STUDY PROGRAM: PhD Program in Mathematics		
SEMESTER: Fall and spring 2023/24 (60 hours),		
COURSE TITLE: Bellman function method in analysis and probability		
CLASSES TYPE	**#hours (weekly)**	**Professor/Lecturer**
lectures	**2**	izv. prof. dr. sc. Vjekoslav Kovač
ECTS POINTS: to be filled in by the administration staff		

COURSE GOALS:

The goal of the course is to familiarize students with the so-called Bellman function technique, especially through numerous examples from mathematical analysis and probability theory. We will cover majority of books [1] and [2]. More precisely, book [1] will be used to familiarize with the basic idea, examples from analysis (e.g., embedding theorems, estimates for integral operators) and tricks for finding Bellman functions, while book [2] will complement the material with examples from probability (e.g., martingale estimates). A mathematician who masters this technique will know how to methodically approach many diverse types of problems and can gain a huge advantage in problem solving compared to the majority of scientists who still find this method "mystical" or think that the solution can be found by mere guessing.

COURSE DESCRIPTION:

The technique we will study is a very useful idea in theoretical mathematics. It is named after an applied mathematician Richard E. Bellman, who influenced it with his work in dynamic programming and stochastic optimal control. In probability theory, the idea was first used by Donald L. Burkholder in the 1980s, so it is sometimes called *Burkholder's method*. The method was applied in mathematical analysis and further developed into a systematic theory by Fedor L. Nazarov, Sergei R. Treil and Alexander L. Volberg in a whole series of books and papers from the late 1990s to the present day. The same people coined the term *Bellman function method*. It is used to prove/disprove and strengthen many classical and new results in probability, analysis, partial differential equations and elsewhere, and it is one of the most famous, original, and spectacular techniques today.

Here is a very rough outline of the method.

1. Choose a problem with internal self-similarity.
2. Reduce the problem to bounding an “invariant” quantity.
3. Write an inequality for the above invariant quantity using the self-similarity of the problem.

4. Work backwards: control the invariant quantity assuming that we already have a solution to the inequality.
5. Find a solution to the inequality.

The above scheme will start making sense only after the reader familiarizes themselves with concrete examples. For example, it is generally not clear what kind of self-similarity we are referring to, whether the inequality is algebraic or differential, what solutions we are looking for, etc. Precisely because of its generality, this scheme is applicable in an unimaginably wide variety of situations.

COURSE CONTENTS:

1. Introductory examples. Examples of exact Bellman functions. (12 hours)
2. Connections to the stochastic optimal control. (12 hours)
3. Dyadic models. Martingale inequalities in discrete time. (12 hours)
4. Estimates for singular integrals. (12 hours)
5. Maximal estimates. Estimates for square functions. (12 hours)

STUDENTS' OBLIGATIONS:

Lecture attendance (in class or online), solving homework problems, presenting a seminar or writing an essay.

FINAL EXAM MINIMUM REQUIREMENTS: Lecture attendance.

FINAL EXAM: In order to pass the course students will have to solve *several homeworks* and, either give a *seminar presentation*, or prepare a *written essay* on an advanced topic (confirmed beforehand with the instructor).

PREREQUISITES: Strictly speaking, there is no prerequisite knowledge required for following the course. However, dexterity in the differential and integral calculus of several variable functions is desirable, as well as familiarity with the basic concepts of measure and integration and the basics of probability theory.

LITERATURE:

[1] V. Vasyunin, A. Volberg, *The Bellman function technique in harmonic analysis*, Cambridge Studies in Advanced Mathematics **186**, Cambridge University Press, Cambridge, 2020.

[2] A. Osękowski, *Sharp martingale and semimartingale* inequalities, Mathematics Institute of the Polish Academy of Sciences, Mathematical Monographs (New Series) **72**, Birkhäuser/Springer Basel AG, Basel, 2012.

SUPPLEMENTARY LITERATURE:

[3] F. Nazarov, S. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz **8** (1996), no. 5, 32–162 (na ruskom); English translation: St. Petersburg Math. J. **8** (1997), no. 5, 721–824.

[4] V. Kovač, K. A. Škreb, *Bellman functions and* L^p *estimates for paraproducts*, Probab. Math. Statist. **38** (2018), no. 2, 459–479.

[5] V. Kovač, K. A. Škreb, *Bilinear embedding in Orlicz spaces for divergence-form operators with complex coefficients*, J. Funct. Anal. **284** (2023), no. 9, Paper no. 109884.

[6] K. A. Škreb, *The Bellman function technique for multilinear martingale estimates*, doctoral dissertation, University of Zagreb, 2017.

[7] V. Kovač, *Applications of the Bellman Function Technique in Multilinear and Nonlinear Harmonic Analysis*, doctoral dissertation, University of California, Los Angeles, 2011.