

AN IMPROVED BOUND IN ERDŐS PROBLEM #1054

ABSTRACT. Written by Vjeko after Terry's proof sketch.

Theorem 1. For $M \geq 1$, $X \geq 1$, and $\delta > 0$, we have

$$\#\{N \leq X : f(N) \leq \delta N\} \ll_M \delta^M X.$$

Consequently, the upper density of the set

$$\{N \in \mathbb{N} : f(N) \leq \delta N\}$$

is $O_M(\delta^M)$ for every $M \geq 1$.

Let $\sigma_j(m)$ denote the sum of the divisors of m excluding the j largest ones.

Lemma 2. For an integer $q \geq 2$ and $x \geq 1$ one has

$$\sum_{m \leq x} \sum_{j \geq 0} \left(\frac{\sigma_j(m)}{m} \right)^q \ll_q x.$$

Proof of Lemma 2. We write (\dots) for the GCD and $[\dots]$ for the LCM. If $r_1 < r_2 < \dots < r_{\tau(m)}$ are the divisors of m , then “reflecting” that list we get $\sigma_j(m)/m = \sum_{i>j} 1/r_i$. Expanding the q -th power we obtain

$$\left(\frac{\sigma_j(m)}{m} \right)^q = \sum_{\substack{a_1, \dots, a_q | m \\ a_1, \dots, a_q > r_j}} \frac{1}{a_1 \cdots a_q}$$

(the condition $> r_0$ being void), so the summation in j gives

$$\sum_{j \geq 0} \left(\frac{\sigma_j(m)}{m} \right)^q = \sum_{\substack{r, a_1, \dots, a_q | m \\ r \leq \min\{a_1, \dots, a_q\}}} \frac{1}{a_1 \cdots a_q}.$$

(Here we counted for which $j = 0, 1, \dots$ the same term $1/a_1 \cdots a_q$ appears.) Finally, we sum over $m \leq x$, interchange the sums, and bound the number of multiples of $[r, a_1, \dots, a_q]$ not exceeding x by $x/[r, a_1, \dots, a_q]$:

$$\sum_{m \leq x} \sum_{j \geq 0} \left(\frac{\sigma_j(m)}{m} \right)^q \leq x \underbrace{\sum_{\substack{r, a_1, \dots, a_q \geq 1 \\ r \leq \min\{a_1, \dots, a_q\}}} \frac{1}{a_1 \cdots a_q [r, a_1, \dots, a_q]}}_{=S}.$$

It remains to show that the sum S on the right hand side is a finite number.

Substitute

$$g_i = (r, a_i), \quad b_i = \frac{r}{g_i}, \quad c_i = \frac{a_i}{g_i}.$$

Then $(b_i, c_i) = 1$ and the condition $r \leq a_i$ becomes $b_i \leq c_i$. Also,

$$a_i = \frac{r c_i}{b_i}, \quad [r, a_1, \dots, a_q] = r [c_1, \dots, c_q].$$

Hence,

$$S \leq \sum_{r \geq 1} \frac{1}{r^{q+1}} \sum_{b_1, \dots, b_q | r} b_1 \cdots b_q \sum_{\substack{c_1 \geq b_1 \\ \dots \\ c_q \geq b_q}} \frac{1}{c_1 \cdots c_q [c_1, \dots, c_q]}.$$

Estimate roughly $[c_1, \dots, c_q] \geq \max\{c_1, \dots, c_q\} \geq (c_1 \cdots c_q)^{1/q}$, so the innermost sum is

$$\leq \sum_{\substack{c_1 \geq b_1 \\ \dots \\ c_q \geq b_q}} \frac{1}{c_i^{1+1/q}} \ll_q \prod_{i=1}^q \frac{1}{b_i^{1/q}}.$$

Consequently,

$$\begin{aligned} S &\ll_q \sum_{r \geq 1} \frac{1}{r^{q+1}} \sum_{b_1, \dots, b_q | r} \prod_{i=1}^q b_i^{1-1/q} = \sum_{r \geq 1} \frac{1}{r^{q+1}} \left(\sum_{b|r} b^{1-1/q} \right)^q \\ &\leq \sum_{r \geq 1} \frac{1}{r^{q+1}} \left(\tau(r) r^{1-1/q} \right)^q = \sum_{r \geq 1} \frac{\tau(r)^q}{r^2} < \infty \end{aligned}$$

because of $\tau(r) \ll_\varepsilon r^\varepsilon$ for every $\varepsilon > 0$. □

Proof of Theorem 1. Denote the exceptional set that we need to estimate:

$$E_\delta(X) = \{N \leq X : f(N) \leq \delta N\}.$$

For every $N \in E_\delta(X)$ put $m = f(N)$. Then $m \leq \delta N \leq \delta X$, and, by the definition of f , there is some $j \geq 0$ such that $N = \sigma_j(m)$. This pair (m, j) satisfies

$$\frac{\sigma_j(m)}{m} = \frac{N}{m} \geq \frac{1}{\delta}.$$

Counting pairs (m, j) we get

$$\begin{aligned} \#E_\delta(X) &\leq \#\left\{ (m, j) : m \leq \delta X, \frac{\sigma_j(m)}{m} \geq \frac{1}{\delta} \right\} \\ &\leq \delta^q \sum_{m \leq \delta X} \sum_{j \geq 0} \left(\frac{\sigma_j(m)}{m} \right)^q. \end{aligned}$$

By Lemma 2 with $x = \delta X$, the last expression is

$$\ll_q \delta^q \delta X = \delta^{q+1} X,$$

which proves the claim with $M = q + 1$. □