On Erdős problem #521 for Kac polynomials

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Abstract

Let

$$K_n^{(\xi)}(x) = \sum_{j=0}^{n} \xi_j x^j$$

be a Kac polynomial with independent, identically distributed real coefficients, and let $N_n^{(\xi)}$ be the number of its real zeros. Throughout the paper the coefficient law is assumed to satisfy Do's moment hypotheses

$$\mathbb{E}\xi = 0, \qquad \mathbb{E}\xi^2 = 1, \qquad \mathbb{E}|\xi|^{2+\eta} < \infty \quad \text{for some } \eta > 0.$$

Under these standing assumptions, Do's theorem gives the strong law for the number of zeros in $[-1,1]$, and a Kolmogorov–Rogozin argument gives the required edge-strip upper tail at the Gaussian bulk scale. Thus the bulk-gap reduction needs only two additional persistence inputs: a Gaussian-order one-time bulk gap estimate and a two-time correlation estimate for such gaps. When these two inputs hold, for every $0 < \delta < 1/(2\pi)$,

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{N_n^{(\xi)}}{\log n} \le \frac{1}{\pi} + 2\delta\right) > 0,$$

and hence $N_n^{(\xi)}/\log n \to 2/\pi$ cannot hold almost surely. We also prove an independent cone-record criterion: if the associated planar coefficient walk has a divergent cone-survival series, then $\liminf_{n\to\infty} N_n^{(\xi)}/\log n \le 1/\pi$ with positive probability. This applies in particular to Rademacher coefficients. For standard Gaussian coefficients we verify the two bulk-persistence inputs and recover the Gaussian conclusion.

1 Introduction and setup

For a real coefficient sequence $\xi_0, \xi_1, \ldots$, set

$$K_n^{(\xi)}(x) := \sum_{j=0}^{n} \xi_j x^j, \qquad Q_n^{(\xi)}(x) := x^n K_n^{(\xi)}(1/x) = \sum_{j=0}^{n} \xi_{n-j} x^j.$$

Let $N_n^{(\xi)}$ denote the number of real zeros of $K_n^{(\xi)}$, counted without multiplicity. For a polynomial P and a set $A \subset \mathbb{R}$, let $N_P(A)$ denote the number of real zeros of P in A, counted without multiplicity. If $P \equiv 0$, then for the interval sets used below we set $N_P(A) = +\infty$; in particular $N_n^{(\xi)} = +\infty$ on the zero-polynomial event. This convention is harmless under the standing hypothesis, since $\mathbb{P}(\xi = 0) < 1$.

From now on, the coefficient law in all main results is assumed to satisfy Do's moment hypotheses, namely

$$\mathbb{E}\xi = 0, \qquad \mathbb{E}\xi^2 = 1, \qquad \mathbb{E}|\xi|^{2+\eta} < \infty \quad \text{for some } \eta > 0. \tag{H}$$

Some deterministic or auxiliary estimates are stated with only the weaker hypotheses they need. Under (H), Do's theorem [4] gives

$$\frac{N_{K_n^{(\xi)}}([-1,1])}{\log n} \longrightarrow \frac{1}{\pi} \qquad \text{almost surely.} \tag{1.1}$$

For $n \geq 3$, put $L_n := 4 \log\log n$, and define

$$I_n := [-1 + L_n/n,\ 1 - L_n/n], \qquad J_n^+ := [1 - L_n/n, 1],$$

$$J_n^- := [-1, -1 + L_n/n], \qquad J_n := J_n^+ \cup J_n^-.$$

For i.i.d. coefficients with common law ξ, set

$$B_n^{(\xi)} := \{N_{Q_n^{(\xi)}}(I_n) = 0\}, \qquad p_n^{(\xi)} := \mathbb{P}(B_n^{(\xi)}),$$

and, for $\delta > 0$,

$$C_n^{(\xi),\delta} := \{N_{Q_n^{(\xi)}}(J_n) \leq 2\delta \log n\}, \qquad D_n^{(\xi),\delta} := B_n^{(\xi)} \cap C_n^{(\xi),\delta}.$$

Definition 1.1 (bulk-persistent coefficient law)**.** We say that the standing coefficient law ξ is *bulk-persistent* if the following two estimates hold.

(B1) The one-time bulk gap has Gaussian order:

$$p_n^{(\xi)} \asymp \left(\frac{L_n}{n}\right)^{3/8}.$$

(B2) For every $\varepsilon > 0$, there is $C_\varepsilon < \infty$ such that, for all sufficiently large n and all integers $h \geq n^\varepsilon$,

$$\mathbb{P}(B_n^{(\xi)} \cap B_{n+h}^{(\xi)}) \leq C_\varepsilon p_n^{(\xi)} p_h^{(\xi)}.$$

Theorem 1.2 (bulk-persistence Erdős–521 criterion)**.** *If ξ is bulk-persistent, then, for every $0 < \delta < 1/(2\pi)$,*

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{N_n^{(\xi)}}{\log n} \leq \frac{1}{\pi} + 2\delta\right) > 0.$$

Consequently,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N_n^{(\xi)}}{\log n} = \frac{2}{\pi}\right) < 1.$$

The second mechanism uses the cone

$$\mathcal{K} := \{(u,v) \in \mathbb{R}^2 : u \geq |v|\}.$$

For the coefficient law ξ, let

$$q_m^{(\xi)} := \mathbb{P}\left(\sum_{i=0}^{r} (\xi_{2i+1}, \xi_{2i}) \in \mathcal{K} \text{ for every } 0 \leq r \leq m\right), \qquad m \geq 0. \tag{1.2}$$

We call $\sum_m q_m^{(\xi)} = \infty$ the *divergent cone-survival* condition.

Theorem 1.3 (cone-survival Kac polynomials)**.** *Under the standing hypothesis* (H)*, if*

$$\sum_{m=0}^{\infty} q_m^{(\xi)} = \infty,$$

then

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{N_n^{(\xi)}}{\log n} \le \frac{1}{\pi}\right) > 0.$$

Consequently,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N_n^{(\xi)}}{\log n} = \frac{2}{\pi}\right) < 1.$$

Corollary 1.4 (Rademacher coefficients)**.** *Let $\varepsilon_0, \varepsilon_1, \dots$ be independent symmetric Rademacher variables and set*

$$K_n(x) = \sum_{j=0}^{n} \varepsilon_j x^j.$$

If N_n denotes the number of real zeros of K_n, counted without multiplicity, then

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{N_n}{\log n} \le \frac{1}{\pi}\right) > 0.$$

In particular,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N_n}{\log n} = \frac{2}{\pi}\right) < 1.$$

The proof of Theorem 1.2 is given in Section 8. The proofs of Theorem 1.3 and Corollary 1.4 are given in Section 7. The point of the standing hypothesis is that the typical interior-zero input (1.1) is automatic, and the edge-strip input follows from Corollary 6.3. Therefore the bulk-gap route requires only the two rare-event estimates in Definition 1.1(B1)–(B2). These estimates are persistence inputs and are not supplied by Do's strong law alone.

Remark 1.5 (status of (B1)–(B2) under the standing hypothesis)**.** The assumptions (H) are not enough, within the present argument, to verify Definition 1.1(B1)–(B2). Do's theorem is an almost-sure law of large numbers for the typical number of zeros; it gives the strong law (1.1), but it does not estimate the rare event that a growing bulk window contains no zero, nor does it estimate correlations between such rare events at two different degrees. Recent persistence-universality results of Ghosal–Mukherjee [7] show logarithmic-scale universality for no-real-zero probabilities under finite variance assumptions, but those results do not supply the two-sided correct-order bound $p_n \asymp (L_n/n)^{3/8}$ with the present $L_n = 4 \log\log n$ cutoff, and they do not give the two-time estimate (B2).

There is also a structural reason not to import the Gaussian proof verbatim. The proof of Theorem 5.2 uses Gaussian tools in an essential way: a Cameron–Martin barrier estimate for the limiting process and Borell–TIS tails to make the old-block bad event smaller than every prescribed power of h. The moment condition (H) gives only a finite $(2+\eta)$-moment and does not provide these sub-Gaussian uniform tails or a Gaussian comparison principle for persistence with deterministic drift. Thus a proof of (B1)–(B2) under the standing hypothesis would require additional persistence-universality and two-time correlation estimates, beyond both Do's strong law and the arguments below.

Remark 1.6 (Rademacher coefficients and the role of (B1)–(B2))**.** A symmetric Rademacher random variable satisfies the standing hypothesis. The Gaussian-style bulk-persistence route would still require the sharp one-time and two-time bulk persistence inputs Definition 1.1(B1)–(B2). Existing persistence-universality results, such as Ghosal–Mukherjee [7], give logarithmic exponent information but not those correct-order and two-time estimates.

Theorem 1.3 therefore uses a different mechanism. Instead of estimating the probability of a typical Gaussian-order bulk gap, it finds a smaller but still nonsummable family of explicit events on which the reversed polynomial has no root in $(-1,1)\setminus\{0\}$. These events are cone-record events for a planar random walk, with pair correlations controlled by the record decomposition. In the Rademacher case their probabilities are of order $1/(m+1)$, which proves Corollary 1.4. Thus the cone-survival result should be viewed as a bypass of (B1)–(B2), not as a verification of them.

The rest of the paper separates the abstract reduction from ensemble-specific inputs. Sections 2–5 verify the two bulk-persistence assumptions for standard Gaussian coefficients, Section 6 proves the edge-strip estimate under the standing moment hypotheses, and Section 7 proves the general cone-survival theorem and the Rademacher corollary by the independent cone-record argument.

For the Gaussian verification, we specialize to

$$K_n(x)=\sum_{j=0}^{n} g_j x^j, \qquad g_0,g_1,\ldots \overset{\text{i.i.d.}}{\sim} N(0,1),$$

and we write

$$Q_n(x):=x^n K_n(1/x)=\sum_{j=0}^{n} g_{n-j}x^j.$$

For each fixed n, Q_n has the same law as K_n. In this Gaussian case we suppress the superscript (g) and write

$$B_n:=\{N_{Q_n}(I_n)=0\}.$$

By countable intersection, there is a full-probability event Ω_* on which, for every $n\ge 0$, one has $g_n\neq 0$, $K_n(\pm 1)\neq 0$, and all real zeros of K_n are simple. We shall use this event only for the Gaussian pathwise identifications; all probability estimates are unchanged after discarding its null complement. On Ω_*, under the reciprocal map $x\mapsto 1/x$, the zeros of Q_n in $(-1,1)$ correspond exactly, with multiplicity, to the zeros of K_n in $(-\infty,-1)\cup(1,\infty)$.

The Gaussian bulk-persistence verification has two steps.

1. Prove the correct-order one-time estimate

$$\mathbb{P}(B_n)\asymp (L_n/n)^{3/8}.$$

2. Prove the pair bound

$$\mathbb{P}(B_n\cap B_{n+h})\ll_\varepsilon (L_n/n)^{3/8}(L_h/h)^{3/8} \qquad (h\ge n^\varepsilon).$$

The standing strong law (1.1) and the endpoint estimate from Section 6 supply the remaining inputs used in the abstract reduction. Combining these inputs gives, with positive probability, infinitely many n for which the reversed polynomial has no roots in the bulk and only $O(\log n)$ roots in the edge strips. The bulk-persistence theorem then shows that convergence of $N_n^{(\xi)}/\log n$ to $2/\pi$ cannot hold with probability one for any bulk-persistent Kac ensemble. For Rademacher coefficients the cone-record proof in Section 7 gives the conclusion directly, without proving the bulk persistence hypotheses (B1)–(B2).

Throughout, $n^{-\omega(1)}$ denotes a quantity bounded by $C_A n^{-A}$ for every fixed $A>0$, with $C_A<\infty$.

2 The limiting stationary process

Set

$$x_t := \tanh\frac{t}{2}, \qquad t \in \mathbb{R}.$$

Then $x_t \in (-1,1)$ and $x_{-t} = -x_t$. For each $n \geq 3$, define

$$T_n := \log\frac{2n - L_n}{L_n}.$$

A direct calculation gives

$$x_{T_n} = 1 - \frac{L_n}{n}, \qquad x_{-T_n} = -1 + \frac{L_n}{n},$$

so x_t maps $[-T_n, T_n]$ onto I_n.

Let

$$\mathcal{G}(x) := \sum_{j=0}^{\infty} \gamma_j x^j, \qquad \gamma_0, \gamma_1, \ldots \overset{\text{i.i.d.}}{\sim} N(0,1),$$

be the Gaussian power series, and define

$$Y_t := \sqrt{1 - x_t^2}\, \mathcal{G}(x_t), \qquad t \in \mathbb{R}.$$

Then Y is a centered stationary Gaussian process with covariance

$$\mathbb{E}[Y_s Y_t] = \operatorname{sech}\Big(\frac{s-t}{2}\Big).$$

Indeed,

$$\mathbb{E}[Y_s Y_t] = \frac{\sqrt{(1 - x_s^2)(1 - x_t^2)}}{1 - x_s x_t} = \frac{1}{\cosh((s-t)/2)}.$$

Let

$$d(T) := \mathbb{P}(Y_t > 0\ \forall |t| \leq T), \qquad T \geq 0.$$

The Pfaffian structure of the real zero process of $\mathcal{G}$ was identified by Forrester [6] and by Matsumoto–Shirai [8]. We use the sharp gap asymptotic of FitzGerald, Tribe and Zaboronski [5, Section 3.1, equation (17)]: after the map

$$u := \Phi(x) := \frac{1}{2}\log\frac{1+x}{1-x} = \operatorname{arctanh} x,$$

the zero process is stationary in the u-coordinate and

$$\log \mathbb{P}(\mathcal{G} \text{ has no real zero in } [a,b]) = -\frac{3}{8}(\Phi(b) - \Phi(a)) + O(1) \qquad (a \downarrow -1,\ b \uparrow 1).$$

In our parametrization $x_t = \tanh(t/2)$, one has $\Phi(x_t) = t/2$. Therefore the event

$$\{Y_t \neq 0\ \forall |t| \leq T\}$$

is exactly the event that $\mathcal{G}$ has no real zero in $[x_{-T}, x_T]$, and

$$\Phi(x_T) - \Phi(x_{-T}) = T.$$

Thus the FTZ formula gives

$$\log \mathbb{P}(Y_t \neq 0\ \forall |t| \leq T) = -\frac{3}{8}T + O(1),$$

and therefore

$$\mathbb{P}(Y_t \neq 0 \ \forall |t| \leq T) \asymp e^{-3T/8}.$$

This is the same normalization as the one-sided statement $\log \mathbb{P}(Y_t \neq 0 \ \forall 0 \leq t \leq T) = -(3/16)T + o(T)$ appearing in [5, equation (21)]; the apparent difference between 3/16 and 3/8 is only that our symmetric interval $[-T, T]$ has t-length $2T$, while the FTZ coordinate $u = \Phi(x)$ satisfies $t = 2u$. Since the paths are continuous and the law is invariant under $Y \mapsto -Y$, the two no-zero sign events have equal probability,

$$d(T) \asymp e^{-3T/8}. \tag{2.1}$$

We also need the elementary fact that a small barrier changes the persistence probability by at most an $e^{O(\delta T)}$ factor.

Lemma 2.1 (barrier stability)**.** *There exists $C < \infty$ such that for all $T \geq 1$ and all $0 \leq \delta \leq 1$,*

$$d(T) \leq \mathbb{P}(Y_t > -\delta \ \forall |t| \leq T) \leq e^{C\delta T} d(T),$$

and

$$e^{-C\delta T} d(T) \leq \mathbb{P}(Y_t > \delta \ \forall |t| \leq T) \leq d(T).$$

Proof. The case $\delta = 0$ is trivial, so assume $0 < \delta \leq 1$. Let $I_T = [-T, T]$ and let K_T be the covariance operator on $L^2(I_T)$,

$$(K_T f)(t) := \int_{-T}^{T} \operatorname{sech}\Big(\frac{t-s}{2}\Big) f(s)\, ds.$$

Set

$$c_0 := \int_0^1 \operatorname{sech}(u/2)\, du > 0, \qquad h_\delta(s) := \frac{\delta}{c_0} \mathbf{1}_{I_T}(s), \qquad m_{T,\delta} := K_T h_\delta.$$

We recall the standard reproducing-kernel Hilbert space identification for this restriction of the process. If

$$W_h := \int_{-T}^{T} h(s) Y_s\, ds, \qquad h \in L^2(I_T),$$

then

$$\mathbb{E}[W_h Y_t] = (K_T h)(t).$$

Hence $K_T h$ belongs to the Cameron–Martin space $\mathcal{H}_T$ of the Gaussian measure induced by $Y|_{I_T}$ on $C(I_T)$. Moreover, with the usual quotient by the null space of K_T,

$$\|K_T h\|_{\mathcal{H}_T}^2 = \operatorname{Var} W_h = \langle h, K_T h \rangle_{L^2(I_T)}.$$

Applying this with $h = h_\delta$, the function $m_{T,\delta} = K_T h_\delta$ is a legitimate Cameron–Martin shift and

$$\|m_{T,\delta}\|_{\mathcal{H}_T}^2 = \langle h_\delta, K_T h_\delta \rangle_{L^2(I_T)}.$$

We write the associated Gaussian linear functional as

$$\langle h_\delta, Y \rangle := \int_{-T}^{T} h_\delta(s) Y_s\, ds.$$

Thus the Cameron–Martin formula applies to the shift $Y \mapsto Y + m_{T,\delta}$ with cost

$$\exp\Big\{ \frac{1}{2} \|m_{T,\delta}\|_{\mathcal{H}_T}^2 \Big\}.$$

After the change of variables $u = t - s$ we have

$$m_{T,\delta}(t) = \frac{\delta}{c_0} \int_{t-T}^{t+T} \operatorname{sech}\Big(\frac{u}{2}\Big)\, du.$$

Let $k(u) := \operatorname{sech}(u/2)$. Then k is even and strictly decreasing on $[0, \infty)$. For $t \in [0, T]$,

$$\frac{d}{dt} \int_{t-T}^{t+T} k(u)\, du = k(t+T) - k(t-T) = k(t+T) - k(T-t) \le 0.$$

By evenness, the same monotonicity holds as $|t|$ increases on $[-T, T]$, so the integral is minimized at $t = \pm T$. Therefore

$$m_{T,\delta}(t) \ge \frac{\delta}{c_0} \int_0^{2T} \operatorname{sech}(u/2)\, du \ge \frac{\delta}{c_0} \int_0^{1} \operatorname{sech}(u/2)\, du = \delta,$$

where we used $T \ge 1$ in the second inequality. Moreover, because $\int_{\mathbb{R}} \operatorname{sech}(u/2)\, du < \infty$,

$$\|m_{T,\delta}\|_{\mathcal{H}_T}^2 \le C_1 \delta^2 T$$

for some absolute $C_1 < \infty$.

Let

$$\mathcal{A}_T := \{f \in C(I_T) : f(t) > 0 \ \forall t \in I_T\}.$$

Since $m_{T,\delta} \ge \delta$ on I_T,

$$\{Y_t > -\delta \ \forall |t| \le T\} \subseteq \{Y + m_{T,\delta} \in \mathcal{A}_T\}.$$

The Cameron–Martin formula gives

$$\mathbb{P}(Y + m_{T,\delta} \in \mathcal{A}_T) = \mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_T}(Y) \exp\Big(\langle h_\delta, Y\rangle - \tfrac{1}{2}\|m_{T,\delta}\|_{\mathcal{H}_T}^2\Big)\Big].$$

Apply Hölder with $p = 1 + \delta$ and $q = (1+\delta)/\delta$. Since $\langle h_\delta, Y\rangle$ is centered Gaussian with variance $\|m_{T,\delta}\|_{\mathcal{H}_T}^2$,

$$\mathbb{P}(Y + m_{T,\delta} \in \mathcal{A}_T) \le d(T)^{1/p} \exp\Big(\frac{q-1}{2}\|m_{T,\delta}\|_{\mathcal{H}_T}^2\Big) \le d(T)^{1/(1+\delta)} e^{C_2 \delta T}.$$

By (2.1), $-\log d(T) \ll T$, so

$$d(T)^{-\delta/(1+\delta)} \le e^{C_3 \delta T}.$$

Therefore

$$\mathbb{P}(Y_t > -\delta \ \forall |t| \le T) \le e^{C\delta T} d(T)$$

for some $C < \infty$. The lower bound is trivial.

For the positive barrier, let

$$\mathcal{B}_{T,\delta} := \{f \in C(I_T) : f(t) > \delta \ \forall t \in I_T\}.$$

Since $m_{T,\delta} \ge \delta$ on I_T,

$$\{Y \in \mathcal{A}_T\} \subseteq \{Y + m_{T,\delta} \in \mathcal{B}_{T,\delta}\}.$$

Hence

$$d(T) = \mathbb{P}(Y \in \mathcal{A}_T) \le \mathbb{P}(Y + m_{T,\delta} \in \mathcal{B}_{T,\delta}).$$

Applying the Cameron–Martin formula again and then Hölder with the same exponents $p = 1 + \delta$ and $q = (1+\delta)/\delta$, we obtain

$$d(T) \le \mathbb{P}(Y + m_{T,\delta} \in \mathcal{B}_{T,\delta}) \le \mathbb{P}(Y_t > \delta \ \forall |t| \le T)^{1/p} e^{C_4 \delta T}.$$

Therefore

$$\mathbb{P}(Y_t > \delta \ \forall |t| \le T) \ge d(T)^p e^{-C_5 \delta T} = d(T)\, d(T)^\delta e^{-C_5 \delta T}.$$

Using again (2.1), we have $d(T)^\delta \ge e^{-C_6 \delta T}$, so

$$\mathbb{P}(Y_t > \delta \ \forall |t| \le T) \ge e^{-C\delta T} d(T)$$

for a possibly larger constant $C < \infty$. The upper bound is trivial. □

3 A uniform approximation on the bulk window

For $|t| \le T_n$, define

$$\Sigma_n(t)^2 := \sum_{j=0}^{n} x_t^{2j} = \frac{1 - x_t^{2n+2}}{1 - x_t^2}, \qquad X_n(t) := \frac{Q_n(x_t)}{\Sigma_n(t)}.$$

Then B_n is exactly the event that X_n is either strictly positive everywhere on $[-T_n, T_n]$ or strictly negative everywhere on $[-T_n, T_n]$.

Lemma 3.1 (approximation by the limiting process)**.** *For each n, one can construct, on a common probability space, a stationary* sech*-process Y and another stationary* sech*-process $\widetilde{Y}^{(n)}$ such that for all $|t| \le T_n$,*

$$X_n(t) = Y_t + \Delta_n(t), \qquad |\Delta_n(t)| \le 2e^{-L_n}|\widetilde{Y}_t^{(n)}| + 2e^{-2L_n}|Y_t|.$$

Consequently, there exist absolute constants $c, C < \infty$ such that for every $M \ge 1$ and all large n,

$$\mathbb{P}\Big(\sup_{|t|\le T_n} |X_n(t) - Y_t| > CMe^{-L_n}\sqrt{\log n}\Big) \le e^{-cM^2 \log n}.$$

Proof. Since only the fixed-n law of Q_n enters this lemma, we may replace the finite coefficient block of Q_n by a fresh block $(a_0, \dots, a_n)$ with the same joint law and realize

$$Q_n(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_0, a_1, \dots \overset{\text{i.i.d.}}{\sim} N(0,1),$$

on the same probability space as the infinite Gaussian power series $\mathcal{G}(x) = \sum_{j=0}^{\infty} a_j x^j$. This coupling preserves exactly the law of the finite polynomial Q_n; the coefficients $a_{n+1}, a_{n+2}, \dots$ are only auxiliary tail variables used to compare Q_n with $\mathcal{G}$. Then

$$Q_n(x) = \mathcal{G}(x) - x^{n+1}\widetilde{\mathcal{G}}_n(x),$$

where

$$\widetilde{\mathcal{G}}_n(x) := \sum_{j=0}^{\infty} a_{n+1+j} x^j.$$

Define

$$\widetilde{Y}_t^{(n)} := \sqrt{1 - x_t^2}\, \widetilde{\mathcal{G}}_n(x_t).$$

Then $\widetilde{Y}^{(n)}$ has the same stationary sech-process law as Y (although, in general, it is not independent of Y). No independence between these two processes is needed here. Since

$$\Sigma_n(t) = \frac{1}{\sqrt{1 - x_t^2}} \sqrt{1 - x_t^{2n+2}},$$

we get

$$X_n(t) = \frac{Y_t - x_t^{n+1}\widetilde{Y}_t^{(n)}}{\sqrt{1 - x_t^{2n+2}}}.$$

Write

$$\Delta_n(t) := X_n(t) - Y_t.$$

On $|t| \le T_n$ one has $|x_t| \le 1 - L_n/n$, so

$$|x_t|^{n+1} \le \Big(1 - \frac{L_n}{n}\Big)^{n+1} \le e^{-L_n}, \qquad |x_t|^{2n+2} \le e^{-2L_n}.$$

For all large n, $e^{-2L_n} \le 1/2$, hence

$$\frac{1}{\sqrt{1-x_t^{2n+2}}} \le 2, \qquad \Big|\frac{1}{\sqrt{1-x_t^{2n+2}}} - 1\Big| \le 2e^{-2L_n}.$$

Substituting this into the formula for $X_n(t)$ gives

$$|\Delta_n(t)| \le 2e^{-L_n}|\widetilde{Y}_t^{(n)}| + 2e^{-2L_n}|Y_t|.$$

For the tail estimate, first consider a stationary Gaussian process Z with covariance

$$\mathbb{E}Z_sZ_t = \operatorname{sech}\Big(\frac{s-t}{2}\Big).$$

Its canonical metric satisfies

$$d_Z(s,t)^2 := \mathbb{E}(Z_s - Z_t)^2 = 2\Big\{1 - \operatorname{sech}\Big(\frac{s-t}{2}\Big)\Big\}.$$

Using $1 - \operatorname{sech} u \le C\min\{u^2, 1\}$, we obtain

$$d_Z(s,t) \le C\min\{|s-t|, 1\}.$$

Consequently,

$$N([-T,T], d_Z, \varepsilon) \le C\frac{2+T}{\varepsilon}, \qquad 0 < \varepsilon \le 1.$$

Dudley's entropy bound gives

$$\mathbb{E}\sup_{|t|\le T}|Z_t| \le C\sqrt{\log(2+T)}.$$

By the Borell–TIS inequality (see, e.g., [1, Section 2.1]), for all $M \ge 1$ and all large n,

$$\mathbb{P}\Big(\sup_{|t|\le T_n}|Z_t| > CM\sqrt{\log n}\Big) \le e^{-cM^2\log n}.$$

Applying this estimate to Y and to $\widetilde{Y}^{(n)}$, and then using the pointwise bound for Δ_n, gives the stated tail estimate. □

Remark 3.2. The same proof applies verbatim to a Gaussian polynomial with any prescribed number of coefficients. Here the subscript m denotes the number of coefficients, not the degree: if

$$P_{m-1}(x) = \sum_{r=0}^{m-1} b_r x^r, \qquad b_0, \ldots, b_{m-1} \overset{\text{i.i.d.}}{\sim} N(0,1),$$

then we define

$$\Sigma_m(t)^2 := \sum_{r=0}^{m-1} x_t^{2r}, \qquad T_m := \log\frac{2m - L_m}{L_m}.$$

With this convention, the normalized process

$$\frac{P_{m-1}(x_t)}{\Sigma_m(t)}$$

admits the same coupling with a stationary sech-process on $[-T_m, T_m]$, with the same proof and with the tail term x_t^m. We shall use this degree-$(h-1)$ variant for the fresh block F_h below.

4 Bulk gap probabilities

Theorem 4.1 (one-time bulk law). *There exist constants $0 < c < C < \infty$ such that for all sufficiently large n,*

$$c\Big(\frac{L_n}{n}\Big)^{3/8} \le \mathbb{P}(B_n) \le C\Big(\frac{L_n}{n}\Big)^{3/8}.$$

Proof. Since B_n is the event that X_n is strictly of one sign on $[-T_n, T_n]$,

$$\mathbb{P}(B_n) = 2\mathbb{P}(X_n(t) > 0 \ \forall |t| \le T_n).$$

Let $c_{\text{app}}, C_{\text{app}} > 0$ be the constants from Lemma 3.1. Fix $M \ge 1$ and set

$$\eta_n := C_{\text{app}} M e^{-L_n}\sqrt{\log n}.$$

By Lemma 3.1, after coupling X_n with Y,

$$\mathbb{P}\Big(\sup_{|t|\le T_n} |X_n(t) - Y_t| > \eta_n\Big) \le e^{-c_{\text{app}} M^2 \log n}.$$

For all sufficiently large n, one has $\eta_n \le 1$. Therefore

$$\begin{aligned}\mathbb{P}(Y_t > \eta_n \ \forall |t| \le T_n) &- e^{-c_{\text{app}} M^2 \log n}\\ &\le \mathbb{P}(X_n(t) > 0 \ \forall |t| \le T_n)\\ &\le \mathbb{P}(Y_t > -\eta_n \ \forall |t| \le T_n) + e^{-c_{\text{app}} M^2 \log n}.\end{aligned}$$

Because

$$\eta_n T_n \ll e^{-L_n}(\log n)^{3/2} = (\log n)^{-5/2} \to 0,$$

and $\eta_n \le 1$ for all sufficiently large n, Lemma 2.1 gives

$$\mathbb{P}(Y_t > \eta_n \ \forall |t| \le T_n) \asymp d(T_n) \asymp \mathbb{P}(Y_t > -\eta_n \ \forall |t| \le T_n).$$

Choose M so large that $c_{\text{app}} M^2 > 1$. Then $e^{-c_{\text{app}} M^2 \log n} = o(d(T_n))$ by (2.1). Hence

$$\mathbb{P}(X_n(t) > 0 \ \forall |t| \le T_n) \asymp d(T_n).$$

Using (2.1),

$$\mathbb{P}(B_n) \asymp e^{-3T_n/8}.$$

Finally,

$$e^{-T_n} = \frac{L_n}{2n - L_n} \asymp \frac{L_n}{n},$$

so

$$\mathbb{P}(B_n) \asymp \Big(\frac{L_n}{n}\Big)^{3/8}.$$

□

5 The bulk pair bound

Fix $n \ge 3$ and $h \ge 3$, and write $m = n + h$. Decompose

$$Q_m(x) = F_h(x) + x^h Q_n(x), \qquad F_h(x) := \sum_{r=0}^{h-1} g_{m-r} x^r.$$

The fresh block F_h is independent of $\mathcal{F}_n := \sigma(g_0, \dots, g_n)$.

For the h-scale bulk window, define

$$T_h := \log \frac{2h - L_h}{L_h}, \qquad \Sigma_h(t)^2 := \sum_{r=0}^{h-1} x_t^{2r}, \qquad |t| \le T_h,$$

and set

$$X_h(t) := \frac{F_h(x_t)}{\Sigma_h(t)}, \qquad \Psi_{n,h}(t) := \frac{x_t^h Q_n(x_t)}{\Sigma_h(t)}.$$

After increasing the lower threshold for n and h, if necessary, we may assume that $r \mapsto L_r/r = 4(\log\log r)/r$ is decreasing for all r under consideration. Since $m > h$,

$$[-1 + L_h/h, 1 - L_h/h] \subseteq [-1 + L_m/m, 1 - L_m/m],$$

so

$$B_m \subseteq \{(X_h + \Psi_{n,h})(t) \neq 0 \text{ for all } |t| \le T_h\}. \tag{5.1}$$

Lemma 5.1 (smallness of the old block)**.** *There exist absolute constants $c, C < \infty$ such that, for all sufficiently large n, h and all $M \ge 1$,*

$$\mathbb{P}\Big(\sup_{|t| \le T_h} |\Psi_{n,h}(t)| > CMe^{-L_h}\sqrt{\log h} \Big) \le e^{-cM^2 \log h}.$$

Proof. Write

$$Q_n(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j := g_{n-j}.$$

Let $(a'_j)_{j \ge 0}$ be i.i.d. $N(0,1)$, independent of $\mathcal{F}_n$, and extend Q_n by an independent Gaussian tail:

$$\mathcal{H}_n(x) := Q_n(x) + x^{n+1} \sum_{j=0}^{\infty} a'_j x^j.$$

Then $\mathcal{H}_n$ is an infinite Gaussian power series with i.i.d. standard normal coefficients, and we can write

$$Q_n(x) = \mathcal{H}_n(x) - x^{n+1} \widetilde{\mathcal{H}}_n(x), \qquad \widetilde{\mathcal{H}}_n(x) := \sum_{j=0}^{\infty} a'_j x^j.$$

Define

$$Z_t := \sqrt{1 - x_t^2}\, \mathcal{H}_n(x_t), \qquad \widetilde{Z}_t := \sqrt{1 - x_t^2}\, \widetilde{\mathcal{H}}_n(x_t),$$

so Z and $\widetilde{Z}$ are stationary sech-processes. They need not be independent. Since

$$\Sigma_h(t) = \frac{1}{\sqrt{1 - x_t^2}} \sqrt{1 - x_t^{2h}},$$

we obtain

$$\Psi_{n,h}(t) = \frac{x_t^h Z_t - x_t^{h+n+1} \widetilde{Z}_t}{\sqrt{1 - x_t^{2h}}}.$$

On $|t| \le T_h$ one has $|x_t| \le 1 - L_h/h$, hence

$$|x_t|^h \le e^{-L_h}, \qquad |x_t|^{h+n+1} \le e^{-L_h}, \qquad \frac{1}{\sqrt{1 - x_t^{2h}}} \le 2$$

for all large h. Therefore

$$|\Psi_{n,h}(t)| \le 2e^{-L_h}(|Z_t| + |\widetilde{Z}_t|).$$

Fix $M \geq 1$ and let

$$A := \Big\{ \sup_{|t|\leq T_h} |\Psi_{n,h}(t)| > CMe^{-L_h}\sqrt{\log h} \Big\}.$$

Since A depends only on Q_n, while the auxiliary tail $(a'_j)_{j\geq 0}$ is independent of Q_n, we may write

$$\mathbb{P}(A) = \mathbb{E}_{Q_n} \mathbf{1}_A(Q_n) = \mathbb{E}_{Q_n,a'} \mathbf{1}_A(Q_n).$$

For every fixed realization of the old block Q_n and every fixed realization of the auxiliary tail, the displayed bound for $\Psi_{n,h}$ gives the pointwise indicator domination

$$\mathbf{1}_A(Q_n) \leq \mathbf{1}_{\{2e^{-L_h}\sup_{|t|\leq T_h}(|Z_t|+|\widetilde{Z}_t|)>CMe^{-L_h}\sqrt{\log h}\}}.$$

The event A itself depends only on Q_n, so adjoining the auxiliary tail does not change its law. Averaging the pointwise inequality first over the auxiliary tail and then over Q_n, or equivalently taking expectation over the enlarged space, we obtain

$$\mathbb{P}(A) \leq \mathbb{P}\Big(\sup_{|t|\leq T_h} (|Z_t| + |\widetilde{Z}_t|) > \tfrac{C}{2} M\sqrt{\log h}\Big).$$

Since Z and $\widetilde{Z}$ are stationary sech-processes, the same entropy estimate and Borell–TIS bound as in Lemma 3.1 (again see [1, Section 2.1]) yield the stated tail estimate after adjusting the constants. □

Theorem 5.2 (bulk pair bound). *Fix $\varepsilon > 0$. There exists $C_\varepsilon < \infty$ such that, for all sufficiently large n and all integers $h \geq n^\varepsilon$,*

$$\mathbb{P}(B_n \cap B_{n+h}) \leq C_\varepsilon \Big(\frac{L_n}{n}\Big)^{3/8} \Big(\frac{L_h}{h}\Big)^{3/8}.$$

Proof. Fix $\varepsilon > 0$. We only need to consider sufficiently large n and integers $h \geq n^\varepsilon$, so in particular $h \geq 3$ and the monotonicity discussion above applies.

Let $c_{\text{app}}, C_{\text{app}} > 0$ be the constants from the degree-$(h-1)$ variant of Lemma 3.1 furnished by Remark 3.2, and let $c_{\text{old}}, C_{\text{old}} > 0$ be the constants from Lemma 5.1. Set

$$c_0 := \min\{c_{\text{app}}, c_{\text{old}}\}, \qquad C_0 := \max\{C_{\text{app}}, C_{\text{old}}\}.$$

Choose M so large that

$$c_0 M^2 > \frac{3}{8}\Big(1 + \frac{1}{\varepsilon}\Big),$$

and define

$$\rho_h := C_0 M e^{-L_h}\sqrt{\log h}.$$

We now make the conditioning and coupling step explicit. Enlarge the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a product space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ by adjoining an auxiliary Gaussian tail, independent of the entire original σ-field $\mathcal{F}$, so that the fresh block F_h becomes the first h coefficients of an infinite Gaussian power series. The degree-$(h-1)$ approximation lemma from Remark 3.2 then yields, on $(\widehat{\Omega}, \widehat{\mathbb{P}})$, a stationary sech-process $Y^{(h)}$ built from the fresh block together with this auxiliary tail. In particular, $Y^{(h)}$ is not independent of X_h, but it is independent of $\mathcal{F}_n$. Define

$$E_h := \Big\{ \sup_{|t|\leq T_h} |X_h(t) - Y_t^{(h)}| \leq \rho_h \Big\}.$$

Then E_h depends only on the fresh block and the auxiliary randomness, is independent of $\mathcal{F}_n$, and satisfies

$$\widehat{\mathbb{P}}(E_h^c \mid \mathcal{F}_n) = \widehat{\mathbb{P}}(E_h^c) \leq e^{-c_0 M^2 \log h}. \tag{5.2}$$

No independence between E_h and B_m is asserted or needed; the argument below uses only the deterministic implication obtained when B_m, E_h, and the old-block good event all occur. Also define the $\mathcal{F}_n$-measurable event

$$G_{n,h} := \Big\{ \sup_{|t|\le T_h} |\Psi_{n,h}(t)| \le \rho_h \Big\}.$$

By Lemma 5.1,

$$\mathbb{P}(G_{n,h}^c) \le e^{-c_0 M^2 \log h}. \tag{5.3}$$

Fix an outcome of $\mathcal{F}_n$ for which $G_{n,h}$ holds. Probabilities below are taken on the enlarged product space, and the original event B_m is identified with its pullback to that space. Since B_m does not depend on the auxiliary randomness, this extension preserves its conditional law:

$$\mathbb{P}(B_m \mid \mathcal{F}_n) = \widehat{\mathbb{P}}(B_m \mid \mathcal{F}_n) \le \widehat{\mathbb{P}}(B_m \cap E_h \mid \mathcal{F}_n) + \widehat{\mathbb{P}}(E_h^c \mid \mathcal{F}_n).$$

If in addition E_h holds and B_m occurs, then by (5.1) the process $X_h + \Psi_{n,h}$ has no zero and therefore is strictly of one sign on $[-T_h, T_h]$. This is a deterministic consequence of the three events B_m, E_h, and $G_{n,h}$, not an independence statement. Hence there exists $\sigma \in \{\pm 1\}$ such that

$$\sigma(X_h(t) + \Psi_{n,h}(t)) > 0 \qquad \text{for all } |t| \le T_h.$$

Since on $E_h \cap G_{n,h}$ one has $|X_h(t) - Y_t^{(h)}| \le \rho_h$ and $|\Psi_{n,h}(t)| \le \rho_h$, it follows that

$$\sigma Y_t^{(h)} \ge \sigma(X_h(t) + \Psi_{n,h}(t)) - |X_h(t) - Y_t^{(h)}| - |\Psi_{n,h}(t)| > -2\rho_h$$

for all $|t| \le T_h$. Therefore, on $G_{n,h}$,

$$\widehat{\mathbb{P}}(B_m \cap E_h \mid \mathcal{F}_n) \le \sum_{\sigma \in \{\pm 1\}} \widehat{\mathbb{P}}(\sigma Y_t^{(h)} > -2\rho_h \ \forall |t| \le T_h) = 2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h).$$

Combining this with (5.2), and using that $Y^{(h)}$ has the same law as Y, we obtain on $G_{n,h}$,

$$\mathbb{P}(B_m \mid \mathcal{F}_n) \le 2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) + e^{-c_0 M^2 \log h}.$$

Since the trivial bound $\mathbb{P}(B_m \mid \mathcal{F}_n) \le 1$ always holds, we may write

$$\mathbb{P}(B_m \mid \mathcal{F}_n) \le \mathbf{1}_{G_{n,h}} \Big[2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) + e^{-c_0 M^2 \log h} \Big] + \mathbf{1}_{G_{n,h}^c}.$$

Multiply by $\mathbf{1}_{B_n}$ and take expectations:

$$\begin{aligned} \mathbb{P}(B_n \cap B_m) &= \mathbb{E}[\mathbf{1}_{B_n} \mathbb{P}(B_m \mid \mathcal{F}_n)] \\ &\le \mathbb{P}(B_n) \Big[2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) + e^{-c_0 M^2 \log h} \Big] + \mathbb{P}(G_{n,h}^c). \end{aligned}$$

Now

$$\rho_h T_h \ll e^{-L_h} (\log h)^{3/2} = (\log h)^{-5/2} \to 0,$$

and $2\rho_h \le 1$ for all sufficiently large h, so Lemma 2.1 gives

$$\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) \ll d(T_h) \asymp \Big(\frac{L_h}{h} \Big)^{3/8}.$$

Also, since $c_0 M^2 > 3/8$,

$$e^{-c_0 M^2 \log h} = h^{-c_0 M^2} = o\Big(\Big(\frac{L_h}{h} \Big)^{3/8} \Big),$$

so the term $\mathbb{P}(B_n)e^{-c_0M^2\log h}$ is absorbed into the main contribution

$$\mathbb{P}(B_n)\Big(\frac{L_h}{h}\Big)^{3/8}.$$

Finally, by (5.3),

$$\mathbb{P}(G_{n,h}^c) \le h^{-c_0M^2}.$$

Put $a = c_0M^2$. Since $h \ge n^\varepsilon$ and $a > \frac{3}{8}(1+1/\varepsilon)$, we have $\varepsilon(a-3/8) > 3/8$, and hence

$$h^{-a} = h^{-3/8}h^{-(a-3/8)} \le h^{-3/8}n^{-\varepsilon(a-3/8)} \le n^{-3/8}h^{-3/8}.$$

Consequently,

$$h^{-c_0M^2} \le C_\varepsilon\Big(\frac{L_n}{n}\Big)^{3/8}\Big(\frac{L_h}{h}\Big)^{3/8}$$

for all large n. Combining this with Theorem 4.1,

$$\mathbb{P}(B_n) \ll \Big(\frac{L_n}{n}\Big)^{3/8},$$

we conclude that

$$\mathbb{P}(B_n \cap B_{n+h}) \ll_\varepsilon \Big(\frac{L_n}{n}\Big)^{3/8}\Big(\frac{L_h}{h}\Big)^{3/8}.$$

□

Corollary 5.3 (bulk gaps infinitely often)**.**

$$\mathbb{P}(B_n \text{ i.o.}) > 0.$$

Proof. Fix $\varepsilon \in (0, 5/8)$. Set

$$p_n := \mathbb{P}(B_n) \asymp \Big(\frac{L_n}{n}\Big)^{3/8}.$$

Then

$$S_N := \sum_{n\le N} p_n \asymp N^{5/8}L_N^{3/8}.$$

To apply the Kochen–Stone lemma [10], it suffices to show

$$\sum_{m,n\le N} \mathbb{P}(B_m \cap B_n) \ll S_N^2.$$

By symmetry,

$$\sum_{m,n\le N} \mathbb{P}(B_m \cap B_n) \ll S_N + \sum_{n\le N}\sum_{1\le h\le N-n} \mathbb{P}(B_n \cap B_{n+h}).$$

Split the inner sum into $h < n^\varepsilon$ and $h \ge n^\varepsilon$. For the small gaps, use the trivial estimate

$$\mathbb{P}(B_n \cap B_{n+h}) \le p_n,$$

which gives

$$\sum_{n\le N}\sum_{1\le h<n^\varepsilon} \mathbb{P}(B_n \cap B_{n+h}) \ll \sum_{n\le N} n^\varepsilon p_n \ll N^{\varepsilon+5/8}L_N^{3/8} = o(S_N^2)$$

because $\varepsilon < 5/8$.

For the large gaps, Theorem 5.2 gives

$$\sum_{n\le N}\sum_{h\ge n^\varepsilon} \mathbb{P}(B_n \cap B_{n+h}) \ll_\varepsilon \sum_{n\le N} p_n \sum_{h\le N} p_h \ll S_N^2.$$

Hence

$$\sum_{m,n\le N} \mathbb{P}(B_m \cap B_n) \ll S_N^2.$$

Since $\sum_n p_n = \infty$, the Kochen–Stone lemma yields

$$\mathbb{P}(B_n \text{ i.o.}) > 0.$$

□

6 A general edge upper tail

The endpoint estimates below are not intrinsically Gaussian. They require only second moments and an anti-concentration estimate for signed sums of the coefficients. We first state the Rolle-theoretic edge estimate with an explicit signed concentration function, and then verify the needed concentration bound from the Kolmogorov–Rogozin inequality under the non-degeneracy already implied by the standing hypothesis. Let

$$P_n(x) := \sum_{j=0}^{n} \xi_j x^j, \qquad \mathbb{E}\xi_j = 0, \qquad \mathbb{E}\xi_j^2 = 1,$$

where $\xi_0, \xi_1, \dots$ are independent copies of a real random variable. For $\theta > 0$, define the signed concentration function

$$\mathfrak{q}_n(\theta) := \sup_{\varepsilon_0,\dots,\varepsilon_n \in \{-1,1\}} \mathbb{P}\left(\left| \sum_{j=0}^{n} \varepsilon_j \xi_j \right| \le \theta\sqrt{n+1} \right).$$

Lemma 6.1 (one-sided edge upper tail)**.** *There is an absolute constant $C < \infty$ such that, for every $n \ge 1$, every $0 \le y < 1$, every integer h with $1 \le h \le n$, and every $\theta > 0$,*

$$\mathbb{P}(N_{P_n}([y,1]) \ge h) \le \mathfrak{q}_n(\theta) + C\theta^{-2} \left(\frac{e\, n(1-y)}{h} \right)^{h}.$$

The same estimate holds for $[-1,-y]$.

Proof. The argument follows the same Rolle–theoretic template as in Can–Nguyen [2], but the proof only uses the variance normalization and the concentration function $\mathfrak{q}_n$.

We first record a deterministic consequence of Rolle's theorem.

Claim. Let $b \in [y,1]$. If $f \in C^h([y,b])$ has at least h distinct zeros in $[y,b]$, then

$$|f(b)| \le \frac{1}{(h-1)!} \int_y^b (b-u)^{h-1} |f^{(h)}(u)|\, du. \tag{6.1}$$

For $h = 1$, this is immediate by integrating from a zero of f in $[y,b]$ to the endpoint b. Assume $h \ge 2$ and that the claim holds for $h-1$. Let z be the largest zero of f in $[y,b]$. For every $s \in [z,b]$, all h zeros of f lie in $[y,s]$, so Rolle's theorem gives at least $h-1$ distinct zeros of f' in $[y,s]$. Applying the induction hypothesis to f' on $[y,s]$, we obtain

$$|f'(s)| \le \frac{1}{(h-2)!} \int_y^s (s-u)^{h-2} |f^{(h)}(u)|\, du.$$

Since $f(z) = 0$,

$$\begin{aligned}
|f(b)| &\le \int_z^b |f'(s)|\, ds \\
&\le \frac{1}{(h-2)!} \int_z^b \int_y^s (s-u)^{h-2} |f^{(h)}(u)|\, du\, ds \\
&= \frac{1}{(h-2)!} \int_y^b |f^{(h)}(u)| \int_{\max\{z,u\}}^b (s-u)^{h-2}\, ds\, du \\
&\le \frac{1}{(h-2)!} \int_y^b |f^{(h)}(u)| \frac{(b-u)^{h-1}}{h-1}\, du,
\end{aligned}$$

which proves (6.1).

Specializing to $b = 1$, we see that if P_n has at least h distinct zeros in $[y, 1]$, then

$$|P_n(1)| \le I_h := \frac{1}{(h-1)!} \int_y^1 (1-u)^{h-1} |P_n^{(h)}(u)| \, du.$$

Hence, for every $\theta > 0$,

$$\mathbb{P}(N_{P_n}([y,1]) \ge h) \le \mathbb{P}\big(|P_n(1)| \le \theta\sqrt{n+1}\big) + \frac{1}{(n+1)\theta^2} \mathbb{E}[I_h^2]. \tag{6.2}$$

The first term is bounded by $\mathfrak{q}_n(\theta)$.

We next estimate $\mathbb{E}[I_h^2]$. By Cauchy–Schwarz,

$$I_h^2 \le \frac{1}{((h-1)!)^2} \left(\int_y^1 (1-u)^{h-1} du \right) \left(\int_y^1 (1-u)^{h-1} |P_n^{(h)}(u)|^2 du \right),$$

hence

$$\mathbb{E}[I_h^2] \le \frac{(1-y)^h}{h!(h-1)!} \int_y^1 (1-u)^{h-1} \mathbb{E}|P_n^{(h)}(u)|^2 \, du.$$

Now

$$P_n^{(h)}(u) = \sum_{j=h}^n (j)_h \xi_j u^{j-h}, \qquad (j)_h := j(j-1)\cdots(j-h+1),$$

and the assumptions $\mathbb{E}\xi_j = 0$, $\mathbb{E}\xi_j^2 = 1$, and independence give

$$\mathbb{E}|P_n^{(h)}(u)|^2 = \sum_{j=h}^n (j)_h^2 u^{2j-2h}.$$

Substituting this and enlarging the u-integral from $[y, 1]$ to $[0, 1]$, we get

$$\mathbb{E}[I_h^2] \le \frac{(1-y)^h}{h!} \sum_{j=h}^n (j)_h^2 \frac{(2j-2h)!}{(2j-h)!}.$$

For each $j \ge h$,

$$(j)_h^2 \frac{(2j-2h)!}{(2j-h)!} = (j)_h \prod_{r=0}^{h-1} \frac{j-r}{2j-h-r} \le (j)_h \le j^h,$$

because $2j - h - r \ge j - r$ for every $0 \le r \le h - 1$. Therefore

$$\mathbb{E}[I_h^2] \le \frac{(1-y)^h}{h!} \sum_{j=h}^n j^h \le \frac{(1-y)^h}{h!} n^{h+1}.$$

Using the crude Stirling bound $h! \ge (h/e)^h$, we get

$$\mathbb{E}[I_h^2] \le n \left(\frac{e\, n(1-y)}{h} \right)^h. \tag{6.3}$$

Combining (6.2) and (6.3) proves the stated bound for $[y, 1]$.

For $[-1, -y]$, apply the same argument to $P_n(-x)$. Its coefficients are $((-1)^j \xi_j)_{j=0}^n$. If the law of ξ is not symmetric, these coefficients need not be identically distributed with the original ones; however, the one-sided proof above used only independence, zero means, unit variances, and the endpoint small-ball estimate. These properties are preserved by deterministic sign changes, the derivative second moments are unchanged, and the endpoint small-ball probability is bounded by the signed concentration function $\mathfrak{q}_n$. □

Lemma 6.2 (signed anti-concentration)**.** *Assume that ξ is not almost surely constant. Then there is a constant $C_\xi < \infty$ such that, for every $n \geq 0$ and every $0 < \theta \leq 1$,*

$$\mathfrak{q}_n(\theta) \leq C_\xi(\theta + (n+1)^{-1/2}). \tag{6.4}$$

Proof. For a real random variable Z, let

$$Q_Z(r) := \sup_{u\in\mathbb{R}} \mathbb{P}(Z \in [u, u+r]), \qquad r > 0,$$

be its concentration function. Since ξ is non-degenerate, there are $\rho > 0$ and $c_\xi > 0$ such that

$$Q_\xi(\rho) \leq 1 - c_\xi.$$

Indeed, since ξ is non-degenerate, there are $a < b$ such that $p_- := \mathbb{P}(\xi \leq a) > 0$ and $p_+ := \mathbb{P}(\xi \geq b) > 0$. Taking $\rho := (b-a)/2$, every interval of length ρ misses at least one of $(-\infty, a]$ and $[b, \infty)$, so $Q_\xi(\rho) \leq 1 - \min(p_-, p_+)$. The same bound holds for $-\xi$.

Fix signs $\varepsilon_0, \dots, \varepsilon_n \in \{-1, 1\}$, put $X_j := \varepsilon_j \xi_j$, and let $S_N := \sum_{j=0}^n X_j$, where $N := n+1$. We use the Kolmogorov–Rogozin inequality in the following form [9]: if $0 < r_j \leq R$, then

$$Q_{S_N}(R) \leq C_{\mathrm{KR}} \frac{R}{\sqrt{\sum_{j=0}^n r_j^2(1 - Q_{X_j}(r_j))}}.$$

Let $R := 2\theta\sqrt{N}$. The event $|S_N| \leq \theta\sqrt{N}$ is contained in an interval of length R, so its probability is at most $Q_{S_N}(R)$. If $R \leq \rho$, choose $r_j = R$. Since $Q_{X_j}(R) \leq Q_{X_j}(\rho) \leq 1 - c_\xi$, Kolmogorov–Rogozin gives

$$Q_{S_N}(R) \leq C_\xi N^{-1/2}.$$

If $R > \rho$, choose $r_j = \rho$. Then

$$Q_{S_N}(R) \leq C_\xi \frac{R}{\sqrt{N}} \leq C_\xi \theta.$$

Combining the two cases and taking the supremum over the signs proves (6.4). ☐

Corollary 6.3 (edge strips under the standing moment hypotheses)**.** *Under the standing hypothesis* (H)*, for every fixed $\delta > 0$,*

$$\mathbb{P}\Big(N_{P_n}(J_n) \geq 2\delta \log n\Big) = o\left(\left(\frac{L_n}{n}\right)^{3/8}\right).$$

The same statement holds with P_n replaced by the reversed polynomial $x^n P_n(1/x)$. Consequently, if in addition $p_n^{(\xi)} \asymp (L_n/n)^{3/8}$, then

$$\mathbb{P}((C_n^{(\xi),\delta})^c) = o(p_n^{(\xi)}),$$

which is the edge estimate needed in the proof of Theorem 1.2.

Proof. The variance normalization in (H) implies that ξ is not degenerate, so Lemma 6.2 applies. Let $h_n := \lceil \delta \log n \rceil$. Since $h_n / L_n \to \infty$, for all sufficiently large n the quantity

$$\theta_n := \left(C \frac{L_n}{h_n}\right)^{h_n/3}$$

lies in $(0, 1]$, where C is chosen large enough to absorb the absolute constants in Lemma 6.1. Applying Lemma 6.1 with $y = 1 - L_n/n$, and then Lemma 6.2, gives

$$\mathbb{P}\Big(N_{P_n}(J_n^+) \geq h_n\Big) \leq C_\xi(\theta_n + n^{-1/2}) + \left(C \frac{L_n}{h_n}\right)^{h_n/3}.$$

The same bound holds for J_n^-. Since $L_n = 4\log\log n$ and $h_n = \delta \log n + O(1)$,

$$\theta_n = \exp\Big[-\frac{h_n}{3}\log\frac{h_n}{CL_n}\Big] = n^{-\omega(1)}.$$

Thus each one-sided edge probability is $O(n^{-1/2}) + n^{-\omega(1)}$, which is $o((L_n/n)^{3/8})$. To pass from the one-sided estimates to J_n, note the integer threshold: if both J_n^+ and J_n^- contained fewer than h_n zeros, their total would be at most $2h_n - 2 < 2\delta\log n$. Hence $N_{P_n}(J_n) \geq 2\delta \log n$ implies that at least one endpoint strip contains at least h_n zeros. A union bound gives the displayed estimate for J_n. The reversed polynomial has the same coefficient law as P_n, so the same argument applies to it. If $p_n^{(\xi)} \asymp (L_n/n)^{3/8}$, the displayed estimate is exactly $o(p_n^{(\xi)})$. □

7 A cone-record proof for cone-survival coefficients

This section proves Theorem 1.3. The argument is independent of the Gaussian bulk persistence estimates. It uses the elementary observation that a deterministic cone condition on the even and odd partial sums forces a polynomial to be positive at every nonzero point of $(-1, 1)$; it then applies this condition to record events for a two-dimensional coefficient walk.

Recall that

$$\mathcal{K} = \{(u, v) \in \mathbb{R}^2 : u \geq |v|\}.$$

Lemma 7.1 (Abel cone criterion). *Let*

$$P(x) = \sum_{j=0}^{2m+1} a_j x^j, \qquad a_j \in \mathbb{R},$$

and define

$$E_r := \sum_{i=0}^{r} a_{2i}, \qquad O_r := \sum_{i=0}^{r} a_{2i+1}, \qquad 0 \leq r \leq m.$$

If $(E_r, O_r) \in \mathcal{K}$ for every $0 \leq r \leq m$, and if the pairs (a_{2i}, a_{2i+1}), $0 \leq i \leq m$, are not all zero, then

$$P(x) > 0 \qquad (0 < |x| < 1).$$

If, in addition, $(a_0, a_1) \neq (0, 0)$, then $P(x) > 0$ for every $-1 < x < 1$.

Proof. Write $y = x^2$. Put

$$E(y) := \sum_{i=0}^{m} a_{2i} y^i, \qquad O(y) := \sum_{i=0}^{m} a_{2i+1} y^i.$$

For each sign $\sigma \in \{-1, 1\}$, the partial sums of the coefficients $a_{2i} + \sigma a_{2i+1}$ are

$$E_r + \sigma O_r \geq 0.$$

Abel summation gives, for $0 \leq y < 1$,

$$E(y) + \sigma O(y) = (1 - y) \sum_{r=0}^{m-1} (E_r + \sigma O_r) y^r + (E_m + \sigma O_m) y^m \geq 0.$$

Moreover, if $0 < y < 1$, the two quantities $E(y) + O(y)$ and $E(y) - O(y)$ cannot both vanish, unless all their nonnegative Abel-summation terms vanish. That would force $E_r = O_r = 0$ for all r, contrary to the assumption that these pairs are not all zero.

Finally, for $-1 < x < 1$,

$$P(x) = E(x^2) + xO(x^2) = \frac{1+x}{2}(E(x^2) + O(x^2)) + \frac{1-x}{2}(E(x^2) - O(x^2)).$$

For $0 < |x| < 1$, both weights are positive and at least one of the two bracketed terms is positive, so $P(x) > 0$. If $(a_0, a_1) \neq (0, 0)$, then the cone condition at $r = 0$ gives $a_0 \geq |a_1|$ and hence $P(0) = a_0 > 0$, proving the last assertion. □

Let $\xi_0, \xi_1, \ldots$ be i.i.d. real random variables. For $i \geq 0$, set

$$Z_i := (\xi_{2i+1}, \xi_{2i}), \qquad W_{-1} := 0, \qquad W_m := \sum_{i=0}^{m} Z_i \quad (m \geq 0).$$

Define the cone-record event

$$A_m^{(\xi)} := \{W_m - W_k \in \mathcal{K} \text{ for every } -1 \leq k < m\}. \tag{7.1}$$

Lemma 7.2 (cone records force no nonzero bulk zeros)**.** *Let*

$$G_m := \{(\xi_0, \xi_1, \ldots, \xi_{2m+1}) \neq 0\}.$$

On $A_m^{(\xi)} \cap G_m$*, the reversed polynomial*

$$Q_{2m+1}^{(\xi)}(x) = \sum_{j=0}^{2m+1} \xi_{2m+1-j} x^j$$

has no zero in $(-1, 0) \cup (0, 1)$.

Proof. For $Q_{2m+1}^{(\xi)}$, the even and odd coefficient sums from Lemma 7.1 are

$$E_r = \sum_{s=0}^{r} \xi_{2m+1-2s} = \sum_{i=m-r}^{m} \xi_{2i+1}, \qquad O_r = \sum_{s=0}^{r} \xi_{2m-2s} = \sum_{i=m-r}^{m} \xi_{2i}.$$

Thus

$$(E_r, O_r) = W_m - W_{m-r-1},$$

with the convention $W_{-1} = 0$. The event $A_m^{(\xi)}$ says precisely that these vectors belong to $\mathcal{K}$ for all $0 \leq r \leq m$. On G_m the coefficient pairs are not all zero. Lemma 7.1 therefore implies $Q_{2m+1}^{(\xi)}(x) > 0$ for every $0 < |x| < 1$. □

Lemma 7.3 (record decomposition)**.** *For every i.i.d. coefficient law* ξ,

$$\mathbb{P}(A_m^{(\xi)}) = q_m^{(\xi)}.$$

Moreover, if $0 \leq m < \ell$*, then*

$$\mathbb{P}(A_m^{(\xi)} \cap A_\ell^{(\xi)}) = q_m^{(\xi)} q_{\ell-m-1}^{(\xi)}.$$

Proof. By reversing the order of the independent increments, $\mathbb{P}(A_m^{(\xi)})$ is the probability that the walk $S_r := Z_0 + \cdots + Z_r$, $0 \leq r \leq m$, stays in $\mathcal{K}$. This is exactly $q_m^{(\xi)}$.

For $m < \ell$, let

$$B_{m,\ell} := \{W_\ell - W_k \in \mathcal{K} \text{ for every } m \leq k < \ell\}.$$

The event $B_{m,\ell}$ depends only on the increments $Z_{m+1}, \ldots, Z_\ell$, is independent of $A_m^{(\xi)}$, and has probability $q_{\ell-m-1}^{(\xi)}$. Since $\mathcal{K}$ is closed under addition,

$$A_m^{(\xi)} \cap B_{m,\ell} = A_m^{(\xi)} \cap A_\ell^{(\xi)}.$$

Indeed, if $k < m$, then on $A_m^{(\xi)} \cap B_{m,\ell}$,

$$W_\ell - W_k = (W_\ell - W_m) + (W_m - W_k) \in \mathcal{K} + \mathcal{K} \subseteq \mathcal{K},$$

and the case $m \le k < \ell$ is exactly $B_{m,\ell}$. The reverse inclusion is immediate from the definitions. Independence gives the displayed identity. □

Corollary 7.4 (cone records under divergent cone survival)**.** *If $\sum_{m=0}^{\infty} q_m^{(\xi)} = \infty$, then*

$$\mathbb{P}(A_m^{(\xi)} \text{ i.o.}) > 0.$$

Proof. Let $S_N := \sum_{m=0}^{N} q_m^{(\xi)}$. By assumption $S_N \to \infty$. Using Lemma 7.3,

$$\sum_{m,\ell \le N} \mathbb{P}(A_m^{(\xi)} \cap A_\ell^{(\xi)}) \le S_N + 2 \sum_{0 \le m < \ell \le N} q_m^{(\xi)} q_{\ell-m-1}^{(\xi)} \le S_N + 2S_N^2.$$

The Kochen–Stone lemma [10] yields

$$\mathbb{P}(A_m^{(\xi)} \text{ i.o.}) \ge \limsup_{N\to\infty} \frac{S_N^2}{S_N + 2S_N^2} > 0.$$

□

Proof of Theorem 1.3. By Corollary 7.4, with positive probability there are infinitely many m for which $A_m^{(\xi)}$ holds. Since $\mathbb{E}\xi^2 = 1$, we have $\rho := \mathbb{P}(\xi = 0) < 1$, and hence

$$\sum_{m\ge 0} \mathbb{P}(G_m^c) = \sum_{m \ge 0} \rho^{2m+2} < \infty.$$

By Borel–Cantelli, G_m fails only finitely often almost surely. Therefore, on a positive-probability event, there are infinitely many m for which both $A_m^{(\xi)}$ and G_m hold. Lemma 7.2 gives that $Q_{2m+1}^{(\xi)}$ has no zero in $(-1, 0) \cup (0, 1)$ for every such m. The reciprocal identity

$$Q_{2m+1}^{(\xi)}(x) = x^{2m+1} K_{2m+1}^{(\xi)}(1/x)$$

then shows that $K_{2m+1}^{(\xi)}$ has no zero in $(-\infty, -1) \cup (1, \infty)$. Thus, along these indices,

$$N_{2m+1}^{(\xi)} = N_{K_{2m+1}^{(\xi)}}([-1, 1]).$$

Do's theorem [4], applied under the standard hypothesis, gives

$$\frac{N_{K_n^{(\xi)}}([-1,1])}{\log n} \longrightarrow \frac{1}{\pi} \qquad \text{almost surely.}$$

Intersecting this full-probability event with the positive-probability cone-record event proves

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{N_n^{(\xi)}}{\log n} \le \frac{1}{\pi}\right) > 0.$$

Since $1/\pi < 2/\pi$, almost sure convergence to $2/\pi$ is impossible. □

Lemma 7.5 (Rademacher cone survival)**.** *If ξ is a symmetric Rademacher variable, then there are constants $0 < c < C < \infty$ such that, for every $m \ge 0$,*

$$\frac{c}{m+1} \le q_m^{(\xi)} \le \frac{C}{m+1}.$$

Proof. Apply the linear map $(u, v) \mapsto (u + v, u - v)$, which sends $\mathcal{K}$ to the closed quadrant $[0, \infty)^2$. A transformed Rademacher increment is one of

$$(2,0), \quad (-2,0), \quad (0,2), \quad (0,-2),$$

each with probability $1/4$. Conditional on the number N of horizontal moves among the first $m+1$ transformed increments, the two coordinates are independent one-dimensional simple random walks of respective lengths N and $m+1-N$; the order of the horizontal and vertical moves is irrelevant because each coordinate changes only on its own moves. If

$$b_r := \mathbb{P}(\mathsf{S}_1, \dots, \mathsf{S}_r \geq 0)$$

for a one-dimensional simple symmetric walk (S_j) started at zero, then the classical one-dimensional ballot estimate gives

$$b_r = 2^{-r}\binom{r}{\lfloor r/2 \rfloor} \asymp (r+1)^{-1/2}.$$

Consequently,

$$q_m^{(\xi)} = \mathbb{E}[b_N b_{m+1-N}].$$

The upper bound follows from $b_r \leq C(r+1)^{-1/2}$ and the elementary binomial estimate

$$\mathbb{E}\left[\frac{1}{\sqrt{(N+1)(m+2-N)}}\right] \leq \frac{C}{m+1}.$$

For the lower bound, restrict to the event $(m+1)/3 \leq N \leq 2(m+1)/3$, which has probability bounded below by an absolute positive constant; on this event $b_N b_{m+1-N} \geq c/(m+1)$. This proves the claim. $\square$

Proof of Corollary 1.4. A symmetric Rademacher random variable satisfies the standing hypothesis. By Lemma 7.5, $\sum_m q_m^{(\xi)} = \infty$. The corollary follows from Theorem 1.3. $\square$

Remark 7.6 (more general coefficient laws)**.** The cone-record mechanism is not intrinsically tied to the values ± 1. The Abel criterion applies to any real coefficients, and Theorem 1.3 reduces the probabilistic input to the divergent cone-survival condition $\sum_m q_m^{(\xi)} = \infty$ for the planar walk with increments (ξ_{2i+1}, ξ_{2i}). This formulation is useful because it can be checked by standard random-walk-in-cones estimates. For instance, under (H) the increment covariance is the identity, so the Brownian cone exponent of $\mathcal{K}$, a wedge of angle $\pi/2$, is one. Denisov–Wachtel tail asymptotics for random walks in cones [3] therefore give $q_m^{(\xi)} \asymp 1/(m+1)$ whenever the usual cone-entry, non-degeneracy, and lattice or aperiodicity hypotheses needed for those estimates are satisfied. The Rademacher verification above is included because it is short, explicit, and avoids listing those technical support assumptions.

8 Proof of the bulk-persistence theorem and Gaussian conclusion

Proof of Theorem 1.2. Fix $0 < \delta < 1/(2\pi)$. By Corollary 6.3 and Definition 1.1(B1),

$$\mathbb{P}((C_n^{(\xi),\delta})^c) = o(p_n^{(\xi)}).$$

Therefore

$$\mathbb{P}(D_n^{(\xi),\delta}) \geq p_n^{(\xi)} - \mathbb{P}((C_n^{(\xi),\delta})^c) = (1 - o(1))p_n^{(\xi)}.$$

Together with the trivial bound $\mathbb{P}(D_n^{(\xi),\delta}) \le p_n^{(\xi)}$, this implies

$$\sum_n \mathbb{P}(D_n^{(\xi),\delta}) = \infty, \qquad \sum_{n \le N} \mathbb{P}(D_n^{(\xi),\delta}) \asymp N^{5/8} L_N^{3/8}.$$

We claim that $D_n^{(\xi),\delta}$ occurs infinitely often with positive probability. Choose $\varepsilon \in (0, 5/8)$. The diagonal contribution is at most $\sum_{n \le N} p_n^{(\xi)}$, which is $o((\sum_{n \le N} p_n^{(\xi)})^2)$. Since $D_n^{(\xi),\delta} \subseteq B_n^{(\xi)}$, the short-spacing contribution satisfies

$$\sum_{n \le N} \sum_{1 \le h < n^\varepsilon} \mathbb{P}(D_n^{(\xi),\delta} \cap D_{n+h}^{(\xi),\delta}) \le \sum_{n \le N} n^\varepsilon p_n^{(\xi)} = o\left(\left(\sum_{n \le N} p_n^{(\xi)} \right)^2 \right).$$

For $h \ge n^\varepsilon$, Definition 1.1(B2) gives

$$\mathbb{P}(D_n^{(\xi),\delta} \cap D_{n+h}^{(\xi),\delta}) \le \mathbb{P}(B_n^{(\xi)} \cap B_{n+h}^{(\xi)}) \ll_\varepsilon p_n^{(\xi)} p_h^{(\xi)}.$$

Thus

$$\sum_{m,n \le N} \mathbb{P}(D_m^{(\xi),\delta} \cap D_n^{(\xi),\delta}) \ll \left(\sum_{n \le N} p_n^{(\xi)} \right)^2.$$

The Kochen–Stone lemma [10] yields

$$\mathbb{P}(D_n^{(\xi),\delta} \text{ i.o.}) > 0. \tag{8.1}$$

On $D_n^{(\xi),\delta}$, the polynomial $Q_n^{(\xi)}$ has no zero in I_n and at most $2\delta \log n$ zeros in J_n, hence

$$N_{Q_n^{(\xi)}}((-1,1)) \le 2\delta \log n.$$

Every zero x of $K_n^{(\xi)}$ with $|x| > 1$ maps injectively to the zero $1/x \in (-1,1)$ of $Q_n^{(\xi)}$. This one-way injection remains valid even when the leading coefficient vanishes and the degree drops; such a degree drop may create additional zeros of $Q_n^{(\xi)}$ at 0, but those extra zeros can only make the upper bound below easier. Therefore, with

$$O_n^{(\xi)} := N_{K_n^{(\xi)}}((-\infty,-1) \cup (1,\infty)),$$

we have

$$O_n^{(\xi)} \le 2\delta \log n \qquad \text{on } D_n^{(\xi),\delta}. \tag{8.2}$$

Intersecting the positive-probability event in (8.1) with the full-probability event on which (1.1) holds, we obtain a positive-probability event on which there is a subsequence $n_j \to \infty$ with $D_{n_j}^{(\xi),\delta}$ for every j, and also

$$\frac{N_{K_{n_j}^{(\xi)}}([-1,1])}{\log n_j} \to \frac{1}{\pi}.$$

Using (8.2), along this subsequence

$$\frac{N_{n_j}^{(\xi)}}{\log n_j} \le \frac{N_{K_{n_j}^{(\xi)}}([-1,1])}{\log n_j} + 2\delta.$$

Taking $j \to \infty$ proves

$$\liminf_{n \to \infty} \frac{N_n^{(\xi)}}{\log n} \le \frac{1}{\pi} + 2\delta$$

on an event of positive probability. Since $\delta < 1/(2\pi)$, the value $1/\pi + 2\delta$ is strictly smaller than $2/\pi$, so the almost sure convergence $N_n^{(\xi)}/\log n \to 2/\pi$ cannot hold with probability one. □

Corollary 8.1 (Gaussian Kac polynomials)**.** *Let*

$$K_n(x) = \sum_{j=0}^{n} g_j x^j, \qquad g_0, g_1, \ldots \overset{\text{i.i.d.}}{\sim} N(0,1),$$

and let N_n be the number of real zeros of K_n. Then, for every $0 < \delta < 1/(2\pi)$,

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{N_n}{\log n} \le \frac{1}{\pi} + 2\delta\right) > 0.$$

In particular,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{N_n}{\log n} = \frac{2}{\pi}\right) < 1.$$

Proof. The standard Gaussian law satisfies the standing hypothesis. The one-time bulk estimate, Theorem 4.1, gives Definition 1.1(B1). Theorem 5.2 gives the pair estimate in the explicit form

$$\mathbb{P}(B_n \cap B_{n+h}) \ll_\varepsilon \left(\frac{L_n}{n}\right)^{3/8} \left(\frac{L_h}{h}\right)^{3/8}.$$

Using the lower bound in Theorem 4.1, $p_r \gg (L_r/r)^{3/8}$, this is precisely $\mathbb{P}(B_n \cap B_{n+h}) \ll_\varepsilon p_n p_h$, as required in Definition 1.1(B2). Thus the standard Gaussian law is bulk-persistent, and Theorem 1.2 applies. □

Remark 8.2. The bulk-persistence theorem separates the proof into a deterministic–probabilistic reduction and ensemble-specific inputs. Under the standing moment hypotheses, the strong law (1.1) follows from Do's theorem [4], and the edge input follows from the Kolmogorov–Rogozin bound in Lemma 6.2 once the one-time bulk scale Definition 1.1(B1) is known. The one-time and two-time bulk-gap estimates in Definition 1.1(B1)–(B2) are separate ensemble-specific persistence inputs; they are not proved here for general laws satisfying the standing assumptions and are not supplied by Do's theorem alone. The cone-survival theorem bypasses those inputs by using explicit cone-record events instead of Gaussian-order bulk gaps; the Rademacher corollary is the simplest explicit instance.

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