On the Erdős problem #251

ChatGPT 5.4 Pro (orchestrated by Vjeko Kovač)

Abstract

We disprove the assertion that, if p_n is the n-th prime and $(g_n)_{n\geq 1}$ is a sequence of integers with $g_n \geq 2$ and $g_n = o(p_n)$, then

$$\sum_{n=1}^{\infty} \frac{p_n}{g_1 \cdots g_n}$$

must be irrational. In fact, we construct such a sequence for which the sum is exactly 1.

Theorem 1. *There exists a sequence of integers $(g_n)_{n\geq 1}$ with $g_n \geq 2$ and $g_n = o(p_n)$ such that*

$$\sum_{n=1}^{\infty} \frac{p_n}{g_1 \cdots g_n} = 1.$$

Consequently, the above irrationality statement is false.

Proof. Let

$$m_n := 1 + \lfloor \log\log(n+3) \rfloor \qquad (n \geq 1).$$

Then $m_n \to \infty$ and

$$\sum_{k=1}^{n} m_k = O(n \log\log n).$$

By the prime number theorem, $p_n \sim n \log n$, and therefore

$$\sum_{k=1}^{n} m_k = o(p_n).$$

Hence, after choosing N sufficiently large, we may assume that

$$1 + \sum_{k=N}^{n-1} m_k < p_n \qquad (n \geq N). \tag{1}$$

We now define an auxiliary sequence of positive integers $(c_n)_{n\geq 1}$. Set

$$c_1 = \cdots = c_N = 1.$$

For each $n \geq N$, choose c_{n+1} to be the unique integer in the set

$$\{m_n, m_n + 1, \ldots, m_n + c_n - 1\}$$

such that

$$c_{n+1} \equiv -p_n \pmod{c_n}.$$

This is possible because the displayed set consists of c_n consecutive integers, hence contains exactly one representative of each residue class modulo c_n.

We claim that

$$1 \leq c_n \leq 1 + \sum_{k=N}^{n-1} m_k \qquad (n \geq N). \tag{2}$$

For $n = N$ this is immediate, since $c_N = 1$. If the bound holds for c_n, then by construction

$$c_{n+1} \le m_n + c_n - 1 \le m_n + \sum_{k=N}^{n-1} m_k = \sum_{k=N}^{n} m_k,$$

which yields (2) with $n+1$ in place of n. Thus (2) follows by induction. Combining (2) with (1), we obtain

$$c_n < p_n \qquad (n \ge N). \tag{3}$$

Now define

$$g_n := \frac{p_n + c_{n+1}}{c_n} \qquad (n \ge 1).$$

For $n < N$, we have $c_n = c_{n+1} = 1$, so $g_n = p_n + 1$. For $n \ge N$, the congruence defining c_{n+1} shows that g_n is an integer. Moreover, by (3),

$$p_n + c_{n+1} > c_n,$$

so $g_n > 1$. Since g_n is integral, it follows that

$$g_n \ge 2 \qquad (n \ge 1).$$

Next we verify that $g_n = o(p_n)$. Since $c_n \ge m_{n-1}$ for $n \ge N+1$, we have $c_n \to \infty$. Also, from the construction,

$$c_{n+1} \le c_n + m_n - 1,$$

so

$$\frac{g_n}{p_n} = \frac{1}{c_n} + \frac{c_{n+1}}{c_n p_n} \le \frac{1}{c_n} + \frac{1}{p_n} + \frac{m_n}{p_n}.$$

The right-hand side tends to 0 because $c_n \to \infty$, $p_n \to \infty$, and $m_n = o(p_n)$. Therefore $g_n = o(p_n)$.

Finally, let

$$G_n := g_1 g_2 \cdots g_n, \qquad G_0 := 1.$$

From the definition of g_n we have

$$p_n = c_n g_n - c_{n+1},$$

and hence

$$\frac{p_n}{G_n} = \frac{c_n g_n - c_{n+1}}{G_n} = \frac{c_n}{G_{n-1}} - \frac{c_{n+1}}{G_n}.$$

Therefore the partial sums telescope:

$$\sum_{n=1}^{M} \frac{p_n}{G_n} = \sum_{n=1}^{M} \left(\frac{c_n}{G_{n-1}} - \frac{c_{n+1}}{G_n} \right) = c_1 - \frac{c_{M+1}}{G_M} = 1 - \frac{c_{M+1}}{G_M}.$$

Since every $g_n \ge 2$, we have $G_M \ge 2^M$, while (2) gives $c_{M+1} = O(M \log\log M)$. Thus

$$\frac{c_{M+1}}{G_M} \to 0,$$

so

$$\sum_{n=1}^{\infty} \frac{p_n}{g_1 \cdots g_n} = 1.$$

This proves the theorem. □