widehat{k}(\xi) = -4\pi^2\,\|\xi\|_2^2\, e^{-\pi\|\xi\|_2^2}, \qquad \partial_t g_t = (2\pi t)^{-1} k_t,$$

and

$$g_a * g_b = g_{(a^2+b^2)^{1/2}}, \qquad h_a^{(\ell)} * h_b^{(\ell)} = \frac{ab}{a^2+b^2}\, k^{(\ell)}_{(a^2+b^2)^{1/2}}. \tag{3.6}$$

Finally define

$$\rho(x) = d\varphi(x) + x\cdot\nabla\varphi(x).$$

Then $-t\partial_t \varphi_t = \rho_t$, so

$$\sigma_\lambda * \varphi_{a\lambda} - \sigma_\lambda * \varphi_{b\lambda} = \int_a^b \sigma_\lambda * \rho_{t\lambda} \frac{\mathrm{d}t}{t}. \tag{3.7}$$

Moreover $\int \rho = 0$, because $\rho = \operatorname{div}(x\varphi(x))$.

Lemma 3.6 (The multiscale telescoping estimate). *Let $0 \leqslant B \leqslant 1$ be supported in $[0,1]^d$, and put*

$$F_{n-1}(x,y) = \prod_{i=0}^{n-1} B(x+iy).$$

For $a<b$, $\alpha,\gamma \geqslant 2^{-1/2}$, and $\ell \in \{1,\dots,d\}$ define

$$\Lambda^{\alpha,\gamma,\ell}_{a,b} = \int_a^b \int_{(\mathbb{R}^d)^2} \left| \int_{\mathbb{R}^d} F_{n-1}(x,y) h^{(\ell)}_{s\gamma}(y-q)\,\mathrm{d}y \right| g_{s\alpha}(q)\,\mathrm{d}q\,\mathrm{d}x \frac{\mathrm{d}s}{s}.$$

Then

$$\Lambda^{\alpha,\gamma,\ell}_{a,b} \lesssim_{n,d} (\alpha\gamma)^2 \Big(\log\frac{b}{a}\Big)^{1-2^{-n+2}}. \tag{3.8}$$

Proof. The proof is the Cauchy–Schwarz telescoping argument of Durcik and the author [10]. We give the mechanism, since it is the source of the exponent $1-2^{-n+2}$.

For $1 \leqslant k \leqslant n-1$ introduce the partially doubled products

$$F_k(x,y,u_{k+1},\dots,u_{n-1}) = \prod_{i=0}^{k} \prod_{r\in\{0,1\}^{n-k-1}} B\Big(x+iy+\sum_{s=k+1}^{n-1} r_s(i+s-k)u_s\Big).$$

Thus F_{n-1} is the product in the statement. The inductive quantity at level k is obtained from the preceding display by inserting one derivative in the y variable and Gaussian weights in the already created difference variables. In the notation relevant for the induction,

$$\Lambda_{k,l,a,b}^{\alpha,\alpha_k,\dots,\alpha_{n-1}} = \int_a^b \int \left| \iint F_k(x,y,u_{k+1},\dots,u_{n-1}) h_{t\alpha_k}^{(l)}(y-p_k)\,\mathrm{d}y \right| \\ \times g_{t\alpha}(p_k+\dots+p_{n-1}) \prod_{j=k+1}^{n-1} g_{t\alpha_j}(p_j) g_{t\alpha_j}(u_j-p_j)\,\mathrm{d}\Omega \frac{\mathrm{d}t}{t},$$

where $\mathrm{d}\Omega$ denotes integration in the variables $x, p_k, \dots, p_{n-1}, u_{k+1}, \dots, u_{n-1}$. The lemma is the case $k = n-1$.

The claim is proved by induction on k. The basis $k = 1$ is the key positivity step. After changing variables so that the two factors $B(x)$ and $B(x+y)$ are separated, Cauchy–Schwarz writes the relevant expression as the geometric mean of two quantities of the form

$$\widetilde{\Theta}_{l,m,a,b} = \int_a^b \int G(x,u)\, \widetilde{K}_{l,m,t}(u)\,\mathrm{d}x\,\mathrm{d}u \frac{\mathrm{d}t}{t},$$

where $0 \leqslant G \leqslant \mathbb{1}_{[0,1]^d}(x)$ and

$$\widetilde{K}_{l,m,t}(u) = -\int h_{t\alpha_m}^{(l)}(p_m) h_{t\alpha_m}^{(l)}(u_m-p_m) \prod_{j\neq m} g_{t\alpha_j}(p_j) g_{t\alpha_j}(u_j-p_j)\,\mathrm{d}p.$$

The convolution identity (3.6) shows that each $\widetilde{\Theta}_{l,m,a,b} \geqslant 0$: after integrating in all variables except one, it is a square. On the other hand, the heat equation gives the exact telescoping identity

$$\sum_l \sum_m \widetilde{\Theta}_{l,m,a,b} = \pi(\widetilde{\Xi}_a - \widetilde{\Xi}_b),$$

where $\widetilde{\Xi}_t$ is the same integral with every $\widetilde{K}$ replaced by the product of the corresponding Gaussians. Since $0 \leqslant \widetilde{\Xi}_t \leqslant 1$, every nonnegative summand $\widetilde{\Theta}_{l,m,a,b}$ is $O(1)$. This proves the induction basis.

Assume now that the estimate has been proved at level $k-1$ and consider level $k \geqslant 2$. Split the product defining F_k into the $i = 0$ factor and the remaining factors. Cauchy–Schwarz gives

$$\Lambda_k \leqslant S_k^{1/2} T_k^{1/2}.$$

The easy factor has no derivative and is bounded by

$$S_k \leqslant \int_a^b \frac{\mathrm{d}t}{t} = \log(b/a).$$

For the second factor, expanding the square introduces a new difference variable u_k; after the harmless changes $x \mapsto x-y$ and $p_k \mapsto p_k + y$, the expression becomes a positive form $\Theta_{k,l,k,a,b}$ with two first derivatives joined into one second derivative. More generally, for $m = k, \dots, n-1$ one obtains nonnegative forms $\Theta_{k,l,m,a,b} \geqslant 0$. The same Gaussian heat identity as in the basis gives

$$\sum_l \Big(\Psi_{k,l,a,b} + \sum_{m=k}^{n-1} \Theta_{k,l,m,a,b} \Big) = \pi(\Xi_{k,a} - \Xi_{k,b}),$$

where $0 \leqslant \Xi_{k,t} \leqslant 1$. The term Ψ is the only one not visibly positive; it contains a second derivative in the outer Gaussian. Using (3.6), this second derivative is split into two first derivatives at scale $2^{-1/2}t$, and the induction hypothesis at level $k-1$ gives

$$|\Psi_{k,l,a,b}| \lesssim_{n,d} (\alpha\alpha_k \cdots \alpha_{n-1})^4 \Big(\log \frac{b}{a}\Big)^{1-2^{-k+2}}.$$

Since the Θ's are nonnegative and their sum is bounded by $1 + \sum_l |\Psi_{k,l,a,b}|$, each $\Theta_{k,l,m,a,b}$ satisfies the same bound. Returning to $\Lambda_k \leqslant S_k^{1/2} T_k^{1/2}$ gives

$$\Lambda_k \lesssim_{n,d} (\alpha\alpha_k \cdots \alpha_{n-1})^2 \Big(\log \frac{b}{a}\Big)^{1-2^{-k+1}}.$$

This closes the induction. Setting $k = n-1$ gives (3.8). □

Proposition 3.7 (Error part [10]). *There exists $F = F(n,p,d)$ such that, whenever $\lambda_j \in (2^{-j}, 2^{-j+1}]$, $j = 1, \ldots, J$,*

$$\sum_{j=1}^{J} \Big|\mathcal{N}^{\varepsilon}_{\lambda_j}(A) - \mathcal{N}^{1}_{\lambda_j}(A)\Big| \leqslant \varepsilon^{-F} J^{1-2^{-n+2}}$$

for all $0 < \varepsilon \leqslant 1/2$ and all measurable $A \subseteq [0,1]^d$.

Proof. Choose complex numbers κ_j of modulus at most 1 so that the left side is the absolute value of the signed sum. It suffices to prove

$$\left|\sum_{j=1}^{J} \kappa_j (\mathcal{N}^{\varepsilon}_{\lambda_j}(A) - \mathcal{N}^{1}_{\lambda_j}(A))\right| \lesssim \varepsilon^{-F} J^{1-2^{-n+2}}. \tag{3.9}$$

By (3.7),

$$\sigma_{\lambda_j} * \varphi_{\varepsilon\lambda_j} - \sigma_{\lambda_j} * \varphi_{\lambda_j} = \int_{\varepsilon}^{1} \sigma_{\lambda_j} * \rho_{t\lambda_j} \frac{\mathrm{d}t}{t}.$$

The cancellation $\int \rho = 0$ allows us to insert a continuous Littlewood–Paley identity. Choose a Schwartz function θ satisfying

$$-\int_0^{\infty} \widehat{\theta}(u\xi)\widehat{k}(u\xi) \frac{\mathrm{d}u}{u} = 1, \qquad \xi \neq 0.$$

Applying this to the mean-zero kernel $\rho_{t\lambda_j}$ and using the triangle inequality reduces (3.9) to bounding

$$\sum_{j=1}^{J} \int_0^{\infty} \int_{\varepsilon}^{1} \left| \iint F_{n-1}(x,y)(\sigma_{\lambda_j} * \rho_{t\lambda_j} * \theta_{tu\lambda_j} * k_{tu\lambda_j})(y)\, \mathrm{d}y\, \mathrm{d}x \right| \frac{\mathrm{d}t}{t} \frac{\mathrm{d}u}{u}. \tag{3.10}$$

We now split the second derivative at the scale of the jth gap. Since $\lambda_j \simeq 2^{-j}$,

$$1 = \frac{1}{\log 2} \int_{2^{-j-5}t}^{2^{-j-4}t} \frac{\mathrm{d}s}{s}.$$

For such s set $r_j(s,t) = (t^2\lambda_j^2 - s^2)^{1/2}$; then $s \simeq r_j(s,t) \simeq t\lambda_j$. Formula (3.6) yields

$$k_{tu\lambda_j} = \frac{t^2\lambda_j^2}{sr_j(s,t)} \sum_{\ell=1}^{d} h^{(\ell)}_{r_j(s,t)u} * h^{(\ell)}_{su},$$

with a coefficient bounded above and below by constants. Expanding one of the two first derivatives and moving absolute values outward, (3.10) is bounded by a constant times

$$\sum_{j=1}^{J}\sum_{\ell=1}^{d}\int_0^\infty\int_\varepsilon^1\int_{2^{-j-5}t}^{2^{-j-4}t}\iint\left|\int F_{n-1}(x,y)h^{(\ell)}_{su}(y-q)\,\mathrm{d}y\right|\left|L^{(\ell)}_{j,t,u,s}(q)\right|\,\mathrm{d}q\,\mathrm{d}x\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\frac{\mathrm{d}u}{u}, \tag{3.11}$$

where

$$L^{(\ell)}_{j,t,u,s} = \sigma_{\lambda_j} * \rho_{t\lambda_j} * \theta_{tu\lambda_j} * h^{(\ell)}_{r_j(s,t)u}.$$

We next dominate this harmless kernel by Gaussians at the common scale su. The Fourier transform of ρ vanishes at the origin and is rapidly decreasing. Hence, for any large N,

$$\left|(\rho_{u^{-1}} * \theta * h^{(\ell)}_{r_j(s,t)/(t\lambda_j)})(z)\right| \lesssim_{N,d} \min\{u,u^{-1}\}(1+\|z\|_2)^{-N}.$$

Convolving with σ and using $t \geqslant \varepsilon$ gives, after rescaling and choosing $N = d+3$,

$$\left|L^{(\ell)}_{j,t,u,s}(q)\right| \lesssim_{p,d} \varepsilon^{-d-3}\min\{u,u^{-1}\}\int_1^\infty g_{\beta su}(q)\frac{\mathrm{d}\beta}{\beta^4}.$$

Substitute this estimate into (3.11), interchange the integrals, and then replace s by su. The sum over j fills a single interval of s-scales, namely

$$2^{-J-5}tu \leqslant s \leqslant 2^{-5}tu.$$

Lemma 3.6 gives, for each fixed t, u, β, ℓ,

$$\int_{2^{-J-5}tu}^{2^{-5}tu}\iint\left|\int F_{n-1}(x,y)h^{(\ell)}_s(y-q)\,\mathrm{d}y\right| g_{\beta s}(q)\,\mathrm{d}q\,\mathrm{d}x\frac{\mathrm{d}s}{s} \lesssim_{n,d} \beta^2 J^{1-2^{-n+2}}.$$

Therefore (3.10) is at most

$$C_{n,p,d}\varepsilon^{-d-3}J^{1-2^{-n+2}}\int_1^\infty\beta^2\frac{\mathrm{d}\beta}{\beta^4}\int_0^\infty\min\{u,u^{-1}\}\frac{\mathrm{d}u}{u}\int_\varepsilon^1\frac{\mathrm{d}t}{t}.$$

The β integral is finite. The u integral is finite after the sharper form $\min\{u,u^{-1}\}\,\mathrm{d}u/u = \min\{1,u^{-2}\}\,\mathrm{d}u$ that results from the preceding rescaling. Finally $\int_\varepsilon^1 \mathrm{d}t/t \leqslant \varepsilon^{-1}$. All fixed powers of ε^{-1} are absorbed into ε^{-F}, proving the proposition. □

Proposition 3.8 (Uniform part [8, 10]). *If $d \geqslant D(n,p)$, then*

$$\left|\mathcal{N}^0_\lambda(A) - \mathcal{N}^\varepsilon_\lambda(A)\right| \lesssim_{n,p,d} \varepsilon^{1/3}$$

for all $\lambda, \varepsilon \in (0,1]$ and all measurable $A \subseteq [0,1]^d$.

Proof. We use Euclidean Gowers norms. For a compactly supported measurable f on $\mathbb{R}^d$ put

$$\|f\|_{U^n(\mathbb{R}^d)}^{2^n} = \int_{(\mathbb{R}^d)^{n+1}} \prod_{\omega\in\{0,1\}^n} \mathcal{C}^{|\omega|} f(x+\omega\cdot h)\,\mathrm{d}x\,\mathrm{d}h,$$

where $\mathcal{C}$ denotes complex conjugation. We shall use two elementary facts: the scaling law

$$\|f_\lambda\|_{U^n} = \lambda^{-d(1-(n+1)2^{-n})}\,\|f\|_{U^n}$$

and the generalized von Neumann inequality

$$\left|\int\!\!\int \prod_{i=0}^{n-1} \mathbb{1}_A(x+iy)g(y)\,\mathrm{d}y\,\mathrm{d}x\right| \lesssim_{n,d} \lambda^{d(1-(n+1)2^{-n})}\,\|g\|_{U^n(\mathbb{R}^d)} \tag{3.12}$$

whenever g is supported in a cube of sidelength $O(\lambda)$. To prove (3.12), define for $1 \leqslant k \leqslant n$ the iterated expressions

$$\Lambda_k = \int \prod_{i=0}^{k-1}\Big(\prod_{m=1}^{n-k} \Delta_{(i+m)h_m}\mathbb{1}_A(x+iy)\Big)\Big(\prod_{m=1}^{n-k}\Delta_{h_m} g(y)\Big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}h_1\cdots\mathrm{d}h_{n-k}.$$

The case $k=n$ is the left side of (3.12). Splitting off the $i=0$ factor and applying Cauchy–Schwarz gives

$$|\Lambda_k| \leqslant L_k^{1/2} M_k^{1/2}.$$

The support of g forces all h_m to have size $O_n(\lambda)$, so $L_k \lesssim_{n,d} \lambda^{d(n-k)}$. Expanding M_k, changing variables, and relabelling the new difference variable turns it into Λ_{k-1}. Starting from $k=1$, where the second Cauchy–Schwarz factor is exactly the U^n norm of g, induction gives (3.12).

It remains to estimate the U^n norm of the difference between the sharp and smoothed ℓ^p spheres. We first record the one-dimensional oscillatory estimate

$$\left\|\mathbb{1}_{[-3,3]}(x)e^{2\pi i u|x|^p}\right\|_{U^n(\mathbb{R})} \lesssim_{n,p} (1+|u|)^{-2/D(n,p)}. \tag{3.13}$$

For $|u|\leqslant 1$ this follows from the trivial $L^{2^n/(n+1)}$ bound. For $|u|>1$ split the interval into $(-\eta,\eta)$, $[\eta,3]$, and $[-3,-\eta]$. The middle interval is representative. Expanding the U^n norm reduces the 2^nth power to integrals of

$$I_h(u) = \int_a^b e^{2\pi i u\Phi_h(x)}\,\mathrm{d}x,$$

where

$$\Phi_h(x) = \sum_{r\in\{0,1\}^{n-1}} (-1)^{|r|}\,|x+r\cdot h|^p\,.$$

If some $|h_i|\leqslant\eta$, we use $|I_h(u)|\leqslant 3$, and the set of such h has measure $O_n(\eta)$. Otherwise every $|h_i|>\eta$. Since $p\notin\{1,\dots,n-1\}$, the repeated finite difference is not identically zero, and the fundamental theorem of calculus gives

$$\Phi_h'(x) = c_{n,p}\,h_1\cdots h_{n-1}\int_{[0,1]^{n-1}}(x+t\cdot h)^{p-n}\,\mathrm{d}t,$$

with $c_{n,p} \neq 0$, and a similar formula for Φ_h''. On the interval $x \geqslant \eta$ this yields

$$|\Phi_h'(x)| \gtrsim_{n,p} \min\{\eta^{p-1}, \eta^{n-1}\}, \qquad |\Phi_h''(x)| \lesssim_{n,p} \max\{\eta^{p-n-1}, 1\}.$$

One integration by parts therefore gives

$$|I_h(u)| \lesssim_{n,p} \eta^{-4(n+p)+1}|u|^{-1}.$$

After squaring and integrating in h, one obtains

$$\left\|\mathbb{1}_{[\eta,3]}e^{2\pi iu|x|^p}\right\|_{U^n} \lesssim_{n,p} \left(\eta + \eta^{-8(n+p)+1}|u|^{-2}\right)^{2^{-n}}.$$

The small interval contributes $O(\eta^{(n+1)2^{-n}})$, and choosing $\eta = |u|^{-1/(4(n+p))}$ proves (3.13).

Now let $0 < \eta < t < 1$. We claim

$$\|\sigma^\eta * \rho_t\|_{U^n(\mathbb{R}^d)} \lesssim_{n,p,d} t^{1/3}. \tag{3.14}$$

Choose an intermediate thickness $t < \tau < 1$ and write

$$\sigma^\eta * \rho_t = \sigma^\tau * \rho_t + (\sigma^\eta - \sigma^\tau) * \rho_t.$$

For the first term, use $\rho = \operatorname{div}(x\varphi)$, hence $\rho_t = t\sum_m \partial_m v_t^{(m)}$ with $v^{(m)}(x) = x_m\varphi(x)$. The convolution inequality for U^n norms and the bound $\|f\|_{U^n} \leqslant \|f\|_{L^{2^n/(n+1)}}$ give

$$\|\sigma^\tau * \rho_t\|_{U^n} \lesssim t\sum_m \|\partial_m\sigma^\tau\|_{L^{2^n/(n+1)}} \lesssim_{p,d} t\tau^{-2}.$$

For the second term, Fourier inversion in the radial variable gives

$$\sigma^\eta(x) - \sigma^\tau(x) = \int_{\mathbb{R}} (\widehat{\psi}(\eta u) - \widehat{\psi}(\tau u))e^{2\pi iu(\|x\|_p^p-1)}\,\mathrm{d}u.$$

On the fixed support $[-3,3]^d$ the phase separates into a product of one-dimensional phases. By the tensor property of the U^n norm and (3.13),

$$\|\sigma^\eta - \sigma^\tau\|_{U^n} \lesssim_{n,p} \int_{\mathbb{R}} \left|\widehat{\psi}(\eta u) - \widehat{\psi}(\tau u)\right| (1+|u|)^{-2d/D(n,p)}\,\mathrm{d}u.$$

Splitting the u-integral into $|u| < 1$, $1 \leqslant |u| < \tau^{-1}$, and $|u| \geqslant \tau^{-1}$, and using $d \geqslant D(n,p)$, gives

$$\|\sigma^\eta - \sigma^\tau\|_{U^n} \lesssim_{n,p,d} \tau.$$

Convolution with ρ_t costs only $\|\rho_t\|_1 = \|\rho\|_1$, so

$$\|\sigma^\eta * \rho_t\|_{U^n} \lesssim t\tau^{-2} + \tau.$$

Choosing $\tau = t^{1/3}$ proves (3.14).

We now estimate the uniform term. For $0 < \vartheta < \varepsilon$, (3.7), vague convergence of σ^η to σ, and Fubini give

$$\mathcal{N}_\lambda^\vartheta(A) - \mathcal{N}_\lambda^\varepsilon(A) = \int_\vartheta^\varepsilon \lim_{\eta\to 0^+} \int\int F(x,y)(\sigma_\lambda^\eta * \rho_{t\lambda})(y)\,\mathrm{d}y\,\mathrm{d}x\frac{\mathrm{d}t}{t}.$$

By scaling and (3.14),

$$\left\|\sigma_\lambda^\eta * \rho_{t\lambda}\right\|_{U^n} \lesssim_{n,p,d} \lambda^{-d(1-(n+1)2^{-n})} t^{1/3}.$$

The support of $\sigma_\lambda^\eta * \rho_{t\lambda}$ is contained in a cube of side $O(\lambda)$, so (3.12) cancels the power of λ and yields

$$\left|\int\int F(x,y)(\sigma_\lambda^\eta * \rho_{t\lambda})(y)\,\mathrm{d}y\,\mathrm{d}x\right| \lesssim_{n,p,d} t^{1/3},$$

uniformly in η. Letting $\eta \to 0^+$ and integrating,

$$\left|\mathcal{N}_\lambda^\vartheta(A) - \mathcal{N}_\lambda^\varepsilon(A)\right| \lesssim_{n,p,d} \int_\vartheta^\varepsilon t^{1/3}\frac{\mathrm{d}t}{t} \lesssim \varepsilon^{1/3}.$$

Finally let $\vartheta \to 0^+$, using $\mathcal{N}_\lambda^\vartheta(A) \to \mathcal{N}_\lambda^0(A)$. □

3.6 Obtaining the interval of scales

The three estimates above do not say that the error term is small at every scale. They say something weaker but sufficient: a long family of separated bad scales cannot exist. This is the compact analogue of the lacunary contradiction in Proposition 2.1.

Proof of Theorem 3.2 from the three estimates. Let $\Theta = \Theta_{n,p,d}(\delta)$ denote the structured lower bound from Proposition 3.5. Thus, with

$$L_\delta = \log(e/\delta), \qquad L_\delta^{(2)} = \log(eL_\delta),$$

one has

$$\Theta = \begin{cases} \exp\left(-EL_\delta^6 L_\delta^{(2)}\right), & n=3, \\ \exp(-\delta^{-E}), & n=4, \\ \exp\left(-\exp(L_\delta^E)\right), & n \geqslant 5. \end{cases}$$

Let G be the implicit constant in Proposition 3.8. Choose

$$\varepsilon = \left(\frac{\Theta}{3G}\right)^3, \qquad J = \left\lfloor (3\Theta^{-1}\varepsilon^{-F})^{2^{n-2}} \right\rfloor + 1,$$

where F is the exponent in Proposition 3.7. Then

$$G\varepsilon^{1/3} \leqslant \Theta/3, \qquad \varepsilon^{-F} J^{-2^{-n+2}} \leqslant \Theta/3.$$

We claim that some dyadic interval $(2^{-j}, 2^{-j+1}]$, $1 \leqslant j \leqslant J$, has small error at every scale in it:

$$\left|\mathcal{N}_\lambda^\varepsilon(A) - \mathcal{N}_\lambda^1(A)\right| \leqslant \varepsilon^{-F} J^{-2^{-n+2}} \quad \text{for all } \lambda \in (2^{-j}, 2^{-j+1}]. \tag{3.15}$$

If no such interval existed, then for each j one could choose $\lambda_j \in (2^{-j}, 2^{-j+1}]$ violating (3.15). Summing the violations would contradict Proposition 3.7. Hence such a j exists.

For every λ in this good dyadic interval, decomposition (3.5) and the three estimates give

$$\begin{aligned} \mathcal{N}_\lambda^0(A) &\geqslant \mathcal{N}_\lambda^1(A) - \left|\mathcal{N}_\lambda^\varepsilon(A) - \mathcal{N}_\lambda^1(A)\right| - \left|\mathcal{N}_\lambda^0(A) - \mathcal{N}_\lambda^\varepsilon(A)\right| \\ &\geqslant \Theta - \Theta/3 - \Theta/3 = \Theta/3 > 0. \end{aligned}$$

Thus every λ in $(2^{-j}, 2^{-j+1}]$ is an ℓ^p gap length of an n-term progression in A.

It remains only to translate the size of J into the displayed density dependence. Since ε is a fixed power of Θ, the choice of J gives

$$J \lesssim_{n,p,d} \Theta^{-C_0}$$

for some $C_0 = C_0(n,p,d)$. Therefore the good dyadic interval has length at least

$$2^{-J} \geqslant \exp(-C\Theta^{-C}).$$

Evaluating this expression in the three regimes yields exactly the theorem. If $n = 3$, then

$$\Theta^{-C} \leqslant \exp\Big(CL_\delta^6 L_\delta^{(2)}\Big),$$

so the length is at least $\exp[-\exp(CL_\delta^6 L_\delta^{(2)})]$. If $n = 4$, then $\Theta^{-C} \leqslant \exp(\delta^{-C})$, giving $\exp[-\exp(\delta^{-C})]$. Finally, if $n \geqslant 5$, then

$$\Theta^{-C} \leqslant \exp\big(C\exp(L_\delta^E)\big) \leqslant \exp\big(\exp(L_\delta^C)\big)$$

after increasing C, and the interval length is at least $\exp[-\exp(\exp(L_\delta^C))]$. $\square$

3.7 Open problem

Problem 3.9 (All large ℓ^p arithmetic progressions). Let $n \geqslant 4$ and

$$p \in [1,\infty) \setminus \{1,2,\dots,n-1\}.$$

One expects that, in sufficiently high dimension, every set of positive upper Banach density contains n-term arithmetic progressions with every sufficiently large ℓ^p gap. The compact interval theorem proved in this chapter gives a density-dependent interval of gaps inside the unit cube [10], but the corresponding all-large-scales theorem remains open in this generality.

Density theorems for configurations of prescribed volume

This chapter has two connected parts. The first concerns configurations whose scale is not specified by a common dilation factor but by a volume constraint: right simplices, rectangular boxes, parallelograms, and product-one hypercube graph embeddings. The second part turns to planar hyperbolic configurations, where the constraint is again a fixed area but the natural parameter is multiplicative rather than Euclidean.

The prescribed-volume material is drawn from the author's coloring and density theorems for configurations of a given volume [30]. The motivating questions go back to Graham and to Erdős–Mauldin-type problems in Euclidean Ramsey theory [22, 16, 23]. The hyperbolic-corner and unit-area-triangle results are due to Bulj and the author [5]. We first discuss positive density theorems, then the coloring and infinite-measure obstructions, and finally the analytic scale method for hyperbolic corners and triangles.

4.1 Right simplices

Theorem 4.1 (Quantitative density theorem for right simplices [30]). *For every integer $m \geqslant 2$ there is $C_m < \infty$ such that for every $d \geqslant m+1$:*

(a) If $A \subseteq [0,R]^d$ is measurable and

$$\frac{|A|}{R^d} \geqslant \left(\frac{C_m}{\log R}\right)^{1/(9m^2)},$$

then A contains the $m+1$ vertices of a right-angled m-simplex of unit m-volume.

(b) If $[0,R]^d$ is measurably colored in r colors and

$$R \geqslant \exp(C_m r^{9m^2}),$$

then some color class contains such a simplex.

The simplex can be chosen with $m-1$ perpendicular edges parallel to $e_1, \ldots, e_{m-1}$ and the last perpendicular edge lying in $\operatorname{span}(e_m, \ldots, e_d)$.

Corollary 4.2 (Positive upper Banach density version [30]). *Let $m \geqslant 2$ and $d \geqslant m+1$. If $A \subseteq \mathbb{R}^d$ is measurable with $\overline{\delta}(A) > 0$, then for every $V > 0$ the set A contains a right-angled m-simplex of m-volume V. Moreover the ratio of any two perpendicular edge lengths may be bounded by*

$$\exp\left(C'_m \overline{\delta}(A)^{-9m^2}\right).$$

Proof of the theorem. It is enough to prove the density statement in dimension $d = m+1$. Indeed, if $d > m+1$, then by Fubini at least one coordinate $(m+1)$-plane parallel to $\mathrm{span}(e_1, \dots, e_{m+1})$ has section-density at least $|A|/R^d$, and the lower-dimensional result applied to that section gives the required simplex in the original cube. The coloring statement follows afterwards from the density statement by applying it to a color class of measure at least R^d/r.

Write a point of $\mathbb{R}^{m+1}$ as (x, y) with $x \in \mathbb{R}^{m-1}$ and $y \in \mathbb{R}^2$. Let $A \subseteq [0, R]^{m+1}$ have density

$$\delta := \frac{|A|}{R^{m+1}} > 0, \qquad \theta := m^{-1} 2^{-m^2-m-1} \delta^{m+1}.$$

For $\lambda > 0$ define an exact counting form

$$\mathcal{N}^0_\lambda(A; R) := R^{-m-1} \lambda^{-m+1} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^2} \mathbb{1}_A(x, y) \prod_{k=1}^{m-1} \mathbb{1}_A(x + u_k e_k, y)$$
$$\times \int_{\mathbb{R}^2} \mathbb{1}_A(x, y + v)\, \mathrm{d}\sigma_{m!|u_1 \cdots u_{m-1}|^{-1}}(v) \prod_{k=1}^{m-1} \mathbb{1}_{\theta\lambda \leqslant |u_k| \leqslant \lambda}\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}u,$$

where σ_r is normalized arclength measure on the circle of radius r in $\mathbb{R}^2$. If $\mathcal{N}^0_\lambda(A; R) > 0$, then for some x, y, u, v all points

$$(x, y), \quad (x + u_1 e_1, y), \dots, (x + u_{m-1} e_{m-1}, y), \quad (x, y + v)$$

belong to A. They form a right simplex, because the displayed edge vectors are mutually orthogonal, and its m-volume is

$$\frac{|u_1 \cdots u_{m-1}|\, |v|}{m!} = \frac{|u_1 \cdots u_{m-1}|}{m!}\, m! |u_1 \cdots u_{m-1}|^{-1} = 1.$$

Thus the task is to make one exact count positive.

Let g_ε be a standard normalized Gaussian at scale ε and define $\mathcal{N}^\varepsilon_\lambda(A; R)$ by replacing σ_r by $\sigma_r * g_{\varepsilon r}$. Dominated convergence gives $\mathcal{N}^\varepsilon_\lambda \to \mathcal{N}^0_\lambda$ as $\varepsilon \to 0^+$. We use the decomposition

$$\mathcal{N}^0_\lambda = \mathcal{N}^1_\lambda + \big(\mathcal{N}^\varepsilon_\lambda - \mathcal{N}^1_\lambda\big) + \big(\mathcal{N}^0_\lambda - \mathcal{N}^\varepsilon_\lambda\big).$$

The three terms are controlled as follows.

First, if $R^{-1/(m-1)} \leqslant \lambda \leqslant R$, then

$$\mathcal{N}^1_\lambda(A; R) \geqslant c_m \delta^{(m+1)(2m-1)}. \tag{4.1}$$

Indeed, when $\theta\lambda \leqslant |u_k| \leqslant \lambda$, the smoothed circle kernel at radius $m!|u_1 \cdots u_{m-1}|^{-1}$ is bounded from below on a square of side comparable to λ^{-m+1} by a multiple of $(\theta\lambda)^{2(m-1)}$. Hence $\mathcal{N}^1_\lambda$ dominates, up to constants depending on m, the average number of parallelepiped-shaped boxes

$$I_1 \times \cdots \times I_{m-1} \times Q, \qquad |I_k| = \lambda, \qquad \ell(Q) = \lambda^{-m+1},$$

inside $[0, R]^{m+1}$ for which one point is chosen from A, each coordinate line through that point has another point in A, and the vertical two-dimensional fiber has another point in A. On each such box Hölder's inequality gives

$$\int \mathbb{1}_A(x, y) \prod_{k=1}^{m-1} \Big(\int_{I_k} \mathbb{1}_A(x_1, \dots, x_{k-1}, x'_k, x_{k+1}, \dots, y)\, \mathrm{d}x'_k \Big)$$

$$\times \Big(\int_Q \mathbb{1}_A(x,y')\,\mathrm{d}y'\Big)\,\mathrm{d}x\,\mathrm{d}y \gtrsim_m |A\cap(I_1\times\cdots\times I_{m-1}\times Q)|^{2m-1}.$$

Summing over the partition and using Jensen's inequality yields the lower bound in (4.1); the small exceptional region where some $|u_k|<\theta\lambda$ is swallowed by the choice of θ.

Second, for every $0<\varepsilon\leqslant 1$,

$$|\mathcal{N}^0_\lambda(A;R)-\mathcal{N}^\varepsilon_\lambda(A;R)|\leqslant C_m\varepsilon^{1/2}. \tag{4.2}$$

To see this, write $f(x,y;u)=\mathbb{1}_A(x,y)\prod_{k<m}\mathbb{1}_A(x+u_ke_k,y)$ and use the heat equation identity

$$\sigma_r-\sigma_r*g_{\varepsilon r}=\int_0^\varepsilon \sigma_r*k_{tr}\,\frac{\mathrm{d}t}{t},$$

where k_t is a cancellative Gaussian derivative. Taking the Fourier transform in the y variable and using the standard decay estimate for circular measure, one obtains

$$\left|\int f(x,y;u)\mathbb{1}_A(x,y+v)(\sigma_r*k_{tr})(v)\,\mathrm{d}v\,\mathrm{d}y\right|\lesssim t^{1/2}R^2.$$

After integrating in x,u and using the normalizing factor $R^{-m-1}\lambda^{-m+1}$, the integral in t is $\int_0^\varepsilon t^{-1/2}\,\mathrm{d}t$, which proves (4.2). This is the usual Fourier-decay estimate for the uniform part in Bourgain's method [3, 8].

Third, the error term satisfies the logarithmic square-function bound

$$\int_{\mathbb{R}}|\mathcal{N}^\varepsilon_{e^\alpha}(A;R)-\mathcal{N}^1_{e^\alpha}(A;R)|^2\,\mathrm{d}\alpha\leqslant C_m\theta^{-4(m-1)}\left(\log\frac{1}{\varepsilon}\right)^2. \tag{4.3}$$

Here one writes $\mathcal{N}^\varepsilon-\mathcal{N}^1$ as $\int_\varepsilon^1$ of the same heat-flow derivative. The cancellative kernel k is split as a sum of products of two first derivatives of Gaussians. Cauchy–Schwarz places one derivative on the product f and the other on the final indicator $\mathbb{1}_A(x,\cdot)$. The first factor is estimated crudely by support size; the second factor is evaluated by Plancherel. After the change of variables $\lambda=e^\alpha$, the interval of admissible Gaussian scales has bounded overlap in the (α,s) plane, and the remaining integral is bounded by $|A|\leqslant R^{m+1}$. This proves (4.3); it is the same square-function mechanism as in the Euclidean Szemerédi proofs of Bourgain and Cook–Magyar–Pramanik [3, 8].

We now finish the proof. Choose

$$\varepsilon=c_m\delta^{2(m+1)(2m-1)}$$

with $c_m>0$ small enough that (4.2) is at most one third of (4.1). By (4.3), if

$$J\geqslant C_m\theta^{-4(m-1)}\delta^{-2(m+1)(2m-1)}\left(\log\frac{1}{\varepsilon}\right)^2,$$

then for some $\beta\in[0,J]$ the error term at $\lambda=e^\beta$ is also at most one third of the structured term. Provided $R\geqslant e^J$, this scale lies in the range where the structured estimate applies. Therefore

$$\mathcal{N}^0_{e^\beta}(A;R)\geqslant\frac{1}{3}\mathcal{N}^1_{e^\beta}(A;R)>0,$$

and hence A contains the desired unit-volume right simplex. Since the displayed value of J is bounded by $C_m\delta^{-9m^2}$ after increasing C_m, the condition $R \geqslant e^J$ follows from

$$\delta \geqslant \left(\frac{C_m}{\log R}\right)^{1/(9m^2)}.$$

This proves the density statement. The coloring statement follows by taking a color class of density at least $1/r$ and applying the density statement when $R \geqslant \exp(C_m r^{9m^2})$. □

Proof of the corollary. It is enough to handle $V = 1$, since the dilation $V^{-1/m}A$ has the same upper Banach density as A, and scaling back by $V^{1/m}$ multiplies the simplex volume by V without changing ratios of perpendicular edge lengths. Put $\delta = \overline{\delta}(A)$ and choose

$$R_0 = \exp(2C_m\delta^{-9m^2}).$$

If some cube Q of side R_0 satisfies

$$\frac{|A \cap Q|}{|Q|} > \left(\frac{C_m}{\log R_0}\right)^{1/(9m^2)},$$

then the theorem applied to $A \cap Q$ gives a unit-volume right simplex in A. Its smallest perpendicular edge length is at least $m!R_0^{-m+1}$ when the product of the m edge lengths is $m!$, and every edge is at most R_0; hence the ratio of any two perpendicular edge lengths is at most $R_0^m/m! \leqslant \exp(C'_m\delta^{-9m^2})$.

If no such cube existed, then every large cube could be tiled by side-R_0 cubes plus a boundary remainder of relative measure $O(R_0/R)$. Taking suprema over translations and then $R \to \infty$ would give

$$\overline{\delta}(A) \leqslant \left(\frac{C_m}{\log R_0}\right)^{1/(9m^2)} < \delta,$$

contradicting the definition of δ. Thus the required cube exists, and the dilation argument gives the statement for every prescribed volume V. □

4.2 Rectangles and rectangular boxes

Theorem 4.3 (No monochromatic unit-area rectangle in 25 colors [30]). *There exists a Jordan-measurable coloring of $\mathbb{R}^2$ in 25 colors such that no color class contains the vertices of a rectangle of area 1.*

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$. For a parallelogram with vertices

$$z, \quad z+u, \quad z+u+v, \quad z+v,$$

consider the alternating complex invariant

$$I = z^2 - (z+u)^2 + (z+u+v)^2 - (z+v)^2 = 2uv.$$

If the parallelogram is a rectangle of area 1, then $u \perp v$ and $|u||v| = 1$, so $|I| = 2$. It is therefore enough to color the plane in such a way that a monochromatic parallelogram never has $|I| = 2$.

For $0 \leqslant j, k \leqslant 4$, define

$$\mathscr{C}_{j,k} = \left\{ z \in \mathbb{C} : z^2 \in \frac{10}{3}\left(\mathbb{Z} + i\mathbb{Z} + \frac{j+ik}{5} + \big[0, \tfrac{1}{5}\big) + i\big[0, \tfrac{1}{5}\big)\right)\right\}.$$

These 25 sets partition $\mathbb{C}$ up to boundaries, and the boundaries are preimages of grid lines under the polynomial map $z \mapsto z^2$, hence have planar measure zero. Thus the coloring is Jordan-measurable in the usual sense.

If all four vertices of a parallelogram are in the same class $\mathscr{C}_{j,k}$, then the four corresponding values of z^2 lie in translates of the same half-open square. Their alternating sum lies in

$$\frac{10}{3}\left(\mathbb{Z} + i\mathbb{Z} + (-\tfrac{2}{5}, \tfrac{2}{5}) + i(-\tfrac{2}{5}, \tfrac{2}{5})\right).$$

The square around the origin has radius at most $(10/3)(2/5)\sqrt{2} = 4\sqrt{2}/3 < 2$, while every other such square is separated from the origin by distance greater than 2. Hence this set is disjoint from the circle $\{|w| = 2\}$. Consequently no monochromatic parallelogram has $|uv| = 1$, and in particular no monochromatic rectangle has area 1. $\square$

Theorem 4.4 (Higher-dimensional box avoidance [30]). *For every positive integer d there exists a finite Jordan-measurable coloring of $\mathbb{R}^d$ such that, for every $m \leqslant d$, no color class contains the 2^m vertices of an m-dimensional rectangular box of m-volume 1.*

Proof. Fix $m \leqslant d$ first; at the end we refine the finitely many colorings obtained for $m = 1, \ldots, d$. If an m-box is based at $p \in \mathbb{R}^d$ and spanned by mutually orthogonal vectors $v_1, \ldots, v_m$, write its vertices as $p + \sum_{j \in T} v_j$, $T \subseteq \{1, \ldots, m\}$. Define

$$J(\mathcal{R}) = \sum_{T \subseteq \{1,\ldots,m\}} (-1)^{m-|T|} \prod_{k=1}^{m} \left(p_k + \sum_{j \in T} v_{j,k} \right).$$

The elementary identity

$$\sum_{T \subseteq \{1,\ldots,m\}} (-1)^{m-|T|} \prod_{k=1}^{m} \left(p_k + \sum_{j \in T} v_{j,k} \right) = \sum_{\pi \in S_m} \prod_{j=1}^{m} v_{j,\pi(j)} \tag{4.4}$$

is proved by expanding the product and observing that every monomial missing some index j cancels after summing over T, while the surviving monomials choose exactly one entry from each row v_j and exactly one coordinate k. In particular, for an axis-parallel box with edge lengths $a_1, \ldots, a_m$, one has $J = a_1 \cdots a_m$.

The same remains approximately true for boxes whose orientation is close to the standard one. Let $\varepsilon_0 = (2^{m+2} m!)^{-1}$. Suppose $v_j = a_j U e_j$ with $U \in SO(d)$ and $\|U - I\|_{\mathrm{op}} < \varepsilon_0$. Then $|v_{j,j} - a_j| < \varepsilon_0 a_j$ and $|v_{j,k}| < \varepsilon_0 a_j$ for $k \neq j$. In (4.4), the identity permutation contributes within $m\varepsilon_0(1+\varepsilon_0)^{m-1} a_1 \cdots a_m$ of $a_1 \cdots a_m$, while every non-identity permutation contains at least one off-diagonal entry and contributes at most $\varepsilon_0(1+\varepsilon_0)^{m-1} a_1 \cdots a_m$. Hence

$$|J(\mathcal{R}) - a_1 \cdots a_m| \leqslant \frac{1}{4} a_1 \cdots a_m.$$

For a unit-volume box this implies

$$J(\mathcal{R}) \in (-5/4, -3/4) \cup (3/4, 5/4),$$

up to the harmless sign coming from the choice of base vertex and parity convention.

Now partition $\mathbb{R}^d$ into

$$\mathscr{S}_l = \left\{ x \in \mathbb{R}^d : x_1 \cdots x_m \in \frac{3}{2}\left(\mathbb{Z} + \left[\tfrac{l}{3\cdot 2^m}, \tfrac{l+1}{3\cdot 2^m}\right)\right) \right\}, \qquad 0 \leqslant l < 3 \cdot 2^m.$$

If all vertices of a near-standard unit-volume m-box belonged to the same $\mathscr{S}_l$, then the alternating sum defining J would lie in

$$\frac{3}{2}\mathbb{Z} + (-1/4, 1/4),$$

which is disjoint from $(-5/4, -3/4) \cup (3/4, 5/4)$. Therefore this coloring forbids all unit-volume m-boxes whose orientation lies in a fixed neighborhood of the standard orientation.

For arbitrary orientations, compactness of $SO(d)$ supplies finitely many rotations $U_1, \ldots, U_L$ such that every $U \in SO(d)$ has $U_i^{-1}U$ within operator distance ε_0 of I for some i. Refine the rotated colorings $U_i\mathscr{S}_l$ over all i. If a unit-volume m-box were monochromatic for the refinement, rotating it back by the appropriate U_i^{-1} would produce a forbidden near-standard monochromatic box for the basic coloring. Finally, refine the finitely many colorings corresponding to $m = 1, \ldots, d$. This gives one finite Jordan-measurable coloring excluding all unit-volume rectangular boxes of all dimensions $m \leqslant d$. □

The negative result for fixed volume does not rule out sufficiently large volumes. In higher ambient dimension one has a positive density theorem.

Theorem 4.5 (Large rectangular boxes [30]). *Let m, d be positive integers with $d \geqslant m + 1$.*

(a) If $A \subseteq \mathbb{R}^d$ is measurable with $\overline{\delta}(A) > 0$, then there exists $V_0(A) > 0$ such that for every $V \geqslant V_0(A)$, A contains the vertices of an m-dimensional rectangular box of volume V.

(b) For every finite measurable coloring of $\mathbb{R}^d$, some color class contains such boxes of every sufficiently large volume.

The boxes can be chosen with $m-1$ edges in the first coordinate directions and the remaining edge in the span of the remaining coordinates.

Proof. It suffices to prove the density statement. The coloring statement follows from the fact that one color class in a finite measurable coloring has positive upper Banach density. Suppose, toward a contradiction, that A has positive upper Banach density and omits boxes of volumes λ_j^m for a sequence $\lambda_j \to \infty$. Passing to a subsequence, assume $\lambda_{j+1} \geqslant 2\lambda_j$. Choose a large cube $x_0 + [0, R]^d$ on which A has density at least $\delta > 0$; after translating, set $A_R = (A - x_0) \cap [0, R]^d$ and assume $R \geqslant \lambda_J$ for a large integer J to be chosen.

Write points as $(x, y) \in \mathbb{R}^{m-1} \times \mathbb{R}^{d-m+1}$. For $\lambda > 0$ define

$$\begin{aligned}\mathcal{N}_\lambda^0(A_R; R) = R^{-d}\lambda^{-m+1} \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{d-m+1}} \int_{\mathbb{R}^{m-1}} \prod_{r \in \{0,1\}^m} \mathbb{1}_{A_R}(x_1 + r_1 u_1, \ldots, x_{m-1} + r_{m-1}u_{m-1}, y + r_m v) \\ \times \, \mathrm{d}\sigma_{\lambda^m |u_1 \cdots u_{m-1}|^{-1}}(v) \prod_{k=1}^{m-1} \mathbb{1}_{\theta\lambda \leqslant |u_k| \leqslant \lambda} \, \mathrm{d}u \, \mathrm{d}y \, \mathrm{d}x,\end{aligned}$$

where $\theta = m^{-1}2^{-2^m d}\delta^{2^m}$. A positive value of $\mathcal{N}_\lambda^0$ gives a rectangular box whose first $m-1$ edge lengths are $|u_k|$ and whose last edge length is $\lambda^m |u_1 \cdots u_{m-1}|^{-1}$, hence whose m-volume is exactly λ^m.

Smooth the spherical measure as before and decompose $\mathcal{N}^0 = \mathcal{N}^1 + (\mathcal{N}^\varepsilon - \mathcal{N}^1) + (\mathcal{N}^0 - \mathcal{N}^\varepsilon)$. Three estimates are needed. First, for $0 < \lambda \leqslant R$,

$$\mathcal{N}^1_\lambda(A_R; R) \geqslant c_{m,d}(\delta) > 0. \tag{4.5}$$

Indeed, for $\theta\lambda \leqslant |u_k| \leqslant \lambda$ the smoothed spherical kernel dominates a constant multiple of $\lambda^{-d+m-1}\mathbb{1}_{[-\lambda,\lambda]^{d-m+1}}$. Partition $[0, R]$ in the first $m-1$ coordinates into intervals of length λ and the remaining coordinate block into cubes of side λ. On each product cell Q the Gowers-box Cauchy–Schwarz inequality, the product-form analogue of Proposition A.5, gives

$$\int_{Q^{\{0,1\}^m}} \prod_{r\in\{0,1\}^m} \mathbb{1}_{A_R}(z_r)\,\mathrm{d}z \geqslant |Q|^{2-2^m}|A_R \cap Q|^{2^m}.$$

Summing in Q and applying Jensen's inequality yields a lower bound of size $(R\lambda)^d\delta^{2^m}$. The exceptional region where one $|u_k| < \theta\lambda$ contributes at most $m\theta(R\lambda)^d$ and is smaller by the definition of θ, giving (4.5).

Second,

$$|\mathcal{N}^0_\lambda(A_R; R) - \mathcal{N}^\varepsilon_\lambda(A_R; R)| \leqslant C_{m,d}\varepsilon^{1/2}. \tag{4.6}$$

This is the same Fourier-decay estimate as for right simplices: after fixing x, u, Fourier transform in the y variable and use the spherical estimate for $\sigma * k_t$ in dimension $d-m+1 \geqslant 2$; the normalizations cancel the sizes of the x and u regions, leaving $\int_0^\varepsilon t^{-1/2}\,\mathrm{d}t$.

Third, for separated scales,

$$\sum_{j=1}^{J} |\mathcal{N}^\varepsilon_{\lambda_j}(A_R; R) - \mathcal{N}^1_{\lambda_j}(A_R; R)| \leqslant C_{m,d,\delta,\varepsilon}. \tag{4.7}$$

The proof is a soft singular Brascamp–Lieb argument. The heat-flow representation of $\mathcal{N}^\varepsilon - \mathcal{N}^1$ replaces one Gaussian by a cancellative derivative. Splitting the derivative into two first derivatives and applying Cauchy–Schwarz reduces the scale sum to a positive cubical form

$$\Theta^{(k)}_\gamma(F) = -\int_0^\infty \int \prod_{r\in\{0,1\}^m} F(z_r) k_{s\gamma_k}(z^0_k - z^1_k) \prod_{i\neq k} g_{s\gamma_i}(z^0_i - z^1_i)\,\mathrm{d}z\,\frac{\mathrm{d}s}{s},$$

with $F = \mathbb{1}_{A_R}$. Positivity follows from $-k_s = 2\sum_l h^{(l)}_{s/\sqrt{2}} * h^{(l)}_{s/\sqrt{2}}$. Also, differentiating the fully Gaussian cubical average in s gives

$$\sum_{k=1}^{m} \Theta^{(k)}_\gamma(F) = 2\pi\,\|F\|^{2^m}_{L^{2^m}}.$$

Thus every $\Theta^{(k)}_\gamma(F)$ is at most $2\pi|A_R| \leqslant 2\pi R^d$. The intervals of derivative scales associated with the separated λ_j have bounded overlap depending on ε; after the prefactor R^{-d} this proves (4.7). This is the cubical singular Brascamp–Lieb estimate in its heat-flow form, related to [9, 14].

Now choose $\varepsilon = \varepsilon(\delta) > 0$ so small that (4.6) is at most one third of (4.5), and then choose $J = J(\delta, \varepsilon)$ so large that (4.7) and the pigeonhole principle provide some $j \leqslant J$ for which the error term is also at most one third of the structured term. For this j,

$$\mathcal{N}^0_{\lambda_j}(A_R; R) \geqslant \frac{1}{3}\mathcal{N}^1_{\lambda_j}(A_R; R) > 0,$$

giving such a box inside A_R and hence inside the original set A. This contradicts the assumed absence of boxes of volume λ_j^m. Therefore a positive-density set contains boxes of every sufficiently large volume. $\square$

4.3 Hypercube graph obstruction

Theorem 4.6 (Avoiding hypercube graph embeddings with product-one edge lengths [30]). *For every positive integer n there exists a Jordan-measurable finite coloring of $\mathbb{R}^2$ such that no color class contains an embedding*

$$z + r_1 u_1 + \cdots + r_n u_n, \qquad (r_1, \ldots, r_n) \in \{0,1\}^n,$$

of the 1-skeleton of an n-box with edge lengths $a_i = |u_i|$ satisfying

$$a_1 a_2 \cdots a_n = 1.$$

Proof. The one-variable specialization of (4.4) is

$$\sum_{T \subseteq \{1,\ldots,n\}} (-1)^{n-|T|} \left(z + \sum_{j \in T} u_j \right)^n = n! u_1 u_2 \cdots u_n.$$

Color $\mathbb{C}$ by

$$\mathscr{C}_{j,k} = \left\{ z \in \mathbb{C} : z^n \in 2n! \left(\mathbb{Z} + i\mathbb{Z} + \frac{j+ik}{2^{n+1}} + [0, 2^{-n-1}) + i[0, 2^{-n-1}) \right) \right\},$$

where $0 \leqslant j, k < 2^{n+1}$. If all 2^n vertices $z + \sum_{j \in T} u_j$ have the same color, then the alternating sum of their nth powers belongs to

$$2n! \left(\mathbb{Z} + i\mathbb{Z} + (-1/2, 1/2) + i(-1/2, 1/2) \right).$$

After division by $n!$, this says

$$u_1 \cdots u_n \in 2(\mathbb{Z} + i\mathbb{Z}) + (-1/2, 1/2) + i(-1/2, 1/2).$$

The latter set is disjoint from the unit circle: its central square lies strictly inside the unit disk, while every other square has distance greater than 1 from the origin. Hence $|u_1 \cdots u_n| \neq 1$, so the product of edge lengths cannot equal 1. □

4.4 Hyperbolic corners

The following sections are based on Bulj and the author [5]. The proof keeps Bourgain's scale-pigeonholing philosophy [3, 4], but the uniform estimate is supplied by a hyperbolic variant of the trilinear smoothing theory of Christ–Durcik–Roos [6].

A hyperbolic corner is a triple

$$(x, y), \quad (x + t, y), \quad (x, y + t^{-1}), \qquad t > 0.$$

It is an upward axis-aligned right triangle of area 1/2, but the parameter t makes the horizontal and vertical scales reciprocal.

Theorem 4.7 (Extremal size of sets without hyperbolic corners [5]). *Let $M(R)$ be the supremum of $|A|$ over measurable $A \subseteq [0, R]^2$ containing no hyperbolic corner. For $R \geqslant 10$,*

$$R \log R \lesssim M(R) \lesssim R^2 \left(\frac{\log \log R}{\log R} \right)^{1/4}.$$

Proof. We first prove the lower bound. Let $m = \lfloor R/4 \rfloor$ and let

$$S_j = \left\{ (x,y) \in [0,R]^2 : R - 4j \leqslant x + y \leqslant R - 4j + \frac{1}{8j} \right\}, \qquad 1 \leqslant j \leqslant m.$$

Set $A_R = \bigcup_{j=1}^m S_j$. The bands are disjoint, and the area of a strip $a \leqslant x + y \leqslant b$ in the relevant triangle is $(b^2 - a^2)/2$. Hence

$$|A_R| = \frac{1}{2} \sum_{j=1}^m \left(\left(R - 4j + \frac{1}{8j} \right)^2 - (R - 4j)^2 \right) = \frac{1}{8} R \log R + O(R).$$

If $(x,y) \in S_j$ and both $(x+t, y)$ and $(x, y+t^{-1})$ were in the same band S_j, then $t \leqslant 1/(8j)$ and $t^{-1} \leqslant 1/(8j)$, impossible. If both moved to earlier bands S_k, $k < j$, the gaps of size almost 4 between bands would force $t > 3$ and $t^{-1} > 3$, also impossible. If one point stays in S_j and the other moves to an earlier band, then, for instance, $t \leqslant 1/(8j)$ while $t^{-1} < 4j$, again impossible. The remaining mixed case is symmetric. Thus A_R contains no hyperbolic corner and has measure $\gtrsim R \log R$.

For the upper bound, let $A \subseteq [0,R]^2$ be corner-free and put $\delta = |A|/R^2$. Fix smooth nonnegative functions ζ and ϕ with ζ supported in $[1/2, 2]$, $\int \phi = 1$, and ϕ bounded below on a fixed neighborhood of the origin. For $\lambda > 0$ define

$$\mathcal{N}_\lambda^0(A) = \int_0^\infty \int_{\mathbb{R}^2} \mathbb{1}_A(x,y) \mathbb{1}_A(x + \lambda u, y) \mathbb{1}_A \left(x, y + \frac{1}{\lambda u} \right) \zeta(u) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}u.$$

If A is corner-free, then $\mathcal{N}_\lambda^0(A) = 0$ for every $\lambda > 0$. Define the smoothed form

$$\mathcal{N}_\lambda^\varepsilon(A) = \mathcal{N}_\lambda^0 \big(\mathbb{1}_A, \mathbb{1}_A *_1 \phi_{\lambda\varepsilon}, \mathbb{1}_A *_2 \phi_{\lambda^{-1}\varepsilon} \big),$$

where $*_1$ and $*_2$ denote convolution in the first and second coordinate. The decomposition is

$$\mathcal{N}_\lambda^0 = \mathcal{N}_\lambda^1 + (\mathcal{N}_\lambda^\varepsilon - \mathcal{N}_\lambda^1) + (\mathcal{N}_\lambda^0 - \mathcal{N}_\lambda^\varepsilon).$$

The structured term obeys, for $\lambda \in [1/R, R]$,

$$\mathcal{N}_\lambda^1(A) \geqslant c \frac{|A|^3}{R^4} = c R^2 \delta^3. \tag{4.8}$$

Indeed, $\mathcal{N}_\lambda^1$ has a positive kernel $K_\lambda^+(x - x', y - y')$ satisfying

$$K_\lambda^+ \gtrsim \mathbb{1}_{[-\lambda,\lambda] \times [-\lambda^{-1}, \lambda^{-1}]}.$$

Partition $[0,R]^2$ into rectangles $Q = I \times J$ with $|I| \leqslant \lambda$ and $|J| \leqslant \lambda^{-1}$. On each Q, Hölder gives

$$\int_{I \times I \times J \times J} \mathbb{1}_A(x,y) \mathbb{1}_A(x',y) \mathbb{1}_A(x,y') \, \mathrm{d}x \, \mathrm{d}x' \, \mathrm{d}y \, \mathrm{d}y' \geqslant |A \cap Q|^3.$$

Since the number of rectangles is $O(R^2)$ and $\sum_Q |A \cap Q| = |A|$, Jensen's inequality gives (4.8).

The error term satisfies

$$\left(\int_0^\infty |\mathcal{N}_\lambda^\varepsilon(A) - \mathcal{N}_\lambda^1(A)|^2 \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/2} \leqslant C|A| \left(\log \frac{1}{\varepsilon} \right)^{1/2}. \tag{4.9}$$

To prove it, expand the difference between the two smoothing scales as a sum of a horizontal and a vertical term:

$$(\phi_{\lambda\varepsilon}-\phi_\lambda)\phi_{\lambda^{-1}\varepsilon}+\phi_\lambda(\phi_{\lambda^{-1}\varepsilon}-\phi_{\lambda^{-1}}).$$

For the horizontal part, Cauchy–Schwarz in (x,y) gives

$$|E_1(\lambda)| \lesssim |A|^{1/2} \, \|\mathbb{1}_A *_1 (\phi_{\lambda\varepsilon}-\phi_\lambda)\|_2 \, .$$

Integrating $|E_1(\lambda)|^2$ in $\mathrm{d}\lambda/\lambda$ and using Plancherel reduces the estimate to

$$\sup_{\xi\neq 0}\int_0^\infty |\widehat{\phi}(\varepsilon\lambda\xi)-\widehat{\phi}(\lambda\xi)|^2\frac{\mathrm{d}\lambda}{\lambda}\lesssim \log\frac{1}{\varepsilon}.$$

This follows by the change of variables $s=\lambda|\xi|$, smoothness at the origin, and rapid decay at infinity. The vertical part is identical after replacing λ by λ^{-1}. This proves (4.9).

For the uniform term, a hyperbolic trilinear smoothing estimate gives some $\sigma>0$ such that

$$|\mathcal{N}^0_\lambda(A)-\mathcal{N}^\varepsilon_\lambda(A)|\leqslant C|A|\varepsilon^\sigma = CR^2\delta\varepsilon^\sigma. \tag{4.10}$$

The local smoothing input is

$$|\mathcal{N}(f_0,f_1,f_2)|\lesssim \|f_0\|_\infty \, \|f_1\|_{H^{-\sigma,0}} \, \|f_2\|_{H^{0,-\sigma}} \, ,$$

for functions supported on a fixed square. To derive (4.10), anisotropically rescale by $(x,y)=(\lambda x',\lambda^{-1}y')$, so all λ become 1. Decompose $f-f *_1 \phi_\varepsilon$ and $f-f*_2\phi_\varepsilon$, localize by a bounded-overlap partition of unity, and apply the smoothing estimate on each bounded window. Since

$$\|f-f*_1\phi_\varepsilon\|_{H^{-\sigma,0}}\lesssim \varepsilon^\sigma\,\|f\|_2\,, \qquad \|f-f*_2\phi_\varepsilon\|_{H^{0,-\sigma}}\lesssim \varepsilon^\sigma\,\|f\|_2\,,$$

the localized square sum is bounded by $\varepsilon^\sigma\,\|f\|_2^2=\varepsilon^\sigma|A|$. This proves (4.10); the underlying smoothing theorem is the hyperbolic analogue of Christ–Durcik–Roos [6].

Now choose $\varepsilon=c_0\delta^{2/\sigma}$ with c_0 small. Since $\mathcal{N}^0_\lambda(A)=0$, (4.10) implies

$$\mathcal{N}^\varepsilon_\lambda(A)\leqslant \frac{1}{2}cR^2\delta^3\leqslant\frac{1}{2}\mathcal{N}^1_\lambda(A) \qquad (1/R\leqslant\lambda\leqslant R).$$

Therefore $|\mathcal{N}^\varepsilon_\lambda(A)-\mathcal{N}^1_\lambda(A)|\gtrsim R^2\delta^3$ throughout this interval. Squaring, integrating in $\mathrm{d}\lambda/\lambda$, and applying (4.9) gives

$$R^4\delta^6\log R\lesssim R^4\delta^2\log\frac{1}{\varepsilon}\lesssim R^4\delta^2\left(1+\log\frac{1}{\delta}\right).$$

Hence $\delta^4\log R\lesssim 1+\log(1/\delta)$, and solving this elementary inequality gives

$$\delta\lesssim\left(\frac{\log\log R}{\log R}\right)^{1/4}.$$

Thus $|A|=\delta R^2\lesssim R^2(\log\log R/\log R)^{1/4}$, as claimed. $\square$

4.5 Unit-area triangles

Theorem 4.8 (Sets without unit-area triangles [5]). *Let $M_\Delta(R)$ be the supremum of $|A|$ over measurable $A \subseteq [0,R]^2$ containing no triple of points spanning a triangle of area 1. For $R \geqslant 10$,*

$$M_\Delta(R) \lesssim R^2 \left(\frac{\log\log R}{\log R}\right)^{1/2}.$$

Proof. After a fixed dilation it is equivalent to exclude triangles of area 1/2. Define the horizontal form

$$\overrightarrow{\mathcal{M}}_\lambda^0(A) = \int_0^\infty \int_{\mathbb{R}^3} \mathbb{1}_A(x,y)\mathbb{1}_A(x+\lambda u, y)\mathbb{1}_A\left(x', y + \frac{1}{\lambda u}\right) \zeta(u)\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}u,$$

and average it over rotations:

$$\mathcal{M}_\lambda^0(A) = \frac{1}{2\pi}\int_0^{2\pi} \overrightarrow{\mathcal{M}}_\lambda^0(\mathcal{R}_\theta A)\,\mathrm{d}\theta.$$

A positive value detects three points spanning area 1/2, because the two non-horizontal vertices have vertical separation $(\lambda u)^{-1}$ while the horizontal base has length λu. Thus, if A has no such triangle, $\mathcal{M}_\lambda^0(A) = 0$ for all $\lambda > 0$. Define $\mathcal{M}_\lambda^\varepsilon$ by the same one-dimensional smoothings as in the hyperbolic-corner proof.

The structured estimate is stronger than for corners:

$$\mathcal{M}_\lambda^1(A) \geqslant c\frac{|A|^3}{R^3} = cR^3\delta^3, \qquad \delta = |A|/R^2, \tag{4.11}$$

whenever $1/R \leqslant \lambda \leqslant R$. It suffices to prove the bound for the horizontal form and then average rotations. The positive kernel again dominates $\mathbb{1}_{[-\lambda,\lambda]\times[-\lambda^{-1},\lambda^{-1}]}$. Partition the x-axis into intervals I of length λ and the y-axis into intervals J of length λ^{-1}. Since the third point has an independent horizontal coordinate x', the contribution of I, J contains

$$|A \cap ([0,R]\times J)| \int_J \sum_I \left(\int_I \mathbb{1}_A(x,y)\,\mathrm{d}x\right)^2 \mathrm{d}y.$$

Cauchy–Schwarz in x, then Jensen in J, gives the lower bound $|A|^3/R^3$, proving (4.11).

The error term is controlled by the Riesz energy

$$\mathcal{E}(A) = \iint \frac{\mathbb{1}_A(z)\mathbb{1}_A(z')}{|z-z'|}\,\mathrm{d}z\,\mathrm{d}z'.$$

More precisely,

$$\left(\int_0^\infty |\mathcal{M}_\lambda^\varepsilon(A) - \mathcal{M}_\lambda^1(A)|^2 \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/2} \leqslant C\mathcal{E}(A)\left(\log\frac{1}{\varepsilon}\right)^{1/2}. \tag{4.12}$$

For the horizontal form, let $G(y) = \int \mathbb{1}_A(x,y)\,\mathrm{d}x$. The same Plancherel calculation as in (4.9) gives

$$\left(\int_0^\infty |\overrightarrow{\mathcal{M}}_\lambda^\varepsilon(A) - \overrightarrow{\mathcal{M}}_\lambda^1(A)|^2 \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/2} \lesssim \left(\log\frac{1}{\varepsilon}\right)^{1/2} \int_{\mathbb{R}} G(y)^2\,\mathrm{d}y.$$

Apply this to every rotation of A and average in the angle. Expanding the square of the X-ray transform and changing variables from (θ, x, x', y) to the two planar points z, z' gives the Jacobian $|z - z'|$. Consequently

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} G_\theta(y)^2 \,\mathrm{d}y\,\mathrm{d}\theta \lesssim \iint \frac{\mathbb{1}_A(z)\mathbb{1}_A(z')}{|z-z'|}\,\mathrm{d}z\,\mathrm{d}z',$$

which proves (4.12).

The uniform term satisfies, for the same smoothing exponent $\sigma > 0$ as before,

$$|\mathcal{M}_\lambda^0(A) - \mathcal{M}_\lambda^\varepsilon(A)| \leqslant C\varepsilon^\sigma R|A| = C\varepsilon^\sigma R^3\delta. \tag{4.13}$$

For the horizontal form, replace the third indicator by the horizontal-section average

$$g(x,y) = \mathbb{1}_{[0,R]^2}(x,y)\frac{1}{R}\int_0^R \mathbb{1}_A(x',y)\,\mathrm{d}x'.$$

Then $\overrightarrow{\mathcal{M}}$ is R times a corner-type form with inputs $\mathbb{1}_A, \mathbb{1}_A, g$. The proof of (4.10) applies to these inputs and gives $R\varepsilon^\sigma \|\mathbb{1}_A\|_2 \|g\|_2 \lesssim R\varepsilon^\sigma|A|$. Averaging rotations gives (4.13).

Choose $\varepsilon = c_0\delta^{2/\sigma}$ so that the uniform term is at most one half of the structured term. Since $\mathcal{M}_\lambda^0(A) = 0$, estimates (4.11)–(4.12) imply

$$R^6\delta^6\log R \lesssim \mathcal{E}(A)^2\left(1 + \log\frac{1}{\delta}\right). \tag{4.14}$$

Hardy–Littlewood–Sobolev alone would give $\mathcal{E}(A) \lesssim |A|^{3/2} = R^3\delta^{3/2}$ and only the weaker exponent $1/3$. To obtain the stated exponent, we improve the energy estimate by induction on the side length.

Let

$$\eta(R) = \left(\frac{\log\log R}{\log R}\right)^{1/2}.$$

We prove, for sufficiently large absolute constants C and R_0, that every triangle-free $A \subseteq [0,R]^2$, $R \geqslant R_0$, satisfies $|A| \leqslant CR^2\eta(R)$. Assume this is already known for all side lengths between R_0 and $R/2$. For a fixed $z \in A$, use the layer-cake identity

$$\int \frac{\mathbb{1}_A(z')}{|z-z'|}\,\mathrm{d}z' \leqslant \int_0^{2R}\frac{|A \cap \mathrm{B}(z,r)|}{r^2}\,\mathrm{d}r + \frac{|A|}{2R}.$$

For $r < R_0$ we use the trivial area bound, for $R_0 \leqslant r \leqslant R/4$ we apply the induction hypothesis to $A \cap \mathrm{B}(z,r)$ inside a square of side comparable to r, and for $r \geqslant R/4$ we use $|A| = \delta R^2$. This gives

$$\int \frac{\mathbb{1}_A(z')}{|z-z'|}\,\mathrm{d}z' \leqslant C'R\big(\delta + C\eta(R)\big).$$

Integrating in $z \in A$ yields

$$\mathcal{E}(A) \leqslant C'R^3\delta\big(\delta + C\eta(R)\big). \tag{4.15}$$

Combining (4.14) and (4.15) gives

$$\delta^2 \leqslant C''\frac{(1+\log(1/\delta))^{1/2}}{(\log R)^{1/2}}\big(\delta + C\eta(R)\big).$$

If C is chosen sufficiently large, this inequality forces $\delta \leqslant C\eta(R)$; otherwise the left side dominates both terms on the right for large R. This closes the induction. The finitely many base scales $R \leqslant R_0$ are absorbed by increasing C. Therefore $|A| \lesssim R^2(\log\log R/\log R)^{1/2}$. □

4.6 Open problems

Problem 4.9 (Rectangular boxes in the critical dimension). For m-dimensional rectangular boxes of sufficiently large volume, the positive theorem assumes ambient dimension $d \geqslant m+1$. The critical case $d = m \geqslant 2$ remains open [30]. Analytically, the associated multilinear form lies just beyond the singular-integral and Brascamp–Lieb estimates available in the positive theorem.

Problem 4.10 (Parallelograms of prescribed area). It is open whether every finite coloring of $\mathbb{R}^2$ has a color class containing the vertices of a parallelogram of every prescribed area [30]. The partial negative theorem shows that any positive proof must deal with nearly degenerate parallelograms and with infinitely many directions, rather than only with a compact family of shapes.

Problem 4.11 (Unit-area triangles in sets of large finite measure). Erdős asked whether there is an absolute constant C such that every planar measurable set of area greater than C contains the vertices of a triangle of area 1 [16]. The upper bounds in this chapter are certain progress, but they remain far from the conjectural $O(1)$ threshold.

Density theorems for large point configurations

After the prescribed-volume examples, we return to arbitrary finite point patterns. The guiding question is how dense a set must be in order to force every sufficiently large copy of every pattern of a fixed size. Here we are mainly interested in negative results. We refine Bourgain's annular obstruction [3], but also add a highly nontrivial input from equidistribution theory and Diophantine approximations.

5.1 The threshold problem and polynomial equidistribution

For fixed d and n, let $\rho_{\min}(d,n)$ be the smallest threshold ρ_* such that every measurable $A \subseteq \mathbb{R}^d$ with $\overline{\delta}(A) > \rho_*$ contains, for every n-point pattern $P \subseteq \mathbb{R}^d$, all sufficiently large similar copies of P. The elementary translated-copy argument of Corollary 1.6 gives

$$\rho_{\min}(d,n) \leqslant 1 - \frac{1}{n},$$

and Falconer, the author, and Yavicoli [17] improved this (slighly but with nontrivial effort) to

$$\rho_{\min}(d,n) \leqslant 1 - \frac{1}{n-1}.$$

We recommend [17] to the interested reader, but do not discuss this improvement here. It is natural to wonder about the asymptotic behavior of these numbers as $n \to \infty$. The near-optimal lower bound known at present is the following.

Theorem 5.1 (Near-optimal Euclidean lower bound [32]). *There exists an absolute constant $C > 0$ such that, for every $d \geqslant 1$ and every sufficiently large n, there exist a measurable set $E \subseteq \mathbb{R}^d$, an n-point configuration $P \subseteq \mathbb{R}^d$ contained in a line, and a sequence $\lambda_j \to \infty$ such that*

$$\mathrm{d}(E) \geqslant 1 - C\frac{\log n}{n},$$

and E contains no Euclidean isometric copy of $\lambda_j P$ for any j.

We prove a slightly more general ℓ^p theorem. Its number-theoretic core is a finite polynomial hitting statement on the circle. Write $\|x\|_{\mathbb{T}} = \operatorname{dist}(x,\mathbb{Z})$ for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $e(t) = e^{2\pi i t}$.

Proposition 5.2 (Uniform polynomial hitting). *Fix an integer $p \geqslant 2$. There is a constant $K_p < \infty$ such that, for every sufficiently large n, one can find a set $P \subseteq \mathbb{Z}$ with $|P| = n$ and a real number α with the following property. For every $B_1, \ldots, B_{p-1} \in \mathbb{R}$, the set*

$$\{\alpha k^p + B_{p-1}k^{p-1} + \cdots + B_1 k \bmod 1 : k \in P\}$$

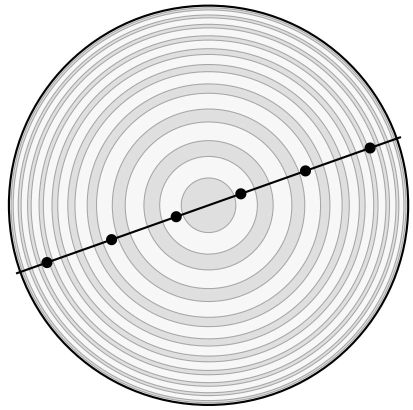

Figure 5.1: A typical annular construction underlying negative density results, together with a collinear pattern that is forced to interact with the annular gaps.

meets every interval in $\mathbb{T}$ *of length*

$$\varepsilon_n = K_p \frac{\log n}{n}.$$

Proof. Choose a prime Q satisfying

$$n^{2^p} < Q < 2n^{2^p},$$

which is possible for all large n by Bertrand's postulate, and put $\alpha = 1/Q$. For $B = (B_1, \dots, B_{p-1}) \in \mathbb{T}^{p-1}$ define

$$x_k(B) = \alpha k^p + B_{p-1}k^{p-1} + \cdots + B_1 k \pmod 1, \qquad 0 \leqslant k < Q.$$

We first show that the full collection $\{x_k(B) : 0 \leqslant k < Q\}$ is well distributed, uniformly in B.

Let $D_Q(B)$ be the discrepancy of this collection. The Erdős–Turán inequality [33] gives, for any $M \geqslant 1$,

$$D_Q(B) \lesssim \frac{1}{M} + \sum_{m=1}^{M} \frac{1}{m} \left| \frac{1}{Q} \sum_{k=0}^{Q-1} e(mx_k(B)) \right|.$$

Take $M = Q - 1$. For $1 \leqslant m \leqslant Q - 1$, the leading coefficient of the polynomial $mx_k(B)$ is m/Q, already in lowest terms because Q is prime. Weyl's inequality [51], applied with denominator Q and with a small exponent loss $\eta = 2^{-p}$, yields

$$\left| \sum_{k=0}^{Q-1} e(mx_k(B)) \right| \lesssim_p Q^{1-2^{-p}},$$

uniformly in m and in all lower-order coefficients. Hence

$$D_Q(B) \lesssim_p \frac{1}{Q} + Q^{-2^{-p}} \sum_{m=1}^{Q-1} \frac{1}{m} \lesssim_p \frac{\log Q}{Q^{2^{-p}}} \lesssim_p \frac{\log n}{n}.$$

Choose K_p so large that $D_Q(B) \leqslant \varepsilon_n/10$ for every B. Then every interval $I \subseteq \mathbb{T}$ of length $9\varepsilon_n/10$ contains at least

$$\left(\frac{9\varepsilon_n}{10} - \frac{\varepsilon_n}{10} \right) Q = \frac{4}{5}\varepsilon_n Q$$

of the points $x_k(B)$.

We now discretize the coefficient torus. For $i = 1, \dots, p-1$, choose a grid $\mathcal{N}_i \subseteq \mathbb{T}$ of mesh

$$\Delta_i = \frac{\varepsilon_n}{100pQ^i},$$

so $|\mathcal{N}_i| \lesssim Q^i/\varepsilon_n$. Put $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_{p-1}$; then

$$|\mathcal{N}| \lesssim_p \frac{Q^{p(p-1)/2}}{\varepsilon_n^{p-1}}.$$

Let $\mathcal{I}$ be the family of intervals of length $9\varepsilon_n/10$ whose left endpoints are integer multiples of $\varepsilon_n/100$. Then $|\mathcal{I}| \lesssim 1/\varepsilon_n$.

For $b \in \mathcal{N}$ and $I \in \mathcal{I}$, set

$$R_{b,I} = \{0 \leqslant k < Q : x_k(b) \in I\}.$$

The discrepancy estimate gives $|R_{b,I}| \geqslant (4/5)\varepsilon_n Q$. Choose an n-element subset $P \subseteq \{0, 1, \dots, Q-1\}$ uniformly at random. The probability that P misses this particular $R_{b,I}$ is at most

$$\frac{\binom{Q-|R_{b,I}|}{n}}{\binom{Q}{n}} \leqslant \left(1 - \frac{|R_{b,I}|}{Q}\right)^n \leqslant \exp\left(-\frac{4}{5}\varepsilon_n n\right).$$

By the union bound, the probability that some pair (b, I) is missed is at most

$$|\mathcal{N}|\,|\mathcal{I}| \exp\left(-\frac{4}{5}\varepsilon_n n\right) \lesssim_p \frac{Q^{p(p-1)/2}}{\varepsilon_n^p} \exp\left(-\frac{4}{5}K_p \log n\right).$$

Since $Q \asymp n^{2^p}$ and $\varepsilon_n = K_p(\log n)/n$, the last expression is < 1 for all large n if K_p is chosen sufficiently large. Therefore there exists a deterministic n-element set P such that

$$\{x_k(b) : k \in P\} \cap I \neq \emptyset \qquad \text{for every } b \in \mathcal{N},\ I \in \mathcal{I}. \tag{5.1}$$

It remains to pass from grid coefficients to arbitrary coefficients. Let $B \in \mathbb{T}^{p-1}$ and let $J \subseteq \mathbb{T}$ be an interval of length ε_n. Choose $b \in \mathcal{N}$ with $\|B_i - b_i\|_{\mathbb{T}} \leqslant \Delta_i$. Since $P \subseteq \{0, \dots, Q-1\}$,

$$\|x_k(B) - x_k(b)\|_{\mathbb{T}} \leqslant \sum_{i=1}^{p-1} \Delta_i Q^i \leqslant \frac{\varepsilon_n}{100} \qquad (k \in P).$$

Let $J' \subseteq J$ be the concentric subinterval of length $96\varepsilon_n/100$. It contains some $I \in \mathcal{I}$. By (5.1), choose $k \in P$ with $x_k(b) \in I \subseteq J'$. The preceding estimate then gives $x_k(B) \in J$. Thus the desired polynomial set meets every interval of length ε_n. □

5.2 Collinear ℓ^p obstructions

We shall use two elementary facts about ℓ^p geometry.

Lemma 5.3 (Collinear copies remain collinear). *Let $1 < p < \infty$, let $\lambda > 0$, and let $P \subseteq \mathbb{R}$ be finite. Suppose that points $y_t \in \mathbb{R}^d$, indexed by $t \in P$, satisfy*

$$\|y_s - y_t\|_p = \lambda|s - t| \qquad (s, t \in P).$$

Then there exist $x, v \in \mathbb{R}^d$ with $\|v\|_p = 1$ such that

$$y_t = x + \lambda t v \qquad (t \in P).$$

Proof. Let $a = \min P$ and $b = \max P$. For any $t \in P$,

$$\|y_b - y_a\|_p = \lambda(b-a) = \lambda(b-t) + \lambda(t-a) = \|y_b - y_t\|_p + \|y_t - y_a\|_p .$$

Thus equality holds in the triangle inequality for the vectors $y_b - y_t$ and $y_t - y_a$. The norm ℓ^p, $1 < p < \infty$, is strictly convex [41, Chapter 5]; therefore these two vectors are nonnegative scalar multiples of the same vector. Hence y_t lies on the line segment from y_a to y_b. Write

$$y_t = y_a + \theta_t(y_b - y_a), \qquad 0 \leqslant \theta_t \leqslant 1.$$

Comparing distances from y_a gives

$$\theta_t = \frac{\|y_t - y_a\|_p}{\|y_b - y_a\|_p} = \frac{t-a}{b-a}.$$

Set

$$v = \frac{y_b - y_a}{\lambda(b-a)}, \qquad x = y_a - \lambda a v.$$

Then $\|v\|_p = 1$ and $y_t = x + \lambda t v$ for every $t \in P$. □

Lemma 5.4 (One-dimensional density of polynomial annuli). *Fix an integer $p \geqslant 2$. For every interval $I \subseteq \mathbb{T}$, every sign $\sigma \in \{-1, 1\}$, and every $R \geqslant 1$,*

$$|\{t \in [-R/2, R/2] : \sigma t^p \bmod 1 \in I\}| = |I|R + O_p(1),$$

uniformly in I, σ, and R.

Proof. By splitting into positive and negative t, and by reflecting I when $\sigma = -1$, it is enough to prove

$$|\{t \in [0, R] : t^p \bmod 1 \in I\}| = |I|R + O_p(1).$$

By subtracting distribution functions it is enough to take $I = [0, a]$, $0 \leqslant a \leqslant 1$. The relevant set is

$$\bigcup_{m \geqslant 0} [m^{1/p}, (m+a)^{1/p}] \cap [0, R].$$

Its measure equals

$$\sum_{m=1}^{\lfloor R^p \rfloor} \big((m+a)^{1/p} - m^{1/p}\big) + O_p(1).$$

For $m \geqslant 1$, Cauchy's mean-value theorem applied to $(1+ax)^{1/p}$ and $(1+x)^{1/p}$ at $x = 1/m$ gives

$$(m+a)^{1/p} - m^{1/p} = a\big((m+1)^{1/p} - m^{1/p}\big) + O_p(m^{-2+1/p}).$$

The error is summable because $p \geqslant 2$. Therefore the sum telescopes:

$$\sum_{m=1}^{\lfloor R^p \rfloor} \big((m+a)^{1/p} - m^{1/p}\big) = a \sum_{m=1}^{\lfloor R^p \rfloor} \big((m+1)^{1/p} - m^{1/p}\big) + O_p(1) = aR + O_p(1).$$

This proves the claim. □

Theorem 5.5 (Collinear ℓ^p lower bound [32]). *Fix integers $d \geqslant 1$ and $p \geqslant 2$. There exists $C_{d,p} < \infty$ such that, for all sufficiently large n, there exist a measurable set $E \subseteq \mathbb{R}^d$, an n-point configuration $P \subseteq \mathbb{R}^d$ contained in a line, and scales $\lambda_j \to \infty$ such that*

$$\underline{\mathrm{d}}(E) \geqslant 1 - C_{d,p}\frac{\log n}{n},$$

and E contains no ℓ^p-isometric copy of $\lambda_j P$ for any j. If p is even, $C_{d,p}$ may be chosen independently of d, and E may be chosen to have an actual density.

Proof. Let $P \subseteq \mathbb{Z}$ and α be supplied by Proposition 5.2, and put

$$\varepsilon_n = K_p \frac{\log n}{n}, \qquad \lambda_j = (\alpha + j)^{1/p}$$

for all sufficiently large positive integers j.

First assume that p is even. Define the annular set

$$E = \left\{ x \in \mathbb{R}^d : \mathrm{dist}(\|x\|_p^p, \mathbb{Z}) < \frac{1-\varepsilon_n}{2} \right\}.$$

For fixed $(x_2, \dots, x_d) \in [-R/2, R/2]^{d-1}$, Lemma 5.4 applied to the interval

$$\left(-\frac{1-\varepsilon_n}{2}, \frac{1-\varepsilon_n}{2} \right) - \sum_{i=2}^{d} x_i^p$$

gives

$$|\{x_1 \in [-R/2, R/2] : (x_1, \dots, x_d) \in E\}| = (1-\varepsilon_n)R + O_p(1).$$

Integrating in the remaining coordinates yields

$$|E \cap [-R/2, R/2]^d| = (1-\varepsilon_n)R^d + O_{d,p}(R^{d-1}),$$

so $\mathrm{d}(E) = 1 - \varepsilon_n$.

Suppose that E contained an ℓ^p-isometric copy of $\lambda_j P$. Since P is collinear, Lemma 5.3 writes the copy as

$$\{x + \lambda_j k v : k \in P\}$$

with $\|v\|_p = 1$. For $k \in P$, the binomial theorem gives

$$\begin{aligned} \|x + \lambda_j k v\|_p^p &= \sum_{i=1}^{d} \sum_{\ell=0}^{p} \binom{p}{\ell} x_i^{p-\ell} \lambda_j^\ell k^\ell v_i^\ell \\ &= (\alpha + j)k^p + B_{p-1}k^{p-1} + \cdots + B_1 k + B_0 \end{aligned}$$

for suitable real coefficients $B_0, \dots, B_{p-1}$; the leading coefficient is $(\alpha + j)\|v\|_p^p = \alpha + j$. Since all copied points lie in E, the values of this polynomial modulo 1 all lie in the interval

$$\left(-\frac{1-\varepsilon_n}{2}, \frac{1-\varepsilon_n}{2} \right) \subseteq \mathbb{T}$$

of length $1-\varepsilon_n$. Subtracting the constant B_0 and the integer jk^p shows that

$$\{\alpha k^p + B_{p-1}k^{p-1} + \cdots + B_1 k \bmod 1 : k \in P\}$$

is contained in an interval of length $1-\varepsilon_n$. The complementary interval has length ε_n, contradicting Proposition 5.2. Thus no such copy exists.

Now assume that p is odd. For each sign vector $\sigma = (\sigma_1, \ldots, \sigma_d) \in \{-1,1\}^d$ set

$$F_\sigma(x) = \sum_{i=1}^{d} \sigma_i x_i^p, \qquad E_\sigma = \left\{ x : \operatorname{dist}(F_\sigma(x), \mathbb{Z}) < \frac{1-\varepsilon_n}{2} \right\},$$

and define

$$E = \bigcap_{\sigma \in \{-1,1\}^d} E_\sigma.$$

By Lemma 5.4, each E_σ has density $1-\varepsilon_n$ in large cubes up to an $O(R^{d-1})$ error. Therefore, by the union bound on complements,

$$\underline{\mathrm{d}}(E) \geqslant 1 - 2^d \varepsilon_n.$$

If an ℓ^p-isometric copy of $\lambda_j P$ lay in E, Lemma 5.3 would again write it as $x + \lambda_j k v$, $k \in P$, with $\|v\|_p = 1$. Choose signs so that $\sigma_i v_i^p = |v_i|^p$ for every i. Then

$$F_\sigma(x + \lambda_j k v) = (\alpha + j)k^p + B_{p-1}k^{p-1} + \cdots + B_1 k + B_0,$$

again because the leading coefficient is

$$\lambda_j^p \sum_i \sigma_i v_i^p = (\alpha + j) \sum_i |v_i|^p = \alpha + j.$$

Since the copied points lie in $E \subseteq E_\sigma$, the same interval-complement argument contradicts Proposition 5.2. This proves the theorem, with $C_{d,p} = K_p$ for even p and $C_{d,p} = 2^d K_p$ for odd p. □

Proof of Theorem 5.1. Take $p = 2$ in Theorem 5.5. The norm ℓ^2 is the Euclidean norm, the constructed pattern is collinear, the constructed set has actual density at least $1 - K_2(\log n)/n$, and the missing ℓ^2-isometric copies are exactly missing Euclidean isometric copies. □

Consequently,

$$\rho_{\min}(d,n) \geqslant 1 - C\frac{\log n}{n}$$

for all sufficiently large n. Together with the upper bound $\rho_{\min}(d,n) \leqslant 1 - 1/(n-1)$ of Falconer, the author, and Yavicoli, this leaves only a logarithmic gap in the density threshold.

5.3 Non-collinear ℓ^p patterns

For $p \neq 2$ the geometry of ℓ^p spaces supplies a simpler obstruction if non-collinear patterns are allowed.

Lemma 5.6 (Equality in Clarkson-type inequalities [7]). *Let $1 < p < \infty$, $p \neq 2$, and let $u, v \in \mathbb{R}^d$. Then*

$$\|u+v\|_p^p + \|u-v\|_p^p \begin{cases} \geqslant 2(\|u\|_p^p + \|v\|_p^p), & p > 2, \\ \leqslant 2(\|u\|_p^p + \|v\|_p^p), & 1 < p < 2. \end{cases}$$

In either case equality holds if and only if u and v have disjoint coordinate supports.

Proof. It is enough to prove the one-coordinate assertion and sum over coordinates. For real a, b, put $s = (a+b)^2$ and $t = (a-b)^2$. If $p > 2$, the map $x \mapsto x^{p/2}$ is strictly convex, and

$$|a+b|^p + |a-b|^p = s^{p/2} + t^{p/2} \geqslant 2\left(\frac{s+t}{2}\right)^{p/2} = 2(a^2+b^2)^{p/2} \geqslant 2(|a|^p + |b|^p).$$

If $1 < p < 2$, the same two inequalities reverse, because $x^{p/2}$ is concave and the ℓ^p norm dominates the ℓ^2 norm in two dimensions. In both cases equality in the first step forces $s = t$, i.e. $ab = 0$, and this condition also gives equality in the second step. Summing over coordinates proves the lemma. □

Theorem 5.7 (Asymptotically sharp ℓ^p obstruction [32]). *Let $d \geqslant 1$ and $p \in (1, \infty)$, $p \neq 2$. For every $n \geqslant 2d+1$ there exist a measurable set $E \subseteq \mathbb{R}^d$, an n-point configuration $P \subseteq \mathbb{R}^d$, and scales $\lambda_j \to \infty$ such that E contains no ℓ^p-isometric copy of $\lambda_j P$, while*

$$\mathrm{d}(E) = 1 - \frac{1}{n - 2d + 2}.$$

Proof. Let $e_1, \dots, e_d$ be the standard basis and set

$$P = \{ke_1 : k = -1, 0, 1, \dots, n-2d\} \cup \{\pm e_2, \dots, \pm e_d\}.$$

The first set has $n - 2d + 2$ points and the second has $2d - 2$ points, so $|P| = n$. Put

$$\varepsilon = \frac{1}{n-2d+2}, \qquad E = \{x \in \mathbb{R}^d : (x_1 + \cdots + x_d) \bmod 1 \in [0, 1-\varepsilon)\}.$$

Fubini in the x_1 variable gives

$$|E \cap [-R/2, R/2]^d| = (1-\varepsilon)R^d + O_d(R^{d-1}),$$

so $\mathrm{d}(E) = 1 - \varepsilon$. Let

$$\lambda_j = j + \varepsilon, \qquad j = 1, 2, \dots.$$

We prove that no ℓ^p-isometric copy of $\lambda_j P$ lies in E.

Assume, for contradiction, that such a copy exists. The collinear part

$$\{\lambda_j k e_1 : k = -1, 0, 1, \dots, n-2d\}$$

must be mapped, by Lemma 5.3, to points

$$y_k = x + \lambda_j k u, \qquad k = -1, 0, 1, \dots, n-2d,$$

where $\|u\|_p = 1$. If $d = 1$, then $u = \pm e_1$. Assume $d \geqslant 2$. Let z_i^+ and z_i^- be the images of $\lambda_j e_i$ and $-\lambda_j e_i$ for $i = 2, \dots, d$, and write

$$z_i^\pm = x + \lambda_j v_i^\pm.$$

Then $\|v_i^\pm\|_p = 1$.

Because the distance between e_i and $-e_i$ is 2, we have

$$\|v_i^+ - v_i^-\|_p = 2 = \|v_i^+\|_p + \|-v_i^-\|_p.$$

Strict convexity of ℓ^p implies equality in the triangle inequality only for positively collinear vectors; hence

$$v_i^- = -v_i^+. \tag{5.2}$$

Next, the model distances

$$\|e_i - e_1\|_p^p = \|e_i - (-e_1)\|_p^p = 2$$

become

$$\|v_i^+ - u\|_p^p = \|v_i^+ + u\|_p^p = 2.$$

Adding the two equalities gives

$$\|v_i^+ - u\|_p^p + \|v_i^+ + u\|_p^p = 4 = 2(\|v_i^+\|_p^p + \|u\|_p^p).$$

By Lemma 5.6, v_i^+ and u have disjoint coordinate supports.

Similarly, for $2 \leqslant i < m \leqslant d$, the model distances

$$\|e_i - e_m\|_p^p = \|e_i + e_m\|_p^p = 2$$

and (5.2) give

$$\|v_i^+ - v_m^+\|_p^p = \|v_i^+ + v_m^+\|_p^p = 2.$$

The equality case in Lemma 5.6 shows that v_i^+ and v_m^+ also have disjoint coordinate supports. Thus the d nonzero unit vectors

$$u, v_2^+, \ldots, v_d^+$$

have pairwise disjoint nonempty supports inside the d coordinate set $\{1, \ldots, d\}$. Each support must therefore be a singleton. In particular,

$$u = \sigma e_\ell$$

for some $\ell \in \{1, \ldots, d\}$ and $\sigma \in \{-1, 1\}$.

All points $y_k = x + \sigma\lambda_j k e_\ell$ lie in E. If $c = x_1 + \cdots + x_d$, then

$$c + \sigma\lambda_j k \bmod 1 \in [0, 1-\varepsilon) \qquad (k = -1, 0, 1, \ldots, n-2d).$$

Since $\lambda_j = j + \varepsilon$, this says that a translate of the set

$$\{\sigma k\varepsilon \bmod 1 : k = -1, 0, 1, \ldots, n-2d\}$$

is contained in an interval of length $1-\varepsilon$. But this is exactly the set of $N = n - 2d + 2$ equally spaced points on the circle, where $\varepsilon = 1/N$. No half-open interval of length $1 - 1/N$ contains all N equally spaced points. This contradiction completes the proof. $\square$

Corollary 5.8 (Sharp ℓ^p threshold [32]). *For $p \in (1, \infty) \setminus \{2\}$, the critical density for forcing all sufficiently large ℓ^p-isometric copies of every n-point configuration satisfies*

$$\rho_{\min}(d, n, p) = 1 - \frac{1}{n} + O_d\left(\frac{1}{n^2}\right).$$

Proof. The translated-copy proposition is valid in every norm, so $\rho_{\min}(d, n, p) \leqslant 1 - 1/n$. Theorem 5.7 gives

$$\rho_{\min}(d, n, p) \geqslant 1 - \frac{1}{n - 2d + 2} = 1 - \frac{1}{n} + O_d\left(\frac{1}{n^2}\right).$$

The two bounds imply the asserted asymptotic formula. $\square$

5.4 Open problem

Problem 5.9 (Sharp density threshold for arbitrary Euclidean patterns). Let $\rho_{\min}(d,n)$ denote the least density threshold forcing all sufficiently large similar copies of every n-point Euclidean pattern in $\mathbb{R}^d$. The best known bounds, due respectively to the author and Santos Sepčić and to Falconer, the author, and Yavicoli [17, 32], are

$$1 - C\frac{\log n}{n} \leqslant \rho_{\min}(d,n) \leqslant 1 - \frac{1}{n-1}.$$

The natural problem is to remove the logarithm in the lower bound or to improve the geometric upper bound. The lower constructions known at present are collinear and number-theoretic, while the upper bound is geometric and uses rotations. Closing the gap likely requires a new idea on one side of this dichotomy.

Preliminaries

This appendix records the technical background used throughout the notes. The basic density notions were introduced in Chapter 1; here we isolate the analytic and combinatorial tools that recur in the proofs.

A.1 Comparison notation

We use standard asymptotic notation throughout the notes. If X and Y are nonnegative quantities, then

$$X \lesssim Y$$

means that $X \leqslant CY$ for a finite constant C independent of the main variables under discussion. A subscript records allowed dependence: for example $X \lesssim_{d,n,\varepsilon} Y$ means that C may depend on d, n, ε, but not on scale parameters such as R or λ unless this is explicitly stated. We write

$$X \gtrsim Y$$

for $Y \lesssim X$, and $X \simeq Y$ when both inequalities hold.

The notation $X = O(Y)$ has the same meaning as $|X| \lesssim Y$; similarly $X = O_{d,n}(Y)$ permits the implicit constant to depend on d and n. For nonnegative quantities, $X = \Omega(Y)$ means $X \gtrsim Y$, again with optional subscripts to indicate permitted parameter dependence. Thus a statement such as $|A| = \Omega_d(R^d)$ asserts that $|A| \geqslant c_d R^d$ for some positive constant depending only on d.

A.2 Upper Banach density and averaging bodies

We first justify two conventions that are used throughout the notes. The definition of upper Banach density contains a genuine limit, not merely a limsup; and the value of that limit does not depend on choosing cubes rather than balls or any other fixed compact convex body. This is the elementary Følner comparison argument [19], specialized to dilates of bounded sets in Euclidean space.

For a bounded measurable set $F \subseteq \mathbb{R}^d$ with $0 < |F| < \infty$, define

$$D_F(E) := \sup_{x \in \mathbb{R}^d} \frac{|E \cap (x + F)|}{|F|}.$$

Thus the cube definition in Chapter 1 is the assertion that $D_{[0,R]^d}(E)$ has a limit as $R \to \infty$.

Lemma A.1 (Comparison of two averaging windows). *Let $F, G \subseteq \mathbb{R}^d$ be bounded measurable sets with positive finite measure. For every measurable $E \subseteq \mathbb{R}^d$,*

$$D_F(E) \leqslant D_G(E) + \eta(F, G), \qquad \eta(F, G) := \sup_{z \in G} \frac{|(F + z) \triangle F|}{|F|}.$$

Proof. Let
$$\mathcal{A}_F f(x) := \frac{1}{|F|}\int_F f(x+y)\ \mathrm{d}y.$$
Then $D_F(E) = \|\mathcal{A}_F \mathbb{1}_E\|_\infty$. For $f = \mathbb{1}_E$ and for every $z \in G$,
$$\big|\mathcal{A}_F f(x+z) - \mathcal{A}_F f(x)\big| \leqslant \frac{|(F+z)\triangle F|}{|F|} \leqslant \eta(F,G).$$
Averaging this inequality over $z \in G$ gives
$$\mathcal{A}_F f(x) \leqslant \frac{1}{|G|}\int_G \mathcal{A}_F f(x+z)\ \mathrm{d}z + \eta(F,G) = \mathcal{A}_F(\mathcal{A}_G f)(x) + \eta(F,G).$$
The operator $\mathcal{A}_F$ is a positive average, so
$$\mathcal{A}_F(\mathcal{A}_G f)(x) \leqslant \|\mathcal{A}_G f\|_\infty = D_G(E).$$
Taking the supremum over x proves the claim. □

Lemma A.2 (Van Hove property of convex dilates). *Let $K, L \subseteq \mathbb{R}^d$ be compact convex sets with nonempty interior. For every fixed $S > 0$,*
$$\eta(RK, SL) \longrightarrow 0 \qquad \text{as } R \to \infty.$$

Proof. Since L is bounded, all $z \in SL$ satisfy $|z| \leqslant C_S$ for some constant C_S. Also
$$\frac{|(RK+z)\triangle RK|}{|RK|} = \frac{|(K+z/R)\triangle K|}{|K|}.$$
The map $u \mapsto \mathbb{1}_{K+u}$ is continuous at $u = 0$ in $L^1(\mathbb{R}^d)$, because translations are continuous in L^1. Hence, for every $\varepsilon > 0$, there is $\delta > 0$ such that
$$|u| < \delta \quad \Longrightarrow \quad |(K+u)\triangle K| < \varepsilon|K|.$$
If $R > C_S/\delta$, then $|z/R| < \delta$ for every $z \in SL$, and the displayed supremum is at most ε. □

Proposition A.3 (Existence of the upper-density limit). *Let $K \subseteq \mathbb{R}^d$ be a compact convex set with nonempty interior. Then, for every measurable $E \subseteq \mathbb{R}^d$, the limit*
$$\lim_{R\to\infty} D_{RK}(E)$$
exists. More precisely,
$$\lim_{R\to\infty} D_{RK}(E) = \inf_{S>0} D_{SK}(E).$$

Proof. Fix $S > 0$ and apply Lemma A.1 with $F = RK$ and $G = SK$:
$$D_{RK}(E) \leqslant D_{SK}(E) + \eta(RK, SK).$$
By Lemma A.2, the error tends to 0 as $R \to \infty$. Therefore
$$\limsup_{R\to\infty} D_{RK}(E) \leqslant D_{SK}(E).$$

Taking the infimum over S gives

$$\limsup_{R\to\infty} D_{RK}(E) \leqslant \inf_{S>0} D_{SK}(E).$$

The quantity on the right is a lower bound for every value $D_{RK}(E)$, and hence it is at most $\liminf_{R\to\infty} D_{RK}(E)$. Thus the limsup and the liminf are equal, and both equal the stated infimum. □

Theorem A.4 (Independence of the averaging body). *Let $K, L \subseteq \mathbb{R}^d$ be compact convex sets with nonempty interior. Then, for every measurable $E \subseteq \mathbb{R}^d$,*

$$\lim_{R\to\infty} D_{RK}(E) = \lim_{R\to\infty} D_{RL}(E).$$

Consequently the upper Banach density may be computed from translates and dilates of any such K:

$$\overline{\delta}(E) = \lim_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{|E\cap(x+RK)|}{R^d|K|}.$$

In particular, cubes, balls, ellipsoids, simplices, and all compact convex bodies with nonempty interior give the same number.

Proof. Write

$$\overline{\delta}_K(E) := \lim_{R\to\infty} D_{RK}(E), \qquad \overline{\delta}_L(E) := \lim_{R\to\infty} D_{RL}(E),$$

which exist by Proposition A.3. Fix $S > 0$ and compare the large window RK with the fixed window SL:

$$D_{RK}(E) \leqslant D_{SL}(E) + \eta(RK, SL).$$

Letting $R \to \infty$ and using Lemma A.2, we obtain

$$\overline{\delta}_K(E) \leqslant D_{SL}(E).$$

Taking the infimum in S and using Proposition A.3 for L gives $\overline{\delta}_K(E) \leqslant \overline{\delta}_L(E)$. Interchanging K and L gives the opposite inequality.

Finally, $|RK| = R^d|K|$, so the displayed formula is exactly the definition of $D_{RK}(E)$. The location of K is irrelevant: replacing K by $K + a$ changes $x + RK$ to $(x + Ra) + RK$, and the translation parameter x already ranges over all of $\mathbb{R}^d$. □

A.3 Surface measures and decay

This section records the Fourier facts about curved surface measure that are used in the uniform estimates in the main text. We use the Fourier normalization

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x.$$

Let $\Sigma \subseteq \mathbb{R}^d$ be a compact C^∞ hypersurface and let σ be a smooth compactly supported surface measure on Σ; in the most important example $\Sigma = S^{d-1}$ and σ is normalized spherical measure. If the Gaussian curvature of Σ is everywhere nonzero on the support of σ, then stationary phase gives

$$|\widehat{\sigma}(\xi)| \lesssim_\Sigma (1+|\xi|)^{-(d-1)/2}. \tag{A.1}$$

For the sphere this estimate can also be read from the Bessel formula for $\widehat{\sigma}$ and the standard asymptotics of Bessel functions; see Abramowitz–Stegun [1]. In this generality it is a classical theorem of Littman and Herz, and it is a standard consequence of the stationary-phase theorem as presented, for instance, in Stein [45], Littman [37], Herz [26], and Sogge [44].

Here is the stationary-phase mechanism in the form needed in these notes. Write $\xi = \rho\theta$, where $\rho = |\xi|$ and $\theta \in S^{d-1}$. On a coordinate patch $u \mapsto \Phi(u) \in \Sigma$,

$$\widehat{\sigma}(\rho\theta) = \int e^{-2\pi i \rho\theta\cdot\Phi(u)} a(u)\,\mathrm{d}u.$$

The critical points are precisely the points at which θ is normal to Σ. Non-vanishing Gaussian curvature says that the Hessian of $u \mapsto \theta \cdot \Phi(u)$ is non-singular at such a point. A partition of unity and the stationary-phase theorem therefore give, uniformly in θ,

$$\widehat{\sigma}(\rho\theta) = \rho^{-(d-1)/2} \sum_{u_c} a_c(\theta) e^{-2\pi i\rho\theta\cdot\Phi(u_c)} + O_\Sigma(\rho^{-(d+1)/2}), \qquad \rho \geqslant 1,$$

where the sum is over the finitely many critical points in the coordinate patches. Away from the critical set, repeated integration by parts gives rapid decay. Together with the trivial bound $|\widehat{\sigma}(\xi)| \leqslant \|\sigma\|$ for $|\xi| \lesssim 1$, this proves (A.1).

The same estimate applies to dilates. If σ_λ denotes the image of σ under $x \mapsto \lambda x$, then

$$\widehat{\sigma_\lambda}(\xi) = \widehat{\sigma}(\lambda\xi), \qquad |\widehat{\sigma_\lambda}(\xi)| \lesssim (1 + \lambda|\xi|)^{-(d-1)/2}.$$

This is the estimate used whenever an exact counting form contains a spherical constraint such as $|y| = \lambda$.

The other recurring ingredient is a smooth approximate identity. Let

$$g(x) = e^{-\pi|x|^2}, \qquad g_t(x) = t^{-d} g(x/t),$$

and let $k = \Delta g$. Then

$$\widehat{g}(\xi) = e^{-\pi|\xi|^2}, \qquad \widehat{k_t}(\xi) = \widehat{k}(t\xi) = -4\pi^2 t^2 |\xi|^2 e^{-\pi t^2|\xi|^2}.$$

Consequently, when σ satisfies (A.1),

$$\sup_{\xi\in\mathbb{R}^d} |\widehat{\sigma}(\lambda\xi)\widehat{k}(t\lambda\xi)| \lesssim t^{\min((d-1)/2,2)}, \qquad 0 < t \leqslant 1.$$

Indeed, with $r = \lambda|\xi|$ the left side is bounded by

$$(1 + r)^{-(d-1)/2}(tr)^2 e^{-\pi t^2 r^2}.$$

The maximum occurs either for $r \lesssim 1$ or for $r \simeq t^{-1}$; the displayed bound follows in both ranges, and the intermediate range is monotone up to harmless constants. In the planar circular estimates used repeatedly in Chapters 2 and 3 this gives the familiar bound

$$\sup_{\xi\in\mathbb{R}^2} |\widehat{\sigma}(\lambda\xi)\widehat{k}(t\lambda\xi)| \lesssim t^{1/2}.$$

A closely related smoothing estimate is used to remove the final mollifier. Since

$$1 - \widehat{g}(\varepsilon\lambda\xi) = 1 - e^{-\pi\varepsilon^2\lambda^2|\xi|^2},$$

one has, for every $0 < \gamma \leqslant \min((d-1)/2, 2)$,

$$\sup_{\xi \in \mathbb{R}^d} |\widehat{\sigma}(\lambda\xi)(1 - \widehat{g}(\varepsilon\lambda\xi))| \lesssim_\gamma \varepsilon^\gamma.$$

This follows by writing $u = \lambda|\xi|$ and bounding

$$(1+u)^{-(d-1)/2} \min(1, \varepsilon^2 u^2).$$

Plancherel then gives the model L^2 estimate

$$\left| \int F(x)\big(F * \sigma_\lambda - F * \sigma_\lambda * g_{\varepsilon\lambda}\big)(x)\,\mathrm{d}x \right| \leqslant C_\gamma \varepsilon^\gamma \, \|F\|_{L^2(\mathbb{R}^d)}^2 ,$$

which is the prototype for the uniform part of Bourgain's method. More complicated configurations require the same idea after freezing all but one variable, or after applying a multilinear smoothing inequality, but the local oscillatory input is still (A.1).

A.4 Gowers norms

The cube notation used in the notes is the continuum analogue of the Gowers uniformity norms introduced in quantitative proofs of Szemerédi's theorem [21]; see also the expositions and related estimates in Green–Tao [24], Tao [48], Shkredov [43], and Eisner–Tao [15]. We record the precise Cauchy–Schwarz inequality invoked in the main text.

For $k \geqslant 1$, put $V_k = \{0,1\}^k$. If $h = (h_1, \ldots, h_k) \in (\mathbb{R}^d)^k$ and $\omega \in V_k$, write

$$\omega \cdot h = \omega_1 h_1 + \cdots + \omega_k h_k, \qquad |\omega| = \omega_1 + \cdots + \omega_k.$$

Let $\mathcal{C}z = \overline{z}$. For a family of bounded compactly supported functions $(f_\omega)_{\omega \in V_k}$ define the Gowers inner product

$$\langle f_\omega : \omega \in V_k \rangle_{U^k} := \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^k} \prod_{\omega \in V_k} \mathcal{C}^{|\omega|} f_\omega(x + \omega \cdot h)\,\mathrm{d}h_1 \cdots \mathrm{d}h_k\,\mathrm{d}x.$$

The U^k seminorm is defined by putting all functions equal:

$$\|f\|_{U^k(\mathbb{R}^d)}^{2^k} = \langle f : \omega \in V_k \rangle_{U^k}.$$

For the nonnegative indicator functions used in the density arguments, the conjugations do not change the value and the formula becomes the positive cube integral

$$\int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^k} \prod_{\omega \in V_k} f(x + \omega \cdot h)\,\mathrm{d}h\,\mathrm{d}x.$$

Proposition A.5 (Gowers–Cauchy–Schwarz inequality). *For every $k \geqslant 1$ and every bounded compactly supported family $(f_\omega)_{\omega \in V_k}$,*

$$\big|\langle f_\omega : \omega \in V_k \rangle_{U^k}\big| \leqslant \prod_{\omega \in V_k} \|f_\omega\|_{U^k(\mathbb{R}^d)} .$$

Proof. The proof is the usual repeated Cauchy–Schwarz argument, written here in a form that is directly compatible with the Euclidean integrals in the notes. The case $k = 1$ is immediate: by Fubini,

$$\langle f_0, f_1\rangle_{U^1} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(x)\overline{f_1(x+h)}\,\mathrm{d}h\,\mathrm{d}x = \left(\int f_0\right)\overline{\left(\int f_1\right)},$$

and this is bounded by $\|f_0\|_{U^1}\,\|f_1\|_{U^1}$.

We first prove a two-function estimate that will be used in the induction. For $k \geqslant 2$ and $s \in \mathbb{R}^d$, write $T_s g(x) = g(x+s)$. We claim that

$$\int_{\mathbb{R}^d} \left\|f\,\overline{T_s g}\right\|_{U^{k-1}}^{2^{k-1}}\,\mathrm{d}s \leqslant \|f\|_{U^k}^{2^{k-1}}\,\|g\|_{U^k}^{2^{k-1}}. \tag{A.2}$$

Expanding the left side gives

$$\int_{s,x,h'} \prod_{\eta\in V_{k-1}} \mathcal{C}^{|\eta|}\big(f(x+\eta\cdot h')\overline{g(x+s+\eta\cdot h')}\big)\,\mathrm{d}h'\,\mathrm{d}x\,\mathrm{d}s,$$

where $h' = (h_1,\dots,h_{k-1})$. Put $y = x+s$. The expression becomes

$$\int_{h'} F(h')\overline{G(h')}\,\mathrm{d}h',$$

where

$$F(h') = \int_{\mathbb{R}^d}\prod_{\eta\in V_{k-1}}\mathcal{C}^{|\eta|}f(x+\eta\cdot h')\,\mathrm{d}x, \qquad G(h') = \int_{\mathbb{R}^d}\prod_{\eta\in V_{k-1}}\mathcal{C}^{|\eta|}g(y+\eta\cdot h')\,\mathrm{d}y.$$

By Cauchy–Schwarz in h' this is at most

$$\left(\int |F(h')|^2\,\mathrm{d}h'\right)^{1/2}\left(\int |G(h')|^2\,\mathrm{d}h'\right)^{1/2}.$$

Finally, after writing the second copy of the base variable as $x + h_k$, the first factor is exactly $\|f\|_{U^k}^{2^{k-1}}$, and similarly the second is $\|g\|_{U^k}^{2^{k-1}}$. This proves (A.2).

We now prove the proposition by induction on k. Assume it has been proved for $k-1$. Write a vertex of V_k as (η,ϵ) with $\eta \in V_{k-1}$ and $\epsilon \in \{0,1\}$. Separating the last difference variable $s = h_k$, the k-dimensional inner product can be written as

$$\int_{\mathbb{R}^d}\big\langle f_{\eta,0}\,\overline{T_s f_{\eta,1}} : \eta\in V_{k-1}\big\rangle_{U^{k-1}}\,\mathrm{d}s.$$

The induction hypothesis bounds the absolute value of the inner product by

$$\prod_{\eta\in V_{k-1}}\left\|f_{\eta,0}\,\overline{T_s f_{\eta,1}}\right\|_{U^{k-1}}.$$

Integrating in s and applying Hölder with the 2^{k-1} equal exponents gives

$$\big|\langle f_\omega : \omega\in V_k\rangle_{U^k}\big| \leqslant \prod_{\eta\in V_{k-1}}\left(\int_{\mathbb{R}^d}\left\|f_{\eta,0}\,\overline{T_s f_{\eta,1}}\right\|_{U^{k-1}}^{2^{k-1}}\,\mathrm{d}s\right)^{1/2^{k-1}}.$$

Applying the two-function estimate (A.2) to each pair $(f_{\eta,0}, f_{\eta,1})$ yields

$$\left|\langle f_\omega : \omega \in V_k \rangle_{U^k}\right| \leqslant \prod_{\eta \in V_{k-1}} \|f_{\eta,0}\|_{U^k} \, \|f_{\eta,1}\|_{U^k} = \prod_{\omega \in V_k} \|f_\omega\|_{U^k} \, .$$

This completes the induction. □

In the main text the proposition is usually applied to indicator functions or to functions bounded by indicators. In that case one may remove all conjugation signs and interpret the conclusion as the statement that any mixed cubical average is controlled by the geometric mean of the corresponding pure cube averages. This is the analytic reason why cubical counting forms have a robust positivity theory.

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