On Erdős problem #521 for Gaussian Kac polynomials

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Abstract

Let

$$K_n(x) = \sum_{j=0}^{n} g_j x^j, \qquad g_0, g_1, \ldots \overset{\text{i.i.d.}}{\sim} N(0,1),$$

and let R_n be the number of real zeros of K_n. We prove that for every $0 < \delta < 1/(2\pi)$,

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{R_n}{\log n} \le \frac{1}{\pi} + 2\delta\right) > 0.$$

Consequently,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{R_n}{\log n} = \frac{2}{\pi}\right) < 1.$$

The proof combines three ingredients: a sharp Gaussian-power-series bulk-gap estimate transferred to the reversed Kac polynomial, a corresponding two-time pair bound, and a strong upper tail showing that the endpoint strips of width L_n/n with $L_n = 4\log\log n$ contain at most $O(\log n)$ roots with overwhelming probability.

1 Setup

Let

$$K_n(x) = \sum_{j=0}^{n} g_j x^j, \qquad g_0, g_1, \ldots \overset{\text{i.i.d.}}{\sim} N(0,1),$$

and let R_n denote the number of real zeros of K_n. Define the reversed polynomial

$$Q_n(x) := x^n K_n(1/x) = \sum_{j=0}^{n} g_{n-j} x^j.$$

For each fixed n, Q_n has the same law as K_n.

For a polynomial P and a set $A \subset \mathbb{R}$, let $N_P(A)$ denote the number of real zeros of P in A, counted without multiplicity. Since Gaussian Kac polynomials have only simple real zeros almost surely, the multiplicity convention does not affect any of the probabilistic statements below.

Fix

$$L_n := 4\log\log n \qquad (n \ge 3),$$

and define the bulk interval and bulk-gap event

$$I_n := [-1 + L_n/n,\, 1 - L_n/n], \qquad B_n := \{N_{Q_n}(I_n) = 0\}.$$

Also write

$$J_n^+ := [1 - L_n/n, 1], \qquad J_n^- := [-1, -1 + L_n/n], \qquad J_n := J_n^+ \cup J_n^-.$$

By countable intersection, there is a full-probability event Ω_* on which, for every $n \geq 0$, one has $g_n \neq 0$, $K_n(\pm 1) \neq 0$, and all real zeros of K_n are simple. We shall use this event for pathwise identifications; all probability estimates are unchanged after discarding its null complement. On Ω_*, under the reciprocal map $x \mapsto 1/x$, the zeros of Q_n in $(-1,1)$ correspond exactly, with multiplicity, to the zeros of K_n in $(-\infty,-1) \cup (1,\infty)$.

The proof has three steps.

1. Prove the correct-order one-time estimate
$$\mathbb{P}(B_n) \asymp (L_n/n)^{3/8}.$$
2. Prove the pair bound
$$\mathbb{P}(B_n \cap B_{n+h}) \ll_\varepsilon (L_n/n)^{3/8}(L_h/h)^{3/8} \qquad (h \geq n^\varepsilon).$$
3. Show that the edge strips $J_n^\pm$ contain at most $\delta \log n$ roots each with probability $1 - n^{-\omega(1)}$.

Combining these three facts gives, with positive probability, infinitely many n for which Q_n has no roots in the bulk and only $O(\log n)$ roots in the edge strips, hence K_n has only $O(\log n)$ roots outside $[-1,1]$. Yen Q. Do's theorem on $[-1,1]$ [3] will then be combined with this positive-probability event to rule out almost sure convergence of $R_n/\log n$ to $2/\pi$.

Throughout, $n^{-\omega(1)}$ denotes a quantity that is $O_A(n^{-A})$ for every fixed $A > 0$.

2 The limiting stationary process

Set
$$x_t := \tanh\frac{t}{2}, \qquad t \in \mathbb{R}.$$
Then $x_t \in (-1,1)$ and $x_{-t} = -x_t$. For each $n \geq 3$, define
$$T_n := \log\frac{2n - L_n}{L_n}.$$
A direct calculation gives
$$x_{T_n} = 1 - \frac{L_n}{n}, \qquad x_{-T_n} = -1 + \frac{L_n}{n},$$
so x_t maps $[-T_n, T_n]$ onto I_n.

Let
$$\mathcal{G}(x) := \sum_{j=0}^{\infty} \xi_j x^j, \qquad \xi_0, \xi_1, \ldots \overset{\text{i.i.d.}}{\sim} N(0,1),$$
be the Gaussian power series, and define
$$Y_t := \sqrt{1 - x_t^2}\, \mathcal{G}(x_t), \qquad t \in \mathbb{R}.$$
Then Y is a centered stationary Gaussian process with covariance
$$\mathbb{E}[Y_s Y_t] = \operatorname{sech}\Big(\frac{s-t}{2}\Big).$$
Indeed,
$$\mathbb{E}[Y_s Y_t] = \frac{\sqrt{(1-x_s^2)(1-x_t^2)}}{1 - x_s x_t} = \frac{1}{\cosh((s-t)/2)}.$$

Let

$$d(T) := \mathbb{P}(Y_t > 0 \ \forall |t| \le T), \qquad T \ge 0.$$

FitzGerald, Tribe and Zaboronski [4] prove that the zero set of the Gaussian power series on $(-1,1)$, after the map

$$u := \Phi(x) := \frac{1}{2} \log \frac{1+x}{1-x} = \operatorname{arctanh} x,$$

is a stationary Pfaffian point process and satisfies the sharp gap estimate

$$\log \mathbb{P}(\mathcal{G} \text{ has no real zero in } [a,b]) = -\frac{3}{8}(\Phi(b) - \Phi(a)) + O(1) \qquad (a \downarrow -1,\ b \uparrow 1).$$

In our parametrization $x_t = \tanh(t/2)$, one has $\Phi(x_t) = t/2$. Therefore the event

$$\{Y_t \ne 0 \ \forall |t| \le T\}$$

is exactly the event that $\mathcal{G}$ has no real zero in $[x_{-T}, x_T]$, and

$$\Phi(x_T) - \Phi(x_{-T}) = T.$$

Hence

$$\log \mathbb{P}(Y_t \ne 0 \ \forall |t| \le T) = -\frac{3}{8} T + O(1),$$

and therefore

$$\mathbb{P}(Y_t \ne 0 \ \forall |t| \le T) \asymp e^{-3T/8}.$$

Since the paths are continuous and the law is invariant under $Y \mapsto -Y$, the two no-zero sign events have equal probability,

$$d(T) \asymp e^{-3T/8}. \tag{2.1}$$

We also need the elementary fact that a small barrier changes the persistence probability by at most an $e^{O(\delta T)}$ factor.

Lemma 2.1 (barrier stability)**.** *There exists $C < \infty$ such that for all $T \ge 1$ and all $0 \le \delta \le 1$,*

$$d(T) \le \mathbb{P}(Y_t > -\delta \ \forall |t| \le T) \le e^{C\delta T} d(T),$$

and

$$e^{-C\delta T} d(T) \le \mathbb{P}(Y_t > \delta \ \forall |t| \le T) \le d(T).$$

Proof. The case $\delta = 0$ is trivial, so assume $0 < \delta \le 1$. Let $I_T = [-T, T]$ and let K_T be the covariance operator on $L^2(I_T)$,

$$(K_T f)(t) := \int_{-T}^{T} \operatorname{sech}\Big(\frac{t-s}{2}\Big) f(s)\, ds.$$

Set

$$c_0 := \int_0^1 \operatorname{sech}(u/2)\, du > 0, \qquad h_\delta(s) := \frac{\delta}{c_0} \mathbf{1}_{I_T}(s), \qquad m_{T,\delta} := K_T h_\delta.$$

The function $m_{T,\delta} = K_T h_\delta$ belongs to the Cameron–Martin space $\mathcal{H}_T$ of the restriction of Y to I_T, and its Cameron–Martin norm is given by

$$\|m_{T,\delta}\|^2_{\mathcal{H}_T} = \langle h_\delta, K_T h_\delta \rangle_{L^2(I_T)}.$$

CLARIFICATION: equivalently, the Gaussian linear functional associated with h_δ is

$$\langle h_\delta, Y \rangle := \int_{-T}^{T} h_\delta(s) Y_s\, ds,$$

and it has variance

$$\mathrm{Var}\langle h_\delta, Y\rangle = \langle h_\delta, K_T h_\delta\rangle_{L^2[-T,T]}.$$

Thus the Cameron–Martin formula applies to the shift $Y \mapsto Y + m_{T,\delta}$ with cost

$$\exp\left\{\frac{1}{2}\|m_{T,\delta}\|^2_{\mathcal{H}_T}\right\}.$$

After the change of variables $u = t - s$ we have

$$m_{T,\delta}(t) = \frac{\delta}{c_0}\int_{t-T}^{t+T} \mathrm{sech}\Big(\frac{u}{2}\Big)\, du.$$

Let $k(u) := \mathrm{sech}(u/2)$. Then k is even and strictly decreasing on $[0,\infty)$. For $t \in [0,T]$,

$$\frac{d}{dt}\int_{t-T}^{t+T} k(u)\, du = k(t+T) - k(t-T) = k(t+T) - k(T-t) \le 0.$$

By evenness, the same monotonicity holds as $|t|$ increases on $[-T,T]$, so the integral is minimized at $t = \pm T$. Therefore

$$m_{T,\delta}(t) \ge \frac{\delta}{c_0}\int_0^{2T} \mathrm{sech}(u/2)\, du \ge \frac{\delta}{c_0}\int_0^{1} \mathrm{sech}(u/2)\, du = \delta,$$

where we used $T \ge 1$ in the second inequality. Moreover, because $\int_{\mathbb{R}} \mathrm{sech}(u/2)\, du < \infty$,

$$\|m_{T,\delta}\|^2_{\mathcal{H}_T} \le C_1\delta^2 T$$

for some absolute $C_1 < \infty$.

Let

$$\mathcal{A}_T := \{f \in C(I_T) : f(t) > 0\ \forall t \in I_T\}.$$

Since $m_{T,\delta} \ge \delta$ on I_T,

$$\{Y_t > -\delta\ \forall |t| \le T\} \subseteq \{Y + m_{T,\delta} \in \mathcal{A}_T\}.$$

The Cameron–Martin formula gives

$$\mathbb{P}(Y + m_{T,\delta} \in \mathcal{A}_T) = \mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_T}(Y)\exp\Big(\langle h_\delta, Y\rangle - \tfrac{1}{2}\|m_{T,\delta}\|^2_{\mathcal{H}_T}\Big)\Big].$$

Apply Hölder with $p = 1+\delta$ and $q = (1+\delta)/\delta$. Since $\langle h_\delta, Y\rangle$ is centered Gaussian with variance $\|m_{T,\delta}\|^2_{\mathcal{H}_T}$,

$$\mathbb{P}(Y + m_{T,\delta} \in \mathcal{A}_T) \le d(T)^{1/p}\exp\Big(\frac{q-1}{2}\|m_{T,\delta}\|^2_{\mathcal{H}_T}\Big) \le d(T)^{1/(1+\delta)}e^{C_2\delta T}.$$

By (2.1), $-\log d(T) \ll T$, so

$$d(T)^{-\delta/(1+\delta)} \le e^{C_3\delta T}.$$

Therefore

$$\mathbb{P}(Y_t > -\delta\ \forall |t| \le T) \le e^{C\delta T} d(T)$$

for some $C < \infty$. The lower bound is trivial.

For the positive barrier, let

$$\mathcal{B}_{T,\delta} := \{f \in C(I_T) : f(t) > \delta\ \forall t \in I_T\}.$$

Since $m_{T,\delta} \ge \delta$ on I_T,

$$\{Y \in \mathcal{A}_T\} \subseteq \{Y + m_{T,\delta} \in \mathcal{B}_{T,\delta}\}.$$

Hence

$$d(T) = \mathbb{P}(Y \in \mathcal{A}_T) \le \mathbb{P}(Y + m_{T,\delta} \in \mathcal{B}_{T,\delta}).$$

Applying the Cameron–Martin formula again and then Hölder with the same exponents $p = 1+\delta$ and $q = (1+\delta)/\delta$, we obtain

$$d(T) \le \mathbb{P}(Y + m_{T,\delta} \in \mathcal{B}_{T,\delta}) \le \mathbb{P}(Y_t > \delta \ \forall |t| \le T)^{1/p} e^{C_4 \delta T}.$$

Therefore

$$\mathbb{P}(Y_t > \delta \ \forall |t| \le T) \ge d(T)^p e^{-C_5 \delta T} = d(T)\, d(T)^\delta e^{-C_5 \delta T}.$$

Using again (2.1), we have $d(T)^\delta \ge e^{-C_6 \delta T}$, so

$$\mathbb{P}(Y_t > \delta \ \forall |t| \le T) \ge e^{-C\delta T} d(T)$$

for a possibly larger constant $C < \infty$. The upper bound is trivial. □

3 A uniform approximation on the bulk window

For $|t| \le T_n$, define

$$\Sigma_n(t)^2 := \sum_{j=0}^n x_t^{2j} = \frac{1 - x_t^{2n+2}}{1 - x_t^2}, \qquad X_n(t) := \frac{Q_n(x_t)}{\Sigma_n(t)}.$$

Then B_n is exactly the event that X_n is either strictly positive everywhere on $[-T_n, T_n]$ or strictly negative everywhere on $[-T_n, T_n]$.

Lemma 3.1 (approximation by the limiting process)**.** *For each n, one can construct, on a common probability space, a stationary* sech*-process Y and another stationary* sech*-process $\widetilde{Y}^{(n)}$ such that for all $|t| \le T_n$,*

$$X_n(t) = Y_t + \Delta_n(t), \qquad |\Delta_n(t)| \le 2e^{-L_n} |\widetilde{Y}_t^{(n)}| + 2e^{-2L_n} |Y_t|.$$

Consequently, there exist absolute constants $c, C < \infty$ such that for every $M \ge 1$ and all large n,

$$\mathbb{P}\Big(\sup_{|t| \le T_n} |X_n(t) - Y_t| > CMe^{-L_n} \sqrt{\log n} \Big) \le e^{-cM^2 \log n}.$$

Proof. Since only the marginal law of Q_n enters this lemma, we may realize

$$Q_n(x) = \sum_{j=0}^n \xi_j x^j$$

on the same probability space as $\mathcal{G}$. Then

$$Q_n(x) = \mathcal{G}(x) - x^{n+1} \widetilde{\mathcal{G}}_n(x),$$

where

$$\widetilde{\mathcal{G}}_n(x) := \sum_{j=0}^\infty \xi_{n+1+j} x^j.$$

Define

$$\widetilde{Y}_t^{(n)} := \sqrt{1 - x_t^2}\, \widetilde{\mathcal{G}}_n(x_t).$$

Then $\widetilde{Y}^{(n)}$ has the same stationary sech-process law as Y (although, in general, it is not independent of Y). No independence between these two processes is needed here. Since

$$\Sigma_n(t) = \frac{1}{\sqrt{1 - x_t^2}} \sqrt{1 - x_t^{2n+2}},$$

we get

$$X_n(t) = \frac{Y_t - x_t^{n+1}\widetilde{Y}_t^{(n)}}{\sqrt{1 - x_t^{2n+2}}}.$$

Write

$$\Delta_n(t) := X_n(t) - Y_t.$$

On $|t| \le T_n$ one has $|x_t| \le 1 - L_n/n$, so

$$|x_t|^{n+1} \le \Big(1 - \frac{L_n}{n}\Big)^{n+1} \le e^{-L_n}, \qquad |x_t|^{2n+2} \le e^{-2L_n}.$$

For all large n, $e^{-2L_n} \le 1/2$, hence

$$\frac{1}{\sqrt{1 - x_t^{2n+2}}} \le 2, \qquad \Big|\frac{1}{\sqrt{1 - x_t^{2n+2}}} - 1\Big| \le 2e^{-2L_n}.$$

Substituting this into the formula for $X_n(t)$ gives

$$|\Delta_n(t)| \le 2e^{-L_n}|\widetilde{Y}_t^{(n)}| + 2e^{-2L_n}|Y_t|.$$

For the tail estimate, both Y and $\widetilde{Y}^{(n)}$ are stationary Gaussian processes with smooth covariance, so standard entropy bounds together with the Borell–TIS inequality (see, e.g., [1, Section 2.1]) give

$$\mathbb{E}\sup_{|t|\le T_n}|Y_t| + \mathbb{E}\sup_{|t|\le T_n}|\widetilde{Y}_t^{(n)}| \ll \sqrt{\log(2+T_n)} \ll \sqrt{\log n}.$$

Applying Borell–TIS separately to Y and $\widetilde{Y}^{(n)}$, and then using the pointwise bound above, gives the stated Gaussian tail for $\sup_{|t|\le T_n}|X_n(t) - Y_t|$.

CLARIFICATION: For the stationary Gaussian process Y with covariance

$$\mathbb{E}Y_sY_t = \operatorname{sech}\Big(\frac{s-t}{2}\Big),$$

the canonical metric satisfies

$$d_Y(s,t)^2 := \mathbb{E}\big(Y_s - Y_t\big)^2 = 2\Big\{1 - \operatorname{sech}\Big(\frac{s-t}{2}\Big)\Big\}.$$

Using $1 - \operatorname{sech} u \le C\min\{u^2, 1\}$, we obtain

$$d_Y(s,t) \le C\min\{|s-t|, 1\}.$$

Consequently, the covering numbers of $[-T,T]$ in the metric d_Y obey

$$N([-T,T], d_Y, \varepsilon) \le C\frac{T}{\varepsilon}, \qquad 0 < \varepsilon \le 1.$$

Dudley's entropy bound therefore gives

$$\mathbb{E}\sup_{|t|\le T}|Y_t| \le C\int_0^1 \sqrt{\log N([-T,T], d_Y, \varepsilon)}\, d\varepsilon \le C\sqrt{\log T}.$$

$\square$

Remark 3.2. The same proof applies verbatim to a Gaussian polynomial with any prescribed number of coefficients. Here the subscript m denotes the number of coefficients, not the degree: if

$$P_{m-1}(x) = \sum_{r=0}^{m-1} b_r x^r, \qquad b_0, \dots, b_{m-1} \overset{\text{i.i.d.}}{\sim} N(0,1),$$

then we define

$$\Sigma_m(t)^2 := \sum_{r=0}^{m-1} x_t^{2r}, \qquad T_m := \log \frac{2m - L_m}{L_m}.$$

With this convention, the normalized process

$$\frac{P_{m-1}(x_t)}{\Sigma_m(t)}$$

admits the same coupling with a stationary sech-process on $[-T_m, T_m]$, with the same proof and with the tail term x_t^m. We shall use this degree-$(h-1)$ variant for the fresh block F_h below.

4 Bulk gap probabilities

Theorem 4.1 (one-time bulk law)**.** *There exist constants $0 < c < C < \infty$ such that for all sufficiently large n,*

$$c\Big(\frac{L_n}{n}\Big)^{3/8} \le \mathbb{P}(B_n) \le C\Big(\frac{L_n}{n}\Big)^{3/8}.$$

Proof. Since B_n is the event that X_n is strictly of one sign on $[-T_n, T_n]$,

$$\mathbb{P}(B_n) = 2\mathbb{P}(X_n(t) > 0 \ \forall |t| \le T_n).$$

Let $c_{\text{app}}, C_{\text{app}} > 0$ be the constants from Lemma 3.1. Fix $M \ge 1$ and set

$$\eta_n := C_{\text{app}} M e^{-L_n} \sqrt{\log n}.$$

By Lemma 3.1, after coupling X_n with Y,

$$\mathbb{P}\Big(\sup_{|t|\le T_n} |X_n(t) - Y_t| > \eta_n\Big) \le e^{-c_{\text{app}} M^2 \log n}.$$

For all sufficiently large n, one has $\eta_n \le 1$. Therefore

$$\begin{aligned}
\mathbb{P}(Y_t > \eta_n \ \forall |t| \le T_n) &- e^{-cM^2 \log n} \\
&\le \mathbb{P}(X_n(t) > 0 \ \forall |t| \le T_n) \\
&\le \mathbb{P}(Y_t > -\eta_n \ \forall |t| \le T_n) + e^{-cM^2 \log n}.
\end{aligned}$$

Because

$$\eta_n T_n \ll e^{-L_n} (\log n)^{3/2} = (\log n)^{-5/2} \to 0,$$

and $\eta_n \le 1$ for all sufficiently large n, Lemma 2.1 gives

$$\mathbb{P}(Y_t > \eta_n \ \forall |t| \le T_n) \asymp d(T_n) \asymp \mathbb{P}(Y_t > -\eta_n \ \forall |t| \le T_n).$$

Choose M so large that $c_{\text{app}} M^2 > 1$. Then $e^{-c_{\text{app}} M^2 \log n} = o(d(T_n))$ by (2.1). Hence

$$\mathbb{P}(X_n(t) > 0 \ \forall |t| \le T_n) \asymp d(T_n).$$

Using (2.1),

$$\mathbb{P}(B_n) \asymp e^{-3T_n/8}.$$

Finally,

$$e^{-T_n} = \frac{L_n}{2n - L_n} \asymp \frac{L_n}{n},$$

so

$$\mathbb{P}(B_n) \asymp \Big(\frac{L_n}{n}\Big)^{3/8}.$$

□

5 The bulk pair bound

Fix $n \geq 3$ and $h \geq 3$, and write $m = n + h$. Decompose

$$Q_m(x) = F_h(x) + x^h Q_n(x), \qquad F_h(x) := \sum_{r=0}^{h-1} g_{m-r} x^r.$$

The fresh block F_h is independent of $\mathcal{F}_n := \sigma(g_0, \ldots, g_n)$.

For the h-scale bulk window, define

$$T_h := \log \frac{2h - L_h}{L_h}, \qquad \Sigma_h(t)^2 := \sum_{r=0}^{h-1} x_t^{2r}, \qquad |t| \leq T_h,$$

and set

$$X_h(t) := \frac{F_h(x_t)}{\Sigma_h(t)}, \qquad \Psi_{n,h}(t) := \frac{x_t^h Q_n(x_t)}{\Sigma_h(t)}.$$

After increasing the initial threshold if necessary, we may assume that $r \mapsto L_r/r = 4(\log\log r)/r$ is decreasing for all r under consideration. Since $m > h$,

$$[-1 + L_h/h, 1 - L_h/h] \subseteq [-1 + L_m/m, 1 - L_m/m],$$

so

$$B_m \subseteq \{X_h + \Psi_{n,h} \text{ has no zero, hence is strictly of one sign, on } [-T_h, T_h]\}. \tag{5.1}$$

Lemma 5.1 (smallness of the old block)**.** *There exist absolute constants $c, C < \infty$ such that, for all sufficiently large n, h and all $M \geq 1$,*

$$\mathbb{P}\Big(\sup_{|t| \leq T_h} |\Psi_{n,h}(t)| > CMe^{-L_h}\sqrt{\log h}\Big) \leq e^{-cM^2 \log h}.$$

Proof. Write

$$Q_n(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j := g_{n-j}.$$

Let $(a'_j)_{j \geq 0}$ be i.i.d. $N(0,1)$, independent of $\mathcal{F}_n$, and extend Q_n by an independent Gaussian tail:

$$\mathcal{H}_n(x) := Q_n(x) + x^{n+1} \sum_{j=0}^{\infty} a'_j x^j.$$

Then $\mathcal{H}_n$ is an infinite Gaussian power series with i.i.d. standard normal coefficients, and we can write

$$Q_n(x) = \mathcal{H}_n(x) - x^{n+1}\widetilde{\mathcal{H}}_n(x), \qquad \widetilde{\mathcal{H}}_n(x) := \sum_{j=0}^{\infty} a'_j x^j.$$

Define

$$Z_t := \sqrt{1 - x_t^2}\, \mathcal{H}_n(x_t), \qquad \widetilde{Z}_t := \sqrt{1 - x_t^2}\, \widetilde{\mathcal{H}}_n(x_t),$$

so Z and $\widetilde{Z}$ are stationary sech-processes. They need not be independent. Since

$$\Sigma_h(t) = \frac{1}{\sqrt{1-x_t^2}}\sqrt{1-x_t^{2h}},$$

we obtain

$$\Psi_{n,h}(t) = \frac{x_t^h Z_t - x_t^{h+n+1}\widetilde{Z}_t}{\sqrt{1-x_t^{2h}}}.$$

On $|t| \le T_h$ one has $|x_t| \le 1 - L_h/h$, hence

$$|x_t|^h \le e^{-L_h}, \qquad |x_t|^{h+n+1} \le e^{-L_h}, \qquad \frac{1}{\sqrt{1-x_t^{2h}}} \le 2$$

for all large h. Therefore

$$|\Psi_{n,h}(t)| \le 2e^{-L_h}(|Z_t| + |\widetilde{Z}_t|).$$

Fix $M \ge 1$ and let

$$A := \Big\{ \sup_{|t|\le T_h} |\Psi_{n,h}(t)| > CMe^{-L_h}\sqrt{\log h} \Big\}.$$

Since A depends only on Q_n, while the auxiliary tail $(a'_j)_{j\ge 0}$ is independent of Q_n, we may write

$$\mathbb{P}(A) = \mathbb{E}_{Q_n,a'}\,\mathbf{1}_A(Q_n).$$

The pointwise bound above implies on the enlarged space that

$$\mathbf{1}_A(Q_n) \le \mathbf{1}_{\{2e^{-L_h}\sup_{|t|\le T_h}(|Z_t|+|\widetilde{Z}_t|) > CMe^{-L_h}\sqrt{\log h}\}}.$$

The left-hand side does not depend on the auxiliary tail. Therefore this pointwise domination remains valid after averaging over the auxiliary tail, and then over Q_n; equivalently, taking expectation over both Q_n and the auxiliary tail, we obtain

$$\mathbb{P}(A) \le \mathbb{P}\Big(\sup_{|t|\le T_h} (|Z_t| + |\widetilde{Z}_t|) > \tfrac{C}{2} M\sqrt{\log h} \Big).$$

Since Z and $\widetilde{Z}$ are stationary sech-processes, the same entropy estimate and Borell–TIS bound as in Lemma 3.1 (again see [1, Section 2.1]) yield the stated tail estimate after adjusting the constants. □

Theorem 5.2 (bulk pair bound). *Fix $\varepsilon > 0$. There exists $C_\varepsilon < \infty$ such that for all sufficiently large n and all integers $h \ge n^\varepsilon$,*

$$\mathbb{P}(B_n \cap B_{n+h}) \le C_\varepsilon \Big(\frac{L_n}{n}\Big)^{3/8} \Big(\frac{L_h}{h}\Big)^{3/8}.$$

Proof. Fix $\varepsilon > 0$. We only need to consider sufficiently large n and integers $h \ge n^\varepsilon$, so in particular $h \ge 3$ and the monotonicity discussion above applies.

Let $c_{\mathrm{app}}, C_{\mathrm{app}} > 0$ be the constants from the degree-$(h-1)$ variant of Lemma 3.1 furnished by Remark 3.2, and let $c_{\mathrm{old}}, C_{\mathrm{old}} > 0$ be the constants from Lemma 5.1. Set

$$c_0 := \min\{c_{\mathrm{app}}, c_{\mathrm{old}}\}, \qquad C_0 := \max\{C_{\mathrm{app}}, C_{\mathrm{old}}\}.$$

Choose M so large that

$$c_0 M^2 > \frac{3}{8}\Big(1 + \frac{1}{\varepsilon}\Big),$$

and define

$$\rho_h := C_0 M e^{-L_h}\sqrt{\log h}.$$

We now make the conditioning and coupling step explicit. Enlarge the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a product space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ by adjoining an auxiliary Gaussian tail, independent of the entire original σ-field $\mathcal{F}$, so that the fresh block F_h becomes the first h coefficients of an infinite Gaussian power series. The degree-$(h-1)$ approximation lemma from Remark 3.2 then yields, on $(\widehat{\Omega}, \widehat{\mathbb{P}})$, a stationary sech-process $Y^{(h)}$ built from the fresh block together with this auxiliary tail. In particular, $Y^{(h)}$ is not independent of X_h, but it is independent of $\mathcal{F}_n$. Define

$$E_h := \Big\{ \sup_{|t|\le T_h} |X_h(t) - Y_t^{(h)}| \le \rho_h \Big\}.$$

Then E_h depends only on the fresh block and the auxiliary randomness, is independent of $\mathcal{F}_n$, and satisfies

$$\widehat{\mathbb{P}}(E_h^c \mid \mathcal{F}_n) = \widehat{\mathbb{P}}(E_h^c) \le e^{-c_0 M^2 \log h}. \tag{5.2}$$

Also define the $\mathcal{F}_n$-measurable event

$$G_{n,h} := \Big\{ \sup_{|t|\le T_h} |\Psi_{n,h}(t)| \le \rho_h \Big\}.$$

By Lemma 5.1,

$$\mathbb{P}(G_{n,h}^c) \le e^{-c_0 M^2 \log h}. \tag{5.3}$$

Fix an outcome of $\mathcal{F}_n$ for which $G_{n,h}$ holds. Probabilities below are taken on the enlarged product space. Since B_m does not depend on the auxiliary randomness,

$$\mathbb{P}(B_m \mid \mathcal{F}_n) = \widehat{\mathbb{P}}(B_m \mid \mathcal{F}_n) \le \widehat{\mathbb{P}}(B_m \cap E_h \mid \mathcal{F}_n) + \widehat{\mathbb{P}}(E_h^c \mid \mathcal{F}_n).$$

If in addition E_h holds and B_m occurs, then by (5.1) the process $X_h + \Psi_{n,h}$ has no zero and therefore is strictly of one sign on $[-T_h, T_h]$. Hence there exists $\sigma \in \{\pm 1\}$ such that

$$\sigma(X_h(t) + \Psi_{n,h}(t)) > 0 \qquad \text{for all } |t| \le T_h.$$

Since on $E_h \cap G_{n,h}$ one has $|X_h(t) - Y_t^{(h)}| \le \rho_h$ and $|\Psi_{n,h}(t)| \le \rho_h$, it follows that

$$\sigma Y_t^{(h)} \ge \sigma(X_h(t) + \Psi_{n,h}(t)) - |X_h(t) - Y_t^{(h)}| - |\Psi_{n,h}(t)| > -2\rho_h$$

for all $|t| \le T_h$. Therefore, on $G_{n,h}$,

$$\widehat{\mathbb{P}}(B_m \cap E_h \mid \mathcal{F}_n) \le \sum_{\sigma\in\{\pm1\}} \widehat{\mathbb{P}}(\sigma Y_t^{(h)} > -2\rho_h \ \forall |t| \le T_h) = 2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h).$$

Combining this with (5.2), and using that $Y^{(h)}$ has the same law as Y, we obtain on $G_{n,h}$,

$$\mathbb{P}(B_m \mid \mathcal{F}_n) \le 2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) + e^{-c_0 M^2 \log h}.$$

Since the trivial bound $\mathbb{P}(B_m \mid \mathcal{F}_n) \le 1$ always holds, we may write

$$\mathbb{P}(B_m \mid \mathcal{F}_n) \le \mathbf{1}_{G_{n,h}}\Big[2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) + e^{-c_0 M^2 \log h}\Big] + \mathbf{1}_{G_{n,h}^c}.$$

Multiply by $\mathbf{1}_{B_n}$ and take expectations:

$$\begin{aligned}\mathbb{P}(B_n \cap B_m) &= \mathbb{E}[\mathbf{1}_{B_n}\mathbb{P}(B_m \mid \mathcal{F}_n)] \\ &\le \mathbb{P}(B_n)\Big[2\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) + e^{-c_0 M^2 \log h}\Big] + \mathbb{P}(G_{n,h}^c).\end{aligned}$$

Now

$$\rho_h T_h \ll e^{-L_h}(\log h)^{3/2} = (\log h)^{-5/2} \to 0,$$

and $2\rho_h \le 1$ for all sufficiently large h, so Lemma 2.1 gives

$$\mathbb{P}(Y_t > -2\rho_h \ \forall |t| \le T_h) \ll d(T_h) \asymp \Big(\frac{L_h}{h}\Big)^{3/8}.$$

Also, since $c_0 M^2 > 3/8$,

$$e^{-c_0 M^2 \log h} = h^{-c_0 M^2} = o\bigg(\Big(\frac{L_h}{h}\Big)^{3/8}\bigg),$$

so the term $\mathbb{P}(B_n)e^{-c_0 M^2 \log h}$ is absorbed into the main contribution

$$\mathbb{P}(B_n)\Big(\frac{L_h}{h}\Big)^{3/8}.$$

Finally, by (5.3),

$$\mathbb{P}(G_{n,h}^c) \le h^{-c_0 M^2}.$$

Since $h \ge n^{\varepsilon}$ and $c_0 M^2 > \frac{3}{8}(1 + 1/\varepsilon)$,

$$h^{-c_0 M^2} \le C_\varepsilon n^{-3/8} h^{-3/8} \le C_\varepsilon \Big(\frac{L_n}{n}\Big)^{3/8}\Big(\frac{L_h}{h}\Big)^{3/8}$$

for all large n. Combining this with Theorem 4.1,

$$\mathbb{P}(B_n) \ll \Big(\frac{L_n}{n}\Big)^{3/8},$$

we conclude that

$$\mathbb{P}(B_n \cap B_{n+h}) \ll_\varepsilon \Big(\frac{L_n}{n}\Big)^{3/8}\Big(\frac{L_h}{h}\Big)^{3/8}.$$

□

Corollary 5.3 (bulk gaps infinitely often)**.**

$$\mathbb{P}(B_n \text{ i.o.}) > 0.$$

Proof. Fix $\varepsilon \in (0, 5/8)$. Set

$$p_n := \mathbb{P}(B_n) \asymp \Big(\frac{L_n}{n}\Big)^{3/8}.$$

Then

$$S_N := \sum_{n \le N} p_n \asymp N^{5/8} L_N^{3/8}.$$

To apply the Kochen–Stone lemma [5], it suffices to show

$$\sum_{m,n \le N} \mathbb{P}(B_m \cap B_n) \ll S_N^2.$$

By symmetry,

$$\sum_{m,n \le N} \mathbb{P}(B_m \cap B_n) \ll S_N + \sum_{n \le N} \sum_{1 \le h \le N-n} \mathbb{P}(B_n \cap B_{n+h}).$$

Split the inner sum into $h < n^{\varepsilon}$ and $h \ge n^{\varepsilon}$. For the small gaps, use the trivial estimate

$$\mathbb{P}(B_n \cap B_{n+h}) \le p_n,$$

which gives

$$\sum_{n\le N}\sum_{1\le h<n^\varepsilon}\mathbb{P}(B_n\cap B_{n+h})\ll\sum_{n\le N}n^\varepsilon p_n\ll N^{\varepsilon+5/8}L_N^{3/8}=o(S_N^2)$$

because $\varepsilon<5/8$.

For the large gaps, Theorem 5.2 gives

$$\sum_{n\le N}\sum_{h\ge n^\varepsilon}\mathbb{P}(B_n\cap B_{n+h})\ll_\varepsilon\sum_{n\le N}p_n\sum_{h\le N}p_h\ll S_N^2.$$

Hence

$$\sum_{m,n\le N}\mathbb{P}(B_m\cap B_n)\ll S_N^2.$$

Since $\sum_n p_n=\infty$, the Kochen–Stone lemma yields

$$\mathbb{P}(B_n\text{ i.o.})>0.$$

□

6 A strong edge upper tail

The final new ingredient is an upper tail showing that the endpoint strips $J_n^\pm$ contain only very few roots with overwhelming probability.

Lemma 6.1 (one-sided edge upper tail)**.** *There exists an absolute constant $C<\infty$ such that for every $n\ge1$, every $0\le y<1$, and every integer h with $1\le h\le n$,*

$$\mathbb{P}(N_{K_n}([y,1])\ge h)\le C\min\left\{1,\left(C\frac{n(1-y)}{h}\right)^{h/3}\right\}.$$

The same estimate holds for $[-1,-y]$ by symmetry.

Proof. The argument follows the same Rolle–theoretic template as in Can–Nguyen [2], but we make the estimate self-contained.

We first record a deterministic consequence of Rolle's theorem.

Claim. Let $b\in[y,1]$. If $f\in C^h([y,b])$ has at least h distinct zeros in $[y,b]$, then

$$|f(b)|\le\frac{1}{(h-1)!}\int_y^b(b-u)^{h-1}|f^{(h)}(u)|\,du. \tag{6.1}$$

For $h=1$, this is immediate by integrating from a zero of f in $[y,b]$ to the endpoint b. Assume $h\ge2$ and that the claim holds for $h-1$. Let z be the largest zero of f in $[y,b]$. For every $s\in[z,b]$, all h zeros of f lie in $[y,s]$, so Rolle's theorem gives at least $h-1$ distinct zeros of f' in $[y,s]$. Applying the induction hypothesis to f' on $[y,s]$, we obtain

$$|f'(s)|\le\frac{1}{(h-2)!}\int_y^s(s-u)^{h-2}|f^{(h)}(u)|\,du.$$

Since $f(z)=0$,

$$\begin{aligned}|f(b)|&\le\int_z^b|f'(s)|\,ds\\&\le\frac{1}{(h-2)!}\int_z^b\int_y^s(s-u)^{h-2}|f^{(h)}(u)|\,du\,ds\\&=\frac{1}{(h-2)!}\int_y^b|f^{(h)}(u)|\int_{\max\{z,u\}}^b(s-u)^{h-2}\,ds\,du\\&\le\frac{1}{(h-2)!}\int_y^b|f^{(h)}(u)|\frac{(b-u)^{h-1}}{h-1}\,du,\end{aligned}$$

which proves (6.1).

Specializing to $b = 1$, we see that if K_n has at least h distinct zeros in $[y, 1]$, then

$$|K_n(1)| \le I_h := \frac{1}{(h-1)!} \int_y^1 (1-u)^{h-1} |K_n^{(h)}(u)| \, du.$$

Since $N_{K_n}([y,1])$ counts real zeros without multiplicity, the event $\{N_{K_n}([y,1]) \ge h\}$ means that K_n has at least h distinct zeros in $[y,1]$. Since $K_n(1) \sim N(0, n+1)$, for every $\theta > 0$,

$$\mathbb{P}(|K_n(1)| \le \theta\sqrt{n}) \le C\theta.$$

Therefore

$$\mathbb{P}(N_{K_n}([y,1]) \ge h) \le C\theta + \frac{1}{n\theta^2}\mathbb{E}[I_h^2]. \tag{6.2}$$

We next estimate $\mathbb{E}[I_h^2]$. By Cauchy–Schwarz,

$$I_h^2 \le \frac{1}{((h-1)!)^2} \left(\int_y^1 (1-u)^{h-1} du \right) \left(\int_y^1 (1-u)^{h-1} |K_n^{(h)}(u)|^2 du \right),$$

hence

$$\mathbb{E}[I_h^2] \le \frac{(1-y)^h}{h!(h-1)!} \int_y^1 (1-u)^{h-1} \mathbb{E}|K_n^{(h)}(u)|^2 \, du.$$

Now

$$K_n^{(h)}(u) = \sum_{j=h}^{n} (j)_h g_j u^{j-h}, \qquad (j)_h := j(j-1)\cdots(j-h+1),$$

so

$$\mathbb{E}|K_n^{(h)}(u)|^2 = \sum_{j=h}^{n} (j)_h^2 u^{2j-2h}.$$

Substituting this and enlarging the u-integral from $[y,1]$ to $[0,1]$, we get

$$\begin{aligned} \mathbb{E}[I_h^2] &\le \frac{(1-y)^h}{h!(h-1)!} \sum_{j=h}^{n} (j)_h^2 \int_0^1 (1-u)^{h-1} u^{2j-2h} \, du \\ &= \frac{(1-y)^h}{h!} \sum_{j=h}^{n} (j)_h^2 \frac{(2j-2h)!}{(2j-h)!}. \end{aligned}$$

For each $j \ge h$,

$$(j)_h^2 \frac{(2j-2h)!}{(2j-h)!} = (j)_h \prod_{r=0}^{h-1} \frac{j-r}{2j-h-r} \le (j)_h \le j^h,$$

because $2j - h - r \ge j - r$ for every $0 \le r \le h-1$. Therefore

$$\mathbb{E}[I_h^2] \le \frac{(1-y)^h}{h!} \sum_{j=h}^{n} j^h \le \frac{(1-y)^h}{h!} n^{h+1}.$$

Using the crude Stirling bound $h! \ge (h/e)^h$, we get

$$\mathbb{E}[I_h^2] \le n \left(\frac{e\,n(1-y)}{h} \right)^h. \tag{6.3}$$

Substituting (6.3) into (6.2) gives

$$\mathbb{P}(N_{K_n}([y,1]) \ge h) \le C\theta + C\theta^{-2} \left(\frac{e\,n(1-y)}{h} \right)^h.$$

If $e\,n(1-y)/h \geq 1$, then the claimed estimate is trivial after enlarging the absolute constant C. Otherwise choose

$$\theta := \left(\frac{e\,n(1-y)}{h}\right)^{h/3},$$

and conclude that

$$\mathbb{P}(N_{K_n}([y,1]) \geq h) \leq C\left(C\frac{n(1-y)}{h}\right)^{h/3}.$$

This proves the lemma. □

Corollary 6.2 (the edge strips contain very few roots)**.** *Fix $\delta > 0$. Then*

$$\mathbb{P}\Big(N_{K_n}(J_n^+) \geq \delta \log n\Big) = n^{-\omega(1)}, \qquad \mathbb{P}\Big(N_{K_n}(J_n^-) \geq \delta \log n\Big) = n^{-\omega(1)}.$$

Hence

$$\mathbb{P}\Big(N_{K_n}(J_n) \geq 2\delta \log n\Big) = n^{-\omega(1)}.$$

The same statement holds with K_n replaced by Q_n.

Proof. Apply Lemma 6.1 with $y = 1 - L_n/n$ and $h_n := \lceil \delta \log n \rceil$. For all sufficiently large n, one has $1 \leq h_n \leq n$, so Lemma 6.1 applies. Then

$$\mathbb{P}\Big(N_{K_n}(J_n^+) \geq \delta \log n\Big) = \mathbb{P}(N_{K_n}(J_n^+) \geq h_n) \leq C\left(C\frac{L_n}{h_n}\right)^{h_n/3}.$$

Since $L_n = 4 \log\log n$ and $h_n = \delta \log n + O(1)$,

$$\log \frac{h_n}{CL_n} = \log\log n - \log\log\log n + O(1) \to \infty,$$

and therefore

$$\left(C\frac{L_n}{h_n}\right)^{h_n/3} = \exp\left[-\frac{h_n}{3}\log\frac{h_n}{CL_n}\right] = n^{-\omega(1)}.$$

The same argument applies to J_n^- by symmetry, and then the union bound gives the two-sided claim. Since Q_n has the same law as K_n, the same bounds hold for Q_n. □

7 Conclusion

Fix $0 < \delta < 1/(2\pi)$ and define

$$C_n^\delta := \{N_{Q_n}(J_n) \leq 2\delta \log n\}, \qquad D_n^\delta := B_n \cap C_n^\delta.$$

On D_n^δ the reversed polynomial Q_n has no roots in the bulk interval I_n and at most $2\delta \log n$ roots in the edge strips J_n. Therefore

$$N_{Q_n}((-1,1)) \leq 2\delta \log n.$$

By the reciprocal-root correspondence discussed in Section 1, on Ω_* the roots of Q_n in $(-1,1)$ are exactly the roots of K_n in $(-\infty,-1) \cup (1,\infty)$. Let

$$O_n := N_{K_n}((-\infty,-1) \cup (1,\infty)).$$

Then

$$O_n \leq 2\delta \log n \qquad \text{on } \Omega_* \cap D_n^\delta. \tag{7.1}$$

By Corollary 6.2,

$$\mathbb{P}((C_n^\delta)^c) = n^{-\omega(1)} = o\Big((L_n/n)^{3/8}\Big) = o(\mathbb{P}(B_n)).$$

Hence, by Theorem 4.1,

$$\mathbb{P}(D_n^\delta) = \mathbb{P}(B_n)(1+o(1)) \asymp (L_n/n)^{3/8}.$$

In particular,

$$\sum_n \mathbb{P}(D_n^\delta) = \infty.$$

Also,

$$\mathbb{P}(D_n^\delta \cap D_m^\delta) \le \mathbb{P}(B_n \cap B_m).$$

Since $\mathbb{P}(D_n^\delta) \sim \mathbb{P}(B_n)$ and the pair probabilities for D_n^δ are dominated by those for B_n, the same denominator estimate as in Corollary 5.3 applies. Hence exactly the same Kochen–Stone computation [5] gives

$$\mathbb{P}(D_n^\delta \text{ i.o.}) > 0. \tag{7.2}$$

ALTERNATIVELY, since

$$\sum_{n=1}^{\infty} \mathbb{P}((C_n^\delta)^c) < \infty,$$

the Borel–Cantelli lemma implies that C_n^δ occurs for all sufficiently large n, almost surely. Hence, up to a null event,

$$\{D_n^\delta \text{ i.o.}\} = \{B_n \cap C_n^\delta \text{ i.o.}\} = \{B_n \text{ i.o.}\}.$$

In particular, by Corollary 5.3,

$$\mathbb{P}(D_n^\delta \text{ i.o.}) = \mathbb{P}(B_n \text{ i.o.}) > 0.$$

Thus, with positive probability, there exist infinitely many n such that the bulk interval I_n contains no zero of Q_n, while the two endpoint intervals contain at most $2\delta \log n$ zeros in total.

Now we use Do's theorem on $[-1,1]$ [3]: for i.i.d. coefficients with mean 0, variance 1, and bounded $(2+\varepsilon)$-moment,

$$\frac{N_{K_n}([-1,1])}{\log n} \to \frac{1}{\pi} \qquad \text{a.s.}$$

The Gaussian coefficients satisfy these hypotheses. Therefore the event

$$\mathcal{E} := \Omega_* \cap \{D_n^\delta \text{ i.o.}\} \cap \left\{\frac{N_{K_n}([-1,1])}{\log n} \to \frac{1}{\pi}\right\}$$

has positive probability by (7.2), because $\mathbb{P}(\Omega_*) = 1$.

On $\mathcal{E}$ there exists a subsequence $n_j \to \infty$ such that $D_{n_j}^\delta$ holds for every j. By (7.1),

$$\frac{R_{n_j}}{\log n_j} = \frac{N_{K_{n_j}}([-1,1]) + O_{n_j}}{\log n_j} \le \frac{N_{K_{n_j}}([-1,1])}{\log n_j} + 2\delta.$$

Taking $j \to \infty$ and using Do's theorem,

$$\liminf_{n\to\infty} \frac{R_n}{\log n} \le \frac{1}{\pi} + 2\delta \qquad \text{on } \mathcal{E}.$$

Since $\mathbb{P}(\mathcal{E}) > 0$, we have proved the following.

Theorem 7.1. *For every* $0 < \delta < 1/(2\pi)$,

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{R_n}{\log n} \le \frac{1}{\pi} + 2\delta\right) > 0.$$

In particular,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{R_n}{\log n} = \frac{2}{\pi}\right) < 1.$$

Thus the almost sure convergence $R_n/\log n \to 2/\pi$ *is false for Gaussian Kac polynomials.*

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